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ALGEBRAIC AND GEOMETRIC THEORY OF THE TOPOLOGICAL RING OF COLOMBEAU GENERALIZED FUNCTIONS

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Abstract We continue the investigation of the algebraic and topological structure of the algebra of Colombeau generalized functions with the aim of building up the algebraic basis for the theory of these functions. This was started in a previous work of Aragona and Juriaans, where the algebraic and topological structure of the Colombeau generalized numbers were studied. Here, among other important things, we determine completely the minimal primes of $\bar{\mathbb{K}}$ and introduce several invariants of the ideals of $\mathcal{G}(\Omega)$. The main tools we use are the algebraic results obtained by Aragona and Juriaans and the theory of differential calculus on generalized manifolds developed by Aragona and co-workers. The main achievement of the differential calculus is that all classical objects, such as distributions, become C^∞-functions. Our purpose is to build an independent and intrinsic theory for Colombeau generalized functions and place them in a wider context.

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1. Introduction

Since Colombeau introduced his definition of generalized functions there has been a great deal of development in the field. The theory has proved to have many useful applications and gives new insight where the classical theory does not (see [**16**]). The algebraic properties of this theory were studied independently by both Aragona and Oberguggenberger, and the theory is given as problem 27.12 in [**16**]. However, it was not until recently that these properties became the subject of a systematic study (see [**2**]). The reason for this was that there did not exist 'good' topologies on the algebras of Colombeau generalized numbers and functions. In [**19**, **20**] Scarpalezos developed natural and Hausdorff topologies. These turned out to be a necessary boost to study of the theory. Taking his work as a starting point, Aragona and Juriaans were able to study many algebraic properties of the now topological ring of Colombeau generalized numbers, $\bar{\mathbb{K}}$. There appeared to be an effective link between the topological and algebraic structures of $\bar{\mathbb{K}}$, which was used in [**2**] to bridge algebra, analysis and topology in the field. This has already proved to be useful: in [**3**] Aragona and Soares used an algebraic result of [**2**] to prove the non-existence of solutions for a certain partial differential equations in the framework of Colombeau theory; this result was generalized by Aragona *et al*. [**4**] using the algebraic theory developed in [**2**] and the differential calculus developed in [**4**]. One of the most important and fundamental algebraic properties of $\bar{\mathbb{K}}$ proved by Aragona and Juriaans in [**2**] is the following: *the unit group of* $\bar{\mathbb{K}}$ *is open and dense and an element* $x \in \bar{\mathbb{K}}$ *is either a unit or a zero divisor*.

In this paper we continue the study of the algebraic properties of the Colombeau generalized numbers. We completely determine its minimal primes. We also start to study the algebraic properties of the topological $\bar{\mathbb{K}}$-module of Colombeau generalized functions $\mathcal{G}(\Omega)$. We are not able to determine completely its maximal and minimal primes but do obtain a wealth of information about its ideals. One interesting fact is that the set of idempotents of $\mathcal{G}(\Omega)$, for Ω open and connected, coincides with that of $\bar{\mathbb{C}}$, which in turn is the same as that of $\bar{\mathbb{R}}$. This fact now gives another explanation of what happens in [**8**, § 2.1]. To prove this we use the important notion of the generalized point value of a generalized function introduced in [**13**], the algebraic results about $\bar{\mathbb{K}}$ obtained in [**2**] and the theory of differential calculus developed in [**4**]. This proves once more how interlinked the topological, algebraic and other properties of these rings are and thus again highlights the importance of the study of the algebraic properties.

The paper is organized as follows: in the next section we recall part of the algebraic theory of [**2**] and also the theory of differential calculus on generalized manifolds introduced in [**4**]. In § 3 we introduce an ordering in $\bar{\mathbb{R}}$ (see also [**12**]) and some of its quotient algebras and develop the machinery that will be used in § 4 to characterize minimal primes of $\bar{\mathbb{K}}$. We also prove the existence of prime ideals which are neither minimal nor maximal primes. This will follow once we prove that $\bar{\mathbb{K}}$ is not von Neumann regular. In § 5 we determine the Boolean algebra of $\mathcal{G}(\Omega)$ and also study its group of units. In § 6 we introduce the notions of *trace and generalized trace of an ideal*, characterize it and use this to study the maximal prime spectrum of $\mathcal{G}(\Omega)$. In § 7 we introduce the *support of an ideal* and once more this is used to study the maximal spectrum of $\mathcal{G}(\Omega)$ and to show the existence of a unique minimal dense ideal. Throughout the paper the geometric and analytic flavour and approach are very clear.

For the theory of Colombeau generalized numbers, functions and their topologies the reader is referred to [**1**, **6**, **7**, **10**, **12**, **16**].

2. Algebraic and geometric theory

In this section we recall the most important algebraic and geometric results. We refer the interested reader to [**2**, **4**] for notation, more details and proofs of the results mentioned here. We start with the algebraic results obtained by Aragona and Juriaans.

The norm of an element $x \in \bar{\mathbb{K}}$ is defined by $\|x\| = D(x, 0)$, where D is the ultra metric in $\bar{\mathbb{K}}$ defined by Scarpalezos. Denote by $\mathrm{Inv}(\bar{\mathbb{K}})$ the unit group of $\bar{\mathbb{K}}$. For details we refer the reader to [**2**].

$\bar{\mathbb{K}}$ is a topologically complete commutative $\mathbb{K}$-algebra which is not local, Artinian, Noetherian or a domain. Its Jacobson and prime radicals are trivial and, as we shall see here, all these properties are also true for $\mathcal{G}(\Omega)$ (some of them were proved in [**2**]). However, the following fundamental fact is true.

Theorem 2.1 (fundamental theorem of $\bar{\mathbb{K}}$ [2]). *Let $x \in \bar{\mathbb{K}}$ be any element. Then one of the following holds:*

(i) $x \in \mathrm{Inv}(\bar{\mathbb{K}})$;

(ii) *there exists an idempotent $e \in \bar{\mathbb{K}}$ such that $x \cdot e = 0$.*

In particular, an element of $\bar{\mathbb{K}}$ is either a unit or a zero divisor. Moreover, $\mathrm{Inv}(\bar{\mathbb{K}})$ *is an open and dense subset of $\bar{\mathbb{K}}$.*

This theorem is very important if we intend to extend geometry, analysis and calculus having $\bar{\mathbb{K}}$ rather than a field as the underlying structure. For example, it tells us that $\mathrm{GL}(\bar{\mathbb{K}}, n)$ is an open and dense subgroup of the $n \times n$ matrix algebra $M_n(\bar{\mathbb{K}})$. The following theorem characterizes invertible elements.

