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SYLOW SUBGROUPS OF TRANSITIVE PERMUTATION GROUPS II

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Abstract

Let G be a transitive permutation group on a finite set of n points, and let P be a Sylow p-subgroup of G for some prime p dividing $|G|$. We are concerned with finding a bound for the number f of points of the set fixed by P. Of all the orbits of P of length greater than one, suppose that the ones of minimal length have length q, and suppose that there are k orbits of P of length q. We show that $f \leqq kp - i_p(n)$, where $i_p(n)$ is the integer satisfying $1 \leqq i_p(n) \leqq p$ and $n + i_p(n) \equiv 0 \pmod p$. This is a generalisation of a bound found by Marcel Herzog and the author, and this new bound is better whenever P has an orbit of length greater than the minimal length q.

Consider the hypothesis:

(*) (i) G is a transitive permutation group on a set Ω of n points;

(ii) P is a Sylow p-subgroup of G for some prime p dividing $|G|$, and P fixes exactly f points of Ω.

This paper continues an investigation, begun in Herzog and Praeger (1976)) and Praeger (1973), of the problem of finding bounds for f. In the previous papers it was shown that $f < \frac{1}{2}n$, (in Praeger (1973)), and that $f \leqq tp - i_p(n)$, where t is the number of long P-oribts, and $i_p(n)$ is the integer satisfying $1 \leqq i_p(n) \leqq p$, $n + i_p(n) \equiv 0 \pmod p$, (in Herzog and Praeger (1976)). There are groups which satisfy these bounds and in Herzog and Praeger (1976) a classification of these groups was given. The result of this paper is a generalisation of the bounds obtained in Herzog and Praeger (1976); the new bounds are better whenever P has long orbits of different lengths. Unfortunately I am not able to describe the groups which attain the new bounds as satisfactorily as Theorem 5 of Herzog and Praeger (1976) describes groups which attain the old bounds. Clearly all the groups described there are examples but there are other groups as well. Most of the notation used is standard, (see Wielandt (1964)). If H has a permutation representation on a set Δ we deote by H^{Δ} the constituent of H on Δ, by $\mathrm{fix}_{\Delta} H$ the set of points of Δ fixed by H, and by a *long* orbit of H we mean any orbit containing at least two points.

THEOREM. *Assume that G, P, Ω satisfy (*). Let p^a be the minimal length of the long orbits of P in Ω, for some $a \geqq 1$, and suppose that P has k orbits of length p^a. Then*

$$f \leqq kp - i_p(n),$$

where $i_p(n)$ is the integer satisfying $1 \leqq i_p(n) \leqq p$, and $n + i_p(n) \equiv 0 \pmod p$.

REMARKS. As we mentioned above, all the groups described in Herzog and Praeger (1976) Theorem 5 satisfy $f = kp - i_p(n)$, but there are other groups for which this bound is attained. In particular if $k = 1$ it follows from the theorem that $f = p - i_p(n)$. The following are examples of groups which are doubly transitive on Ω, and such that P has at least two long orbits, but has only one orbit of minimal length p^a:

(i) $PSL(m, q) \leqq G \leqq P\Gamma L(m, q)$ where $m \geqq 3$ and q is a power of p, in its natural representation.

(ii) $G = M_{12}$ on 12 points with $p = 3$.

(iii) $G \geqq A_n$ for appropriate n.

However I do not know if we have sufficient information to characterise such groups.

PROOF. Assume that hypothesis (*) holds and that P has k orbits of minimal length p^a, $a \geqq 1$. If P has exactly k long orbits then the result follows from Herzog and Praeger (1976) Theorem 1. Also the result is trivial if $f = 0$. So suppose that $f \geqq 1$ and that P has an orbit of length greater than p^a. Thus $|P| > p^a$. Let α be a point in an orbit of P of length p^a, and let $Q = P_\alpha$. Then $|P:Q| = p^a$ and so Q is nontrivial.

LEMMA 1. *A Sylow p-subgroup R of the normaliser of Q, $N(Q)$, has exactly f fixed points, has order $|Q|p^b$ for some $1 \leqq b \leqq a$ and all long orbits of R in* $\mathrm{fix}_\Omega Q$ have length p^b.

