Pythagorean Orthogonality in a Normed Linear Space

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This note presents a proof of the following proposition:

Theorem. *If Pythagorean orthogonality is homogeneous in a normed linear space T then T is an abstract Euclidean space.*

The theorem was originally stated and proved by R. C. James ([**1**], Theorem 5.2) who systematically discusses various characterisations of a Euclidean space in terms of concepts of orthogonality. I came across the result independently and the proof which I constructed is a simplified version of that of James. The hypothesis of the theorem may be stated in the form:

If $\|x\|^2+\|y\|^2=\|x-y\|^2$, *then for all complex numbers* λ, μ

$$\|\lambda x\|^2+\|\mu y\|^2=\|\lambda x-\mu y\|^2. \tag{1}$$

Since a normed linear space is known to be Euclidean if the parallelogram law:

$$\|x-y\|^2+\|x+y\|^2=2(\|x\|^2+\|y\|^2) \tag{2}$$

is valid throughout the space (see [**2**]), it is evidently sufficient to show that (1) implies (2).

Proof. Let $x, y \varepsilon T$; assume that $\|x\| \geqslant \|y\|$, and consider the (continuous) function

$$f(\lambda)=\|x-(y+\lambda x)\|^2-\|x\|^2-\|y+\lambda x\|^2$$

of the real variable λ.

(i) Suppose $f(0)=\|x-y\|^2-\|x\|^2-\|y\|^2 \geqslant 0$.

Then since $f(1)=\|y\|^2-\|x\|^2-\|x+y\|^2 \leqslant 0$, it follows that there exists λ in the interval $[0, 1]$ such that $f(\lambda)=0$.

(ii) Suppose $f(0)<0$.

If $f(-1) \geqslant 0$ then there exists λ in the interval $[-1, 0)$ such that $f(\lambda) = 0$.

On the other hand if $f(-1) = \|2x-y\|^2 - \|x\|^2 - \|x-y\|^2 < 0$ then

$$\begin{aligned} f(-2) &= \|3x-y\|^2 - \|x\|^2 - \|2x-y\|^2 \\ &> \|3x-y\|^2 - 2\|x\|^2 - \|x-y\|^2 \\ &> \|3x-y\|^2 - 3\|x\|^2 - \|y\|^2 \\ &\geqslant \|3x\|^2 + \|y\|^2 - 2\|3x\| . \|y\| - 3\|x\|^2 - \|y\|^2 \\ &= 6\|x\|(\|x\| - \|y\|) \geqslant 0; \end{aligned}$$

and so there exists λ in the interval $[-2, -1)$ such that $f(\lambda) = 0$.

Now let λ^* be any real zero of the function $f(\lambda)$ and write $z = y + \lambda^* x$, so that $\|x\|^2 + \|z\|^2 = \|x-z\|^2$.

Then

$$\begin{aligned} \|x-y\|^2 + \|x+y\|^2 &= \|x-z+\lambda^* x\|^2 + \|x+z-\lambda^* x\|^2 \\ &= \|x(1+\lambda^*)-z\|^2 + \|x(1-\lambda^*)+z\|^2 \\ \text{[by (1)]} \quad &= (1+\lambda^*)^2\|x\|^2 + \|z\|^2 + (1-\lambda^*)^2\|x\|^2 + \|z\|^2 \\ &= 2\|x\|^2 + 2(\lambda^{*2}\|x\|^2 + \|z\|^2) \\ \text{[by (1)]} \quad &= 2\|x\|^2 + 2\|y\|^2; \end{aligned}$$

which establishes (2) and thus completes the proof of the theorem.

REFERENCES.

[1] R. C. James, "Orthogonality in normed linear spaces", *Duke Math. J.*, 12 (1945), 291–302.

[2] P. Jordan and J. von Neumann, "On inner products in linear metric spaces", *Annals of Math.*, 36 (1935), 719–723.

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