Theorem 2.2 (Aragona and Juriaans [2]). *An element $x \in \bar{\mathbb{K}}$ is a unit if and only if there exists $a \geqslant 0$ such that $|\hat{x}(\varepsilon)| \geqslant \varepsilon^a$, for sufficiently small ε.*

Not all of these properties of $\bar{\mathbb{K}}$ carry over to $\mathcal{G}(\Omega)$, as we shall see in the other sections. However, before we study properties of $\mathcal{G}(\Omega)$ we should start by studying the ideals of $\bar{\mathbb{K}}$. It turns out that to do so we must study the characteristic functions of $\boldsymbol{I} :=]0,1]$.

Let $\mathcal{S} := \{S \subset \boldsymbol{I} \mid 0 \in \bar{S} \cap \bar{S}^{\mathrm{c}}\}$, where the bar denotes topological closure. We than denote by $P_*(\mathcal{S})$ the set of all subsets $\mathcal{F}$ of $\mathcal{S}$ which are stable under finite union and such that if $S \in \mathcal{S}$, then either S or S^{c} belongs to $\mathcal{F}$. By $g(\mathcal{F})$ we denote the ideal generated by the characteristic functions of elements of $\mathcal{F}$.

Theorem 2.3 (Aragona and Juriaans [2]). *Let $\mathcal{P} \lhd \bar{\mathbb{K}}$ be a prime ideal. Then*

(i) *there exists $\mathcal{F}_0 \in P_*(\mathcal{S})$ such that $g(\mathcal{F}_0) \subset \mathcal{P}$,*

(ii) *for any $\mathcal{F} \in P_*(\mathcal{S})$, $\overline{g(\mathcal{F})}$ is a maximal ideal of $\bar{\mathbb{K}}$.*

In particular, (ii) describes all maximal ideals of $\bar{\mathbb{K}}$ and hence they are all closed and rare.

If $I \lhd \bar{\mathbb{K}}$ is a maximal ideal, then $\mathbb{K}$ is algebraically closed in $\bar{\mathbb{K}}/I$ and from this it follows that $\mathcal{B}(\bar{\mathbb{K}})$, the set of idempotents of $\bar{\mathbb{K}}$, does not depend on $\mathbb{K}$, i.e. $\mathcal{B}(\bar{\mathbb{C}}) = \mathcal{B}(\bar{\mathbb{R}})$.

Let $r \in \mathbb{R}$. Then $\alpha_r \in \bar{\mathbb{R}}$ is the element having $\varepsilon \to \varepsilon^r$ as a representative. It has the property that $\|\alpha_r\| = e^{-r}$ and $\|\alpha_r x\| = \|\alpha_r\| \cdot \|x\|$ for any $x \in \bar{\mathbb{K}}$. In particular, we have that $\|\alpha_{-\log(\|x\|)}\| = \|x\|$.

Given $n \in \mathbb{N}$, we equip $\bar{\mathbb{K}}^n$ with the product topology. Let $U \subset \bar{\mathbb{K}}^n$ be an open subset, let $f : U \to \bar{\mathbb{K}}$ and let $x_0 \in U$. We say that f is differentiable at x_0 if there exists $z_0 \in \bar{\mathbb{K}}$ such that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0) - z_0(x - x_0)}{\alpha_{-\log\|x - x_0\|}} = 0.$$

Note that

$$\left\| \frac{f(x) - f(x_0) - z_0(x - x_0)}{\alpha_{-\log\|x - x_0\|}} \right\| = \frac{\|f(x) - f(x_0) - z_0(x - x_0)\|}{\|x - x_0\|},$$

and so our definition is a generalization of the Fréchet derivative. If f is differentiable at x_0, we write $D(f)(x_0) = z_0$. One can check that D indeed is a derivation which satisfies all the classical rules of calculus.

Let $\Omega \subset \bar{\mathbb{K}}^n$ be an open subset and let $\boldsymbol{I}_\eta :=]0, \eta[$ for $\eta \in \boldsymbol{I}$. Define

$$\Omega_M = \{(x_\varepsilon) \in \Omega^{\boldsymbol{I}} \mid \exists p > 0,\ \eta > 0 \text{ with } |x_\varepsilon| \leqslant \varepsilon^{-p} \text{ for all } \varepsilon \in \boldsymbol{I}_\eta\},$$

and that $(x_\varepsilon), (y_\varepsilon) \in \Omega_M$ are equivalent, $(x_\varepsilon) \sim (y_\varepsilon)$, if and only if, given any $q > 0$, there exists $\eta > 0$ such that $|x_\varepsilon - y_\varepsilon| \leqslant \varepsilon^q$ for all $\varepsilon \in \boldsymbol{I}_\eta$. Let $\tilde{\Omega} := \Omega_M/{\sim}$. Note that if $\Omega = \mathbb{K}$, then $\tilde{\Omega} = \bar{\mathbb{K}}$ and that $\tilde{\mathbb{K}}^n = \bar{\mathbb{K}}^n$.

An element $x \in \Omega_M$ is said to be *compactly supported* if it has a representative (x_ε) and there exists a compact subset K of Ω such that $x_\varepsilon \in K$ for ε sufficiently small. Define $\tilde{\Omega}_{\mathrm{c}} = \{x \in \tilde{\Omega} \mid x \text{ is compactly supported}\}$.

We can embed Ω into $\tilde{\Omega}_{\mathrm{c}}$ by the mapping $x \in \Omega \longmapsto \mathrm{cl}(x_\varepsilon) \in \tilde{\Omega}_{\mathrm{c}}$, where $x_\varepsilon = x$ for all $\varepsilon \in \boldsymbol{I}$. Note that the image of Ω is a discrete subset of $\bar{\mathbb{K}}^n$.

Let $f \in \mathcal{G}(\Omega)$, let $x \in \tilde{\Omega}_{\mathrm{c}}$ and let (x_ε), $\hat{f}$ be representatives of x and f, respectively. Define $\kappa(f)(x) = \mathrm{cl}(\varepsilon \in \boldsymbol{I} \longmapsto \hat{f}(\varepsilon, x_\varepsilon)) \in \bar{\mathbb{K}}$. This is called the *generalized point value of* f at x. This very important notion was introduced by Kunzinger and Oberguggenberger. The following theorem, due to them, tells us that $\tilde{\Omega}_{\mathrm{c}}$, and not Ω, is the natural domain of f.