PROOF. Let R be a Sylow p-subgroup of $N(Q)$. Then as $R \supseteq Q$, clearly $|R| = |Q|p^b$ for some $1 \leqq b \leqq a$. Let P' be a Sylow p-subgroup of G containing R. Then since all long P'-orbits have length at least p^a and since $|P':R| < |P':Q| = p^a$, it follows that $\mathrm{fix}_\Omega R = \mathrm{fix}_\Omega P'$ and so $|\mathrm{fix}_\Omega R| = f$. Also by the same argument Q is the stabiliser in R of any point in a long R-orbit in $\mathrm{fix}_\Omega Q$. Thus all long R-orbits in $\mathrm{fix}_\Omega Q$ have length p^b.

Without loss of generality we may assume that $N(Q) \cap P = R$ is a Sylow p-subgroup of $N_G(Q)$. Suppose that R has j orbits of length p^b in fix Q.

LEMMA 2. *(a) Q fixes p^b points in each of j of the orbits of P of length p^a.*

(b) The number of distinct conjugates of Q by elements of $N(P)$, namely $t = |N(P) : N(P) \cap N(Q)|$, is of the form $t = t'p^{a-b}$ for some positive integer t'.

(c) The union of the fixed point sets of all the conjugates of Q by elements of $N(P)$ consists of the set $\mathrm{fix}_\Omega P$ and exactly jt' orbits of P of length p^a. In particular then, $jt' \leqq k$.

PROOF. (a) Let Γ be one of the j orbits of P of size p^a in which Q fixes a point. Since $|P : Q| = p^a$ then Q is the stabiliser of a point in the action of P on Γ. Hence by Wielandt (1964) 3.7, $R = N_P(Q)$ is transitive on $\mathrm{fix}_\Gamma Q$, and Q is the stabiliser of a point in this action. Thus $|\mathrm{fix}_\Gamma Q| = |R : Q| = p^b$.

(b) The number of distinct conjugates of Q by elements of $N(P)$ is equal to $t = |N(P) : N(P) \cap N(Q)|$. Since a Sylow p-subgroup of $N(Q) \cap N(P)$ has order $|Q|p^b$ it follows that p^{a-b} divides t and so $t = t'p^{a-b}$ for some integer $t' \geqq 1$.

(c) Let $\Gamma_1, \cdots, \Gamma_j$ be the orbits of P of length p^a which contain points fixed by Q. Then from the proof of part (a) it follows that the union $\bigcup_{1 \leqq i \leqq p^{a-b}} (\mathrm{fix}_\Omega Q^{g_i}) = \mathrm{fix}_\Omega P \cup \Gamma_1 \cup \cdots \cup \Gamma_j$, where $\{Q^{g_i}\}$ is the set of distinct conjugates of Q by elements of P.

It is easily seen that the set of $t = t'p^{a-b}$ distinct conjugates of Q by elements of $N(P)$ is a union of t' conjugacy classes of subgroups of P. From our discussion above, it follows that the union of the fixed point sets of the groups in a given one of these conjugacy classes consists of $\mathrm{fix}_\Omega P$ and j orbits of P of length p^a. Moreover, since the groups are stabilisers of points in these orbits, each conjugacy class fixes a distinct set of j orbits of length p^a, and hence the union of the fixed point sets of all the conjugates of Q by elements of $N(P)$ consists of $\mathrm{fix}_\Omega P$ and jt' orbits of P of length p^a. Thus $jt' \leqq k$.

LEMMA 3. *If $\Sigma_1, \cdots, \Sigma_r$, $r \geqq 1$, are the orbits of $N(Q)$ in $\mathrm{fix}_\Omega Q$ which contain at least one long R-orbit, then $|\Sigma_i \cap \mathrm{fix}_\Omega P| = c_i f/(f, t')$, where the c_i are positive integers such that $\Sigma_{1 \leqq i \leqq r} c_i \leqq (f, t')$. Hence $|\Sigma_i| = j_i p^b + c_i f/(f, t')$ where Σ_i contains j_i long R-orbits, and $\Sigma_{1 \leqq i \leqq r} j_i = j$.*