Theorem 2.4 (Kunzinger and Oberguggenberger [13]). *If $f \in \mathcal{G}(\Omega)$, then $f = 0$ if and only if $\kappa(f)(x) = 0$ for all $x \in \tilde{\Omega}_{\mathrm{c}}$.*

We now come to the following important theorem proved by Aragona *et al*.

Theorem 2.5 (embedding theorem [4]). *Let Ω be an open subset of $\mathbb{R}^n$. The embedding $\kappa : \mathcal{G}(\Omega) \to \mathcal{C}^\infty(\tilde{\Omega}_{\mathrm{c}}, \bar{\mathbb{K}})$ is an injective homomorphism of $\bar{\mathbb{K}}$-algebras. Moreover, κ is continuous and*

$$\kappa\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial(\kappa(f))}{\partial x_i} \quad \textit{for all } f \in \mathcal{G}(\Omega), \quad 1 \leqslant i \leqslant n.$$

It is easily seen that in this new context the composition of generalized functions reduces to the classical one. The theorem tells us that in fact the theory of generalized functions is actually a theory about C^∞-functions. So all irregularities disappear and distributions become C^∞-functions.

Let J be an open interval of $\mathbb{R}$, $f \in \mathcal{G}(J)$ and $a, b \in \tilde{J}_{\mathrm{c}}$. Define

$$\int_a^b \kappa(f) = \left[\varepsilon \longmapsto \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(t)\,\mathrm{d}t\right],$$

where $(a_\varepsilon), (b_\varepsilon)$ and (f_ε) are representatives of a, b and f, respectively, and the second integral is the Riemann integral. Using theorem 2.5, one can show that the *fundamental* *theorem of calculus* is true in this framework, i.e. if J is an open interval of $\mathbb{R}$, $a \in \tilde{J}_{\mathrm{c}}$, $f \in \mathcal{G}(J)$ and F is the function defined on $\tilde{J}_{\mathrm{c}}$ by

$$F(x) = \int_a^x \kappa(f),$$

then F is a differentiable function and

$$F'(x) = D\biggl(\int_a^x \kappa(f)\biggr) = \kappa(f)(x).$$

A *generalized manifold*, or *$\mathcal{G}$-manifold*, of dimension N is a family $\mathcal{A} = ((\mathcal{U}_\lambda, u_\lambda))_{\lambda \in \Lambda}$ whose transition functions take values in $\bar{\mathbb{K}}^N$ and are $\mathcal{C}^\infty$-diffeomorphisms.

Theorem 2.6 (Aragona *et al.* [4]). *Let $(M, \mathcal{A})$ be a $\mathcal{C}^\infty$-differentiable manifold of dimension N, where $\mathcal{A} = ((\mathcal{U}_\lambda, \varphi_\lambda))_{\lambda \in \Lambda}$. There then exists a $\mathcal{G}$-manifold*

$$M^* := \bigcup_{\lambda \in \Lambda} U^\circ_\lambda(\varphi_\lambda)$$

and

$$\mathcal{A}^* := ((U^\circ_\lambda(\varphi_\lambda), \varphi^\circ_\lambda))_{\lambda \in \Lambda}$$

such that any differentiable map defined on M induces a differentiable map defined on M^*.

With this, and with what follows in the next sections, we believe that we have set part of the algebraic, geometric and calculus basis to carry over all, or most, of the classical theorems and constructions of Riemannian and pseudo-Riemannian geometry and calculus on manifolds (see, for example, [**9**]) to the context of Colombeau generalized functions. Just to give some easy examples, let X be a differentiable manifold of dimension N, let $\tilde{X}_{\mathrm{c}}$ be the set of compactly supported generalized points on X and let $\mathcal{G}(X)$ be the Colombeau algebra of generalized functions on the manifold X [**14**]. If $(X^*, \mathcal{A}^*)$ is the $\mathcal{G}$-manifold associated with the manifold $(X, \mathcal{A})$, then $X^* \subset \tilde{X}_{\mathrm{c}}$ and, given $U \in \mathcal{G}(X)$, the function $\tilde{U} : X^* \to \bar{\mathbb{K}}$ defined by $\tilde{U}(\tilde{p}) = U(\tilde{p})$ for all $\tilde{p} \in X^*$ is well defined and is a differentiable function between $\mathcal{G}$-manifolds. Also, if $f \in \mathcal{G}(\mathbb{R})$ is such that $D(f)(x) \in \mathrm{Inv}(\bar{\mathbb{K}})$ for all $x \in \tilde{\mathbb{R}}_{\mathrm{c}}$, then f is injective. Using the fact that $\mathrm{GL}(\bar{\mathbb{K}}, n)$ is an open set this observation extends, in the usual way, to differential maps between subsets of $\bar{\mathbb{K}}^n$. The interested reader should see [**4**] for proofs, definitions and more applications. In the following sections we shall need these tools.

3. Ordering $\bar{\mathbb{R}}$

In this section we introduce a partial order in $\bar{\mathbb{R}}$ which we shall prove to induce a total order in every residual class field. Actually, we prove a stronger result which will allow us to prove, in § 4, that $\bar{\mathbb{K}}$ contains minimal prime ideals. Before we continue, let us make some rules. If $\mathcal{F} \in P_*(\mathcal{S})$, then $g_r(\mathcal{F})$ denotes the ideal of $\bar{\mathbb{R}}$ generated by the characteristic functions of elements of $\mathcal{F}$ and $g(\mathcal{F})$ the ideal of $\bar{\mathbb{C}}$ generated by the same functions (see [**2**, § 4]). When there is no doubt we shall omit the subscript. The next result is the basis for the definition of our order.

Lemma 3.1. *For a given $x \in \bar{\mathbb{R}}$ the following are equivalent:*

(i) *every representative $\hat{x}$ of x satisfies the condition*

$$\forall b > 0, \quad \exists \eta_b \in \boldsymbol{I} \text{ such that } \hat{x}(\varepsilon) \geqslant -\varepsilon^b \text{ whenever } 0 < \varepsilon < \eta_b; \tag{$*$}$$

(ii) *there exists a representative $\hat{x}$ of x satisfying $(*)$;*

(iii) *there exists a representative x_* of x such that $x_*(\varepsilon) \geqslant 0$, for all $\varepsilon \in \boldsymbol{I}$.*

Proof. (ii) $\implies$ (iii). Define $h : \boldsymbol{I} \to \mathbb{R}$ by $h(\varepsilon) = 0$ (respectively, $-\hat{x}(\varepsilon)$) if $\hat{x}(\varepsilon) \geqslant 0$ (respectively, $\hat{x}(\varepsilon) < 0$). Since $\hat{x}$ satisfies $(*)$ it is clear that $h \in \mathcal{N}(\mathbb{R})$; hence, $x_8 := \hat{h} + h$ is a non-negative representative of x.

For (iii) $\implies$ (i), fix any representative $\hat{x}$ of x and $b > 0$. Since $x_* - \hat{x} \in \mathcal{N}(\mathbb{R})$, there is an $\eta \in \boldsymbol{I}$ such that $x_*(\varepsilon) - \hat{x}(\varepsilon) \leqslant |x_*(\varepsilon) - \hat{x}(\varepsilon)| \leqslant \varepsilon^b$ for all $\varepsilon < \eta$; hence, $\hat{x}(\varepsilon) \geqslant x_*(\varepsilon) - \varepsilon^b \geqslant -\varepsilon^b$ whenever $0 < \varepsilon < \eta$. ☐

Definition 3.2. An element $x \in \bar{\mathbb{R}}$ is said to be non-negative or quasi-positive (q-positive) if it has a representative satisfying one of the conditions of Lemma 3.1. We shall denote this by $x \geqslant 0$. We shall say also that x is non-positive or q-negative if $-x$ is q-positive. If $y \in \bar{\mathbb{R}}$ is another element, then we write $x \geqslant y$ if $x - y$ is q-positive and $x \leqslant y$ if $y - x$ is q-positive.

Remark 3.3. This definition is not a total order in $\bar{\mathbb{R}}$. To see this note that $\hat{x}(\varepsilon) = \varepsilon \sin(\varepsilon^{-1})$ gives rise to an element which is neither q-positive nor q-negative. It does, however, define a partial order such that the sum and product of q-positive elements are q-positive.

Let $x \in \bar{\mathbb{K}}$ and $\hat{x}$ one of its representatives. Then it is easily seen that $|\hat{x}|(\varepsilon) := |\hat{x}(\varepsilon)|$ gives rise to an element $|x| \in \bar{\mathbb{K}}$ which depends only on x.

Definition 3.4. Let $x \in \bar{\mathbb{K}}$. The element $|x|$ is called the *absolute value* of x and, when $x \in \bar{\mathbb{R}}$, $x^+ := \frac{1}{2}(x + |x|)$ and $x^- := \frac{1}{2}(x - |x|)$ are respectively called the *q-positive and q-negative parts* of x. Note that x^+ and x^- depend only on x.

Recall that if $\hat{x}$ is a representative of $x \in \bar{\mathbb{K}}$, then $\theta_{\hat{x}}(\varepsilon) := \exp(\mathrm{i} \arg(\hat{x}(\varepsilon)))$ for all $\varepsilon \in \boldsymbol{I}$ (with $\arg(0) := 0$) and $\Theta_{\hat{x}} \in \bar{\mathbb{K}}$ has $\theta_{\hat{x}}$ as a representative (see [**2**, definition 4.8]).

The following proposition is easily proved.

Proposition 3.5. *Let $x, y \in \bar{\mathbb{R}}$.*

(i) *$x = x^+$ if and only if $x = |x|$ if and only if x is q-positive.*

(ii) *$x = x^-$ if and only if $x = -|x|$ if and only if x is q-negative.*

(iii) *$(-x)^+ = -(x^-)$ and $(-x)^- = -(x^+)$.*

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(iv) $|x| \geqslant 0 \leqslant x^+$, $x^- \leqslant 0 \geqslant -x^+$, $|-x| = |x|$ and $|x| \geqslant x$.

(v) $|x+y| \leqslant |x| + |y|$, $||x| - |y|| \leqslant |x-y|$ *(triangular inequality).*

(vi) *If* $x \leqslant y$ *and* $-x \leqslant y$, *then* $|x| \leqslant y$.

(vii) $x^+ = \frac{1}{2}x(1 + \Theta_{\hat{x}})$ *and* $x^- = \frac{1}{2}x(1 - \Theta_{\hat{x}})$.

(viii) *If* $S = \{\varepsilon \in \boldsymbol{I} \mid \hat{x}(\varepsilon) \geqslant 0\}$, *then* $x^+ = x\chi_S$ *and* $x^- = x\chi_{S^c}$.

Remark 3.6. Note that if $z \in \bar{\mathbb{C}}$, then $|z| \in \bar{\mathbb{R}}$, $|z| \geqslant 0$, and so we may apply Proposition 3.5 where possible. In particular, the triangular inequalities hold in this context.

Proposition 3.7 (convexity of ideals). *Let J be an ideal of $\bar{\mathbb{K}}$ and $x, y \in \bar{\mathbb{K}}$. Then*

(i) $x \in J$ *if and only if* $|x| \in J$,

(ii) *if* $x \in J$ *and* $|y| \leqslant |x|$, *then* $y \in J$,

(iii) *If* $\mathbb{K} = \mathbb{R}$, $x \in J$ *and* $0 \leqslant y \leqslant x$, *then* $y \in J$.

Proof. Let $\hat{x}$ be a representative of x. Then $x = \Theta_{\hat{x}}|x|$ and $\Theta_{\hat{x}}$ is a unit in $\bar{\mathbb{K}}$. Hence, (i) follows. To prove (ii) note that, by (i), we may suppose that $y \geqslant 0$ and hence $x \geqslant 0$. So there are non-negative representatives $\hat{x}$ and $\hat{y}$ of x and y such that $\hat{x}(\varepsilon) \geqslant \hat{y}(\varepsilon)$, for ε sufficiently small. Hence, $\hat{y} = u\hat{x}$, with $u(\varepsilon) = \hat{y}(\varepsilon)/\hat{x}(\varepsilon)$ if $\hat{x}(\varepsilon) \neq 0$ and zero otherwise. Since u is bounded, it is also moderate and the result follows. Item (iii) follows by (ii) and Proposition 3.5. □

Remark 3.8. If $z \in \bar{\mathbb{C}}$, then it is clear that we may write $z = x + \mathrm{i}y$, with $x, y \in \bar{\mathbb{R}}$ and i is the class of the constant function $\sqrt{-1}$. Define $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. Clearly, if $\hat{z} = \hat{x} + \sqrt{-1}\hat{y}$ is a representative of z, then $\hat{x}$ and $\hat{y}$ are representatives of x and y, respectively. It is also clear that $\bar{\mathbb{C}}$ is an $\bar{\mathbb{R}}$ module and that the maps $\mathrm{Im}, \mathrm{Re} : \bar{\mathbb{C}} \to \bar{\mathbb{R}}$ are $\bar{\mathbb{R}}$-epimorphisms. Hence, if $J \lhd \bar{\mathbb{C}}$ is an ideal, then its images by these epimorphisms are ideals of $\bar{\mathbb{R}}$ and it is easily seen that they coincide; it will be denoted by J_r and called the *real part* of J. Proposition 3.7 tells us that $J_r \subset J$.