PROOF. First we show that each Σ_i contains at least one point of $\mathrm{fix}_\Omega P$. Let Σ be one of the Σ_i, and suppose that Δ is a long R-orbit in Σ and δ is a point of Δ. Let P' be a Sylow p-subgroup of G_δ containing Q. Since all long P'-orbits have length at least p^a and since $|P' : N(Q) \cap P'| < |P' : Q| = p^a$, it follows that $\mathrm{fix}_\Omega(N(Q) \cap P') = \mathrm{fix}_\Omega P'$. Let R' be a Sylow p-subgroup of $N(Q)$ containing $N(Q) \cap P'$. Then since $\mathrm{fix}_\Omega R' \subseteq \mathrm{fix}_\Omega(N(Q) \cap P') = \mathrm{fix}_\Omega P'$ and since $|\mathrm{fix}_\Omega R'| = |\mathrm{fix}_\Omega P'|$, we have $\mathrm{fix}_\Omega R' = \mathrm{fix}_\Omega P'$, which contains δ. Finally since R and R' are both Sylow p-subgroups of $N(Q)$, then $(R')^g = R$ for some g in $N(Q)$, and $\delta^g \in (\mathrm{fix}\, R' \cap \Sigma)^g = \mathrm{fix}\, R \cap \Sigma = \mathrm{fix}\, P \cap \Sigma$. Thus $|\Sigma \cap \mathrm{fix}\, P| \geqq 1$.

Now by Wielandt (1964) 3.7, $N(P)$ is transitive on $\mathrm{fix}_\Omega P$ of degree f, and since P is a sylow p-subgroup of G and $f > 0$, clearly f is not divisible by p. Thus by Wielandt (1964) 17.1, all orbits of $N(P) \cap N(Q)$ in fix P have length a multiple of $f/(f,t) = f/(f,t')$. It follows that for $1 \leqq i \leqq r$, $|\Sigma_i \cap \mathrm{fix}\, P| = c_i f/(f,t')$ for some positive integer c_i, and $(\Sigma_{i \leqq i \leqq r} c_i) f/(f,t') = |\mathrm{fix}\, P \cap (\bigcup_{1 \leqq i \leqq r} \Sigma_i)| \leqq |\mathrm{fix}\, P| = f$ so that $\Sigma_{1 \leqq i \leqq r} c_i \leqq (f,t')$. Thus by Lemma 1, $|\Sigma_i| = j_i p^b + c_i f/(f,t')$ where Σ_i contains j_i long R-orbits and $\Sigma_{1 \leqq i \leqq r} j_i = j$.

We shall now complete the proof of the theorem. For each $i = 1, \cdots, r$, $N(Q)^{\Sigma_i}$ is a transitive group with a Sylow p-subgroup R^{Σ_i} which has j_i long orbits and $c_i f/(f,t')$ fixed points. Thus by Herzog and Praeger (1976) Theorem 1, $c_i f/(f,t') \leqq j_i p - i_p(c_i f/(f,t'))$ for $i = 1, \cdots, r$. Summing over i we get, $(\Sigma c_i) f/(f,t') \leqq (\Sigma j_i) p - \Sigma i_p (c_i f/(f,t'))$, that is, since $\Sigma j_i = j$,

$$f \leqq j(f,t')p/(\Sigma c_i) - (f,t')(\Sigma i_p(c_i f/(f,t')))/(\Sigma c_i). \tag{1}$$

If $(\Sigma c_i) \geqq 2$ then $f < j(f,t')p/2 \leqq jt'p/2 \leqq kp/2$, by Lemma 2, and $kp/2 \leqq (k-1)p \leqq kp - i_p(n)$ unless $k = 1$. However if $k = 1$ then by Lemma 2, $jt' = 1$, and by Lemma 3, $\Sigma c_i \leqq (f,t') \leqq t' = 1$, a contradiction. Thus if $(\Sigma c_i) \geqq 2$ then the theorem is true, so we assume that $\Sigma c_i = 1$. Then (1) is $f \leqq j(f,t')p - (f,t') i_p(f/(f,t'))$.

By Herzog and Praeger (1976) Lemma 1.1, $(f,t') i_p(f/(f,t')) \geqq i_p(f)$, and as $n \equiv f (\mathrm{mod}\, p)$, so $i_p(f) = i_p(n)$. Also by Lemma 2, $j(f,t') \leqq jt' \leqq k$ and so we have $f \leqq kp - i_p(n)$ and the proof of the theorem is complete.

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