Note that the involution $^{-} : \mathbb{C} \ni z \to \bar{z} \in \mathbb{C}$ extends to an involution $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of $\bar{\mathbb{C}}$. We shall call this involution conjugation. The following result is clear.

Lemma 3.9. *Let $J \lhd \bar{\mathbb{C}}$ be an ideal of $\bar{\mathbb{C}}$. Then*

(i) $J_r \subset J$,

(ii) $J = J_r + \mathrm{i}J_r$ *and J is invariant under conjugation,*

(iii) $J_r = J \cap \bar{\mathbb{R}}$.

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Corollary 3.10. *Let $\mathcal{F} \in P_*(\mathcal{S})$. Then*

(a) *$g_r(\mathcal{F})$ is the real part of $g(\mathcal{F})$,*

(b) *if $z \in \bar{\mathbb{C}}$, then $z \in g(\mathcal{F})$ if and only if $|z| \in g_r(\mathcal{F})$.*

We now turn to the residual class fields of $\bar{\mathbb{K}}$. In what follows we shall concentrate on the quotients $\bar{\mathbb{R}}/g_r(\mathcal{F})$. However, all results that we shall obtain for these quotients also apply to the residual class fields $\bar{\mathbb{R}}/\overline{g_r(\mathcal{F})}$.

Lemma 3.11. *Let $\mathcal{F} \in P_*(\mathcal{S})$ and let $x, y \in \bar{\mathbb{R}}$. Then the following hold.*

(a) *If $x - y \in g_r(\mathcal{F})$ and $x^- \in g_r(\mathcal{F})$, then $y^- \in g_r(\mathcal{F})$.*

(b) *x^+ or x^- belongs to $g_r(\mathcal{F})$.*

Proof. Using the triangular inequality, proposition 3.5, we obtain

$$|x^- - y^-| = \tfrac{1}{2}|(x - |x| - y + |y|)| \leqslant \tfrac{1}{2}(|x - y| + ||y| - |x||) \leqslant |x - y|.$$

Convexity of ideals and our hypothesis gives us (a).

To prove (b) we may suppose that x is neither q-positive nor q-negative and so x has a representative $\hat{x}$ for which $\theta_{\hat{x}} \notin \{\pm 1\}$. This means that if $S := \{\varepsilon \in \boldsymbol{I} \mid \theta_{\hat{x}} = 1\}$, then S or S^{c} belongs to $\mathcal{F}$. The result now follows by Proposition 3.5 (viii). $\square$

Definition 3.12. Let $\alpha \in \bar{\mathbb{R}}/g_r(\mathcal{F})$ be given. We say that α is non-negative, $\alpha \geqslant 0$, if α has a representative $a \in \bar{\mathbb{R}}$ such that $a^- \in g_r(\mathcal{F})$. This gives rise to an ordering in the traditional way.

Lemma 3.11 shows that Definition 3.12 is intrinsic, i.e. it does not depend on the representative. The following lemma is easily shown and should be well known.

Lemma 3.13. *Let $(A, \leqslant)$ be a commutative unitary partially ordered ring and let $a, b \in A$.*

(i) *$(A, \leqslant)$ is totally ordered if and only if, for all $a \in A$, either $a \geqslant 0$ or $-a \geqslant 0$.*

(ii) *If $\geqslant$ is a total order on A, the nil radical $\mathcal{N}(A) = 0$ and $a, b \geqslant 0$ imply $ab \geqslant 0$, then A is an integral domain.*

We now come to the main result of this section.

Theorem 3.14. *Let $\mathcal{F} \in P_*(\mathcal{F})$. Then $\bar{\mathbb{R}}_{\mathcal{F}} := (\bar{\mathbb{R}}/g_r(\mathcal{F}), \leqslant)$ is a totally ordered ring.*

Proof. Symmetry and reflexivity are easily proved (use Proposition 3.5). We next show that transitivity holds: suppose that $(b - a)^-, (c - b)^- \in g_r(\mathcal{F})$. The triangular inequality gives

$$(c-a)^- = \tfrac{1}{2}[(c-a)-|c-a|] = \tfrac{1}{2}[(c-b+b-a)-|c-b+b-a|] \geqslant \tfrac{1}{2}[c-b)^- + (b-a)^-] \in g_r(\mathcal{F})$$

and hence were done. We shall now verify that the right-hand side of condition (i) of Lemma 3.13 holds: let $\alpha \in \bar{\mathbb{R}}_{\mathcal{F}}$ and let $a \in \bar{\mathbb{R}}$ be one of its representatives. Lemma 3.11 tells us that a^+ or a^- belongs to $g_r(\mathcal{F})$ and the result follows by item (iii) of Proposition 3.5. Lemma 3.13 gives us the result. $\square$

As mentioned earlier, all proofs hold for a maximal ideal.

Theorem 3.15. *Let $\mathcal{F} \in P_*(\mathcal{F})$. Then $(\bar{\mathbb{R}}/\overline{g_r(\mathcal{F})}, \leqslant)$ is a totally ordered field.*

4. $\bar{\mathbb{K}}$ revisited

In this section we continue the work on $\bar{\mathbb{K}}$ started in [**2**], where the maximal ideals of $\bar{\mathbb{K}}$ were completely described. Now we completely describe the minimal primes and show that $\bar{\mathbb{K}}$ is not von Neumann regular.

If A is a commutative unitary ring, we denote by $\mathcal{B}(A)$ the set of idempotents of A. Our first result describes completely the idempotents of $\bar{\mathbb{K}}$.

Theorem 4.1. *Let $e \in \bar{\mathbb{K}}$ be a non-trivial idempotent. Then $e = \chi_T$, i.e. e is a characteristic function for some $T \in \mathcal{S}$. In particular $\mathcal{B}(\bar{\mathbb{K}})$ is a discrete subset of $\bar{\mathbb{K}}$.*

Proof. Let $\hat{e} = (e_\varepsilon)$ be a representative of e. Since $e = e^2$ it follows that $|e_\varepsilon(1-e_\varepsilon)| = o(\varepsilon^N)$ for all $N \in \mathbb{N}$. Let $T := \{\varepsilon \in \boldsymbol{I} \mid |e_\varepsilon| > \frac{1}{2}\}$ and let $\hat{u} = \hat{e} - \hat{\chi_T}$. If $\varepsilon \in T$, then $e_\varepsilon > \frac{1}{2}$ and so

$$|\hat{u}(\varepsilon)| = |1 - e_\varepsilon| = |e_\varepsilon|\frac{|1-e_\varepsilon|}{|e_\varepsilon|} \leqslant 2|e_\varepsilon(1-e_\varepsilon)| = o(\varepsilon^N).$$

On the other hand if $\varepsilon \notin T$, then

$$|\hat{u}(\varepsilon)| = |e_\varepsilon| \leqslant \frac{\varepsilon^N}{|1-e_\varepsilon|} \leqslant 2\varepsilon^N = o(\varepsilon^N).$$

Hence, it follows that $|\hat{u}(\varepsilon)| = o(\varepsilon^N)$ for all $N \in \mathbb{N}$ and thus $e = \chi_T$. □

Recall that a unitary ring is said to be von Neumann regular if all its principal ideals are generated by an idempotent (see [**5, 15**] for further results on these rings). For our purpose we state only the following result.

Proposition 4.2. *Let A be a commutative unitary ring and let $\mathcal{N}(A)$ be its nil radical. Then every prime ideal of A is maximal if and only if $A/\mathcal{N}(A)$ is von Neumann regular.*

Lemma 4.3. *Let $\gamma : \boldsymbol{I} \to \mathbb{R} \cup \{\infty\}$ be defined as follows: $\gamma(\varepsilon) = \infty$ if $\varepsilon^{-1} \notin \mathbb{N}$, and $\gamma(\varepsilon) = p$ if $\varepsilon^{-1} \in \mathbb{N}$, where p is the smallest prime dividing ε^{-1}. Let $x(\varepsilon) = \varepsilon^{\gamma(\varepsilon)}$. Then the ideal generated by x is not an idempotent ideal.*

Proof. Suppose that $J = x\bar{\mathbb{K}}$ is an idempotent ideal. By Theorem 4.1 there exists $A \in \mathcal{S}$ such that $J = \chi_A\bar{\mathbb{K}}$. We divide the proof into two cases.

Case 1 ($\gamma(A)$ is finite). Set $\sigma = \max\{z \mid z \in \mathbb{R} \cap \gamma(A)\}$ and fix a prime number $p > \sigma$. Let $\varepsilon_n := p^{-n}$. Then (ε_n) converges to 0 and $\gamma(\varepsilon_n) = p \notin \gamma(A)$ for all $n \in \mathbb{N}$. Hence, $\hat{x}(\varepsilon_n) = \varepsilon_n^p$ for all $n \in \mathbb{N}$ and so $x \cdot \chi_{A^c} \neq 0$. Suppose that $x \in \chi_A\bar{\mathbb{K}}$ and write $x = y\chi_A$. From this we have that $0 = y\chi_A\chi_{A^c} = x\chi_{A^c} \neq 0$, which is a contradiction.

Case 2 ($\gamma(A)$ is infinite). In this case there exists a sequence $(\varepsilon_n) \subset A$ converging to 0 such that $\gamma(\varepsilon_n) \in \mathbb{N}$ is increasing. Suppose that $\chi_A = yx$. Then $\hat{\chi}_A(\varepsilon_n) - y(\varepsilon_n)x(\varepsilon_n) = 1 - y(\varepsilon_n)x(\varepsilon_n) \to 0$ as $n \to \infty$. But, since $\gamma(\varepsilon_n)$ is increasing, it follows easily that y cannot be a moderate function, which is a final contradiction. □

Theorem 4.4. *The topological ring of Colombeau generalized numbers $\bar{\mathbb{K}}$ is not von Neumann regular. In particular, there exists $\mathcal{F} \in P_*(\mathcal{S})$ such that $g(\mathcal{F})$ is not closed and $\bar{\mathbb{K}}$ has a prime ideal which is not maximal.*

Proof. Theorem 4.12 of [**2**] tells us that the nil radical vanishes, i.e. $\mathcal{N}(\bar{\mathbb{K}}) = 0$. Hence, by Proposition 4.2 and Lemma 4.3, $\bar{\mathbb{K}}$ is not von Neumann regular. □

The theorem guarantees that $\bar{\mathbb{K}}$ has a prime ideal $\mathcal{P}$ which is not maximal. On the other hand, we already know that there exists $\mathcal{F} \in P_*(\mathcal{S})$ such that $g(\mathcal{F}) \subset \mathcal{P} \subset \overline{g(\mathcal{F})}$. Our next goal is to prove that, in fact, $\mathcal{P}$ lies between a minimal and a maximal prime.

Theorem 4.5. *For every $\mathcal{F} \in P_*(\mathcal{S})$, $g(\mathcal{F})$ is a prime ideal.*

Proof. We first suppose that $\mathbb{K} = \mathbb{R}$. We need to prove that condition (ii) of lemma 3.13 holds. So let $a, b \in \bar{\mathbb{R}}$ be such that $a^-, b^- \in g(\mathcal{F})$. Then

$$(ab)^- = \tfrac{1}{2}(ab - |a|\,|b|) = \tfrac{1}{2}[(a^+ + a^-)(b^+ + b^-) - (a^+ - a^-)(b^+ - b^-)] = a^+b^- + a^-b^+ \in g(\mathcal{F}).$$

This means that the positive cone is invariant under multiplication. Now let $a \in \bar{\mathbb{R}}$ be such that $a^2 \in g(\mathcal{F})$. From [**2**] we have that there exists $A \in \mathcal{F}$ such that $a^2 = a^2\chi_A$. Hence, $a^2\chi_{A^c} = 0$ and so $a\chi_{A^c} \in \mathcal{N}(\bar{\mathbb{R}}) = 0$. It follows that $a = a\chi_A + a\chi_{A^c} = a\chi_A \in g(\mathcal{F})$. So we are done in this case.

To complete the proof we consider the case when $\mathbb{K} = \mathbb{C}$. Let $x, y \in g(\mathcal{F})$ be such that $xy \in g(\mathcal{F})$. Then $|xy| = |x|\,|y| \in \bar{\mathbb{R}} \cap g(\mathcal{F}) = g_r(\mathcal{F})$. By the first case $|x|$ or $|y|$ belongs to $g_r(\mathcal{F})$. Convexity of ideals now finishes the proof. □

Corollary 4.6. *For every $\mathcal{F} \in P_*(\mathcal{S})$, $\bar{\mathbb{R}}/g_r(\mathcal{F})$ is a totally ordered local integral domain.*

The previous theorem and [**2**, Remark 2.4] give us a complete description of the minimal primes of $\bar{\mathbb{K}}$.

Corollary 4.7. *$\{g_r(\mathcal{F}) \mid \mathcal{F} \in P_*(\mathcal{S})\}$ is the set of minimal prime ideals of $\bar{\mathbb{R}}$ and $\{g(\mathcal{F}) \mid \mathcal{F} \in P_*(\mathcal{S})\}$ that of $\bar{\mathbb{C}}$.*

We finish this section giving a reformulation of theorem 2.2. We use the notation of [**2**].

Theorem 4.8. *An element $x \in \bar{\mathbb{K}}$ is a unit if and only if there exists $r \geqslant 0$ such that $|x| \geqslant \alpha_r$.*

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5. Algebraic properties of $\mathcal{G}(\Omega)$

In this section we start the study of the algebraic properties of $\mathcal{G}(\Omega)$. We start by proving that the Boolean algebra of $\mathcal{G}(\Omega)$ is determined by $\bar{\mathbb{K}}$, its subring of constant functions.

Theorem 5.1. *Let Ω be a connected open subset of $\mathbb{R}^n$. Then $\mathcal{B}(\mathcal{G}(\Omega)) = \mathcal{B}(\bar{\mathbb{C}}) = \mathcal{B}(\bar{\mathbb{R}})$.*

Proof. Let $f \in \mathcal{B}(\mathcal{G}(\Omega))$ be a non-trivial idempotent; we shall prove that it is constant in Ω. We may view f as a differentiable function $f : \tilde{\Omega}_{\rm c} \to \bar{\mathbb{R}}$. Since f is an idempotent, it follows that its image is contained in $\mathcal{B}(\bar{\mathbb{R}})$ and that the latter, by theorem 4.1, is a discrete subset of $\bar{\mathbb{R}}$. Since Ω is connected, it follows by [**4**, Proposition 4.7] that f is constant in $\tilde{\Omega}_{\rm c}$ and hence constant in Ω. So we have proved that there exists an idempotent $\chi_S \in \mathcal{B}(\bar{\mathbb{R}})$ such that $\hat{f}(\varepsilon, x) = \chi_S(\varepsilon)$ for all $\varepsilon \in \boldsymbol{I}$ and for all $x \in \Omega$ and, hence, it is in fact an element of $\bar{\mathbb{R}}$. □

The Heaviside function is not an idempotent and so, as explained in [**8**, § 3.1], all operations in [**8**, § 2.1] are false because the first line is a false statement.

We now continue with the study of the group of units of $\mathcal{G}(\Omega)$. We know that the unit group of $\bar{\mathbb{K}}$ is open and dense. So, in some sense, $\bar{\mathbb{K}}$ behaves like a finite-dimensional algebra. We shall prove that the situation is quite different for $\mathcal{G}(\Omega)$. Note that in [**11**,**17**] partial characterization of the unit group of $\mathcal{G}(\Omega)$ is given.

Lemma 5.2. *Let Ω be an open subset of $\mathbb{R}^n$, let $f \in \mathcal{G}(\Omega)$ and let $\hat{f}$ be a representative of f. The following are equivalent:*

(i) $f \in \operatorname{Inv}(\mathcal{G}(\Omega))$;

(ii) $f(\xi) \in \operatorname{Inv}(\bar{\mathbb{K}})$ *for all* $\xi \in \tilde{\Omega}_{\rm c}$.

Proof. See [**14**]. □

Theorem 5.3. *Let Ω be an open subset of $\mathbb{R}^n$, let $f \in \mathcal{G}(\Omega)$, let $\hat{f}$ be a representative of f and let (Ω_m) be an exhaustive sequence of open subsets of Ω. The following are equivalent:*

(i) $f \in \operatorname{Inv}(\mathcal{G}(\Omega))$;

(ii) $f(\xi) \in \operatorname{Inv}(\bar{\mathbb{K}})$, *for all* $\xi \in \tilde{\Omega}_{\rm c}$;

(iii) $f|_{\Omega_m} \in \operatorname{Inv}(\mathcal{G}(\Omega_m))$ *for all* $m \in \mathbb{N}$.

(iv) $f|_{\bar{\Omega}_m} \in \operatorname{Inv}(\mathcal{G}(\bar{\Omega}_m))$ *for all* $m \in \mathbb{N}$.

(v) $f|_W \in \operatorname{Inv}(\mathcal{G}(W))$ *for any open non-empty relatively compact subset of* Ω.

(vi) *For any non-empty relatively compact subset $W \subset \Omega$, there exists $a \in \mathbb{R}$ such that, for each representative (f_ε) of f, there is $\eta \in \boldsymbol{I}$ verifying $\inf_{x \in W}\{|f_\varepsilon(x)|\} \geqslant \varepsilon^a$, for all $\varepsilon \in \boldsymbol{I}_\eta$.*

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Proof. The previous lemma tells us that (i) and (ii) are equivalent. Since we are working with an exhaustive sequence, the equivalence of (iii), (iv) and (v) is obvious because the generalized functions constitute a sheaf of algebras. That fact that (v) implies (vi) follows from [**2**, Theorem 4.13], which also tells us that (vi) implies (ii) and so we are done. □

We note that in [**14**] the equivalence of (ii) and (vi) is also proved.

Given $f \in \mathcal{G}(\Omega)$, $\xi \in \tilde{\Omega}_{\mathrm{c}}$, where $\hat{f}$ is a a representative of f and $\hat{\xi} = (\xi_\varepsilon)$ is a representative of ξ, define $Z_{\hat{\xi}}(\hat{f}) = \{\varepsilon \in I \mid \hat{f}(\varepsilon, \xi_\varepsilon) = 0\}$ and let $\bar{Z}_{\hat{\xi}}(\hat{f})$ be its closure in $[0, 1]$.

As a consequence of Theorem 5.3 and [**2**, Theorems 4.13 and 2.18] we have the following.

Corollary 5.4. *The following are equivalent:*

(i) $f \in \mathrm{Inv}(\mathcal{G}(\Omega))$;

(ii) *for all* $\xi \in \tilde{\Omega}_{\mathrm{c}}$, *there exists* $a > 0$ *such that* $|f_\varepsilon(\xi_\varepsilon)| \geqslant \varepsilon^a$, *for* ε *small enough;*

(iii) *for all* $\xi \in \tilde{\Omega}_{\mathrm{c}}$, *there exists* $r > 0$ *such that* $|f(\xi)| \geqslant \alpha_r$;

(iv) $0 \notin \bar{Z}_{\hat{\xi}}(\hat{f})$.

The following example shows that $\mathcal{G}(\Omega)$ does not behave like $\bar{\mathbb{K}}$. It shows that there are non-zero divisors which, however, are not units. So an analogue of [**2**, Theorem 4.25] does not hold here.

Example 5.5. We take $\Omega = \mathbb{R}$ and $\hat{f}(\varepsilon, x) = x$. Taking $a = 1$ in the previous corollary, we see easily that f is invertible in the complement of any neighbourhood of 0 and hence cannot be a zero divisor. However, it is not invertible since $f(0) = 0$ is not a unit.

We finish this section showing that $\mathcal{G}(\Omega)$ is a fractal just like $\bar{\mathbb{K}}$.

Note first that the sharp topology on $\mathcal{G}(\Omega)$ is induced by the ultra-metric $d := \sup\{2^{-m} d_m \mid m \in \mathbb{N}\}$, where $d_m := D_{mm}/(1 + D_{mm})$ and $(D_{(m,p)})_{(m,p)\in\mathbb{N}^2}$ is the family of pseudo-ultra metrics defining the sharp topology on $\mathcal{G}(\Omega)$ (see [**2**, Proposition 1.10]). The proof of the next result uses the notation of the proof of [**2**, Theorem 3.5].

Theorem 5.6. $\mathcal{G}(\Omega)$ *is a fractal.*

Proof. Clearly $\mathrm{ind}(\mathcal{G}(\Omega)) = 0$, because $\mathcal{G}(\Omega)$ is an ultra-metric space. Since $\bar{\mathbb{K}}$ is a closed subset of $\mathcal{G}(\Omega)$, it follows that $\dim(\bar{\mathbb{K}}) \leqslant \dim(\mathcal{G}(\Omega))$. Hence, by [**2**, Theorem 3.5], it follows that $\dim(\mathcal{G}(\Omega)) = \infty$. □

6. Traces of ideals

In this section we shall define the notion of the trace of an ideal. Before we do so, however, we establish some basic facts about some bilinear forms defined on $\bar{\mathbb{K}}^n$. Most of the results follow easily by working with representatives.

Let $y = (y_1, \dots, y_n), x = (x_1, \dots, x_n) \in \bar{\mathbb{K}}^n$ and let the $\hat{x}_i$ be representatives of the x_i. For an integer $k \geqslant 1$, define

$$\|x - y\|_\infty = \max\{\|x_i - y_i\| : 1 \leqslant i \leqslant n\} \quad \text{and} \quad \|x - y\|_k = \left[\sum_{i=1}^{n} \|x_i - y_i\|^k\right]^{1/k}.$$

Define $[x]_p = [\sum_{i=1}^n |x_i|^p]^{1/p}$, $p = 1, 2$, where $|x_i|$ is the absolute value of x_i. It is easy to see that for each q-positive x and each $r > 0$ there exists a unique q-positive y, denoted $x^{1/r}$, such that $y^r = x$. Note that $[x]_2$ is induced by the $\bar{\mathbb{R}}$-bilinear map $\langle x \mid y \rangle := \sum x_i \bar{y}_i$. The following lemma sums up the results we shall need in the remainder of the paper. Its proof is easy.

Lemma 6.1. *Let $y = (y_1, \dots, y_n), x = (x_1, \dots, x_n) \in \bar{\mathbb{K}}^n$. Then*

(i) $|\langle x|y\rangle| \leqslant [x]_2[y]_2$ *(generalized Cauchy–Schwarz inequality),*

(ii) $\|\cdot\|_\infty$ *and* $\|\cdot\|_p$, $p = 1, 2$, *induce the same topology on* $\bar{\mathbb{K}}^n$,

(iii) $|x_i| \leqslant [x]_2$ *for all* $i = 1, 2, \dots, n$,

(iv) *there exists a constant* $C = C(n) > 0$ *such that* $[x]_2 \leqslant C[x]_1$.

Let $X \subset \mathbb{R}^n$ be a non-empty subset. Then it makes sense to define $\tilde{X}$ and $\tilde{X}_c$ just as we defined $\tilde{\Omega}_c$ and we have a standard embedding $j_X : \tilde{X} \to \tilde{\mathbb{R}}^n$.

In what follows we shall extend results of [**2**] on ideals of $\mathcal{G}(\Omega)$. Let $\Omega \subset \mathbb{K}^n$, $Y \subset \tilde{\Omega}_c$ and let $I \lhd \bar{\mathbb{K}}$ be an ideal. Given $\xi \in \tilde{\Omega}_c$, define $\nu_\xi : \mathcal{G}(\Omega) \ni f \to f(\xi) \in \bar{\mathbb{K}}$. This is a surjective ring homomorphism (see [**2**, Proposition 2.5]). Define also $\mathcal{G}_{Y,I}(\Omega) = \{f \in \mathcal{G}(\Omega) \mid \nu_\xi(f) \in I \ \ \forall \xi \in Y\}$.

Definition 6.2. Let J be an ideal of $\mathcal{G}(\Omega)$. Define the generalized trace of J by $\mathrm{GTr}(J) = \{\xi \in \tilde{\Omega}_c \mid \nu_\xi(J) \neq \bar{\mathbb{K}}\}$ and its trace by $\mathrm{Tr}(J) = \Omega \cap \mathrm{GTr}(J)$.

Obviously $\mathrm{Tr}(J) \subset \mathrm{GTr}(J)$. Clearly, if $J_1 \subset J_2$ are ideals then $\mathrm{GTr}(J_2) \subset \mathrm{GTr}(J_1)$ and $\mathrm{Tr}(J_2) \subset \mathrm{Tr}(J_1)$.

Example 6.3.

(i) If Y is a non-empty subset of $\tilde{\Omega}_c$ and I is a proper ideal of $\bar{\mathbb{K}}$, then $Y \subset \mathrm{GTr}(\mathcal{G}_{Y,I}(\Omega))$.

(ii) Define $J = \mathcal{G}_c(\Omega) = \{f \in \mathcal{G}(\Omega) \mid \mathrm{supp}(f) \subset\subset \Omega\}$. Then it is easily seen that J is an ideal of $\mathcal{G}(\Omega)$. If $\xi = [(\xi_\varepsilon)] \in \tilde{\Omega}_c$, then there exists a compact subset $K \subset \Omega$ such that $\xi_\varepsilon \in K$ for ε sufficiently small. Choose a Urysohn function ϕ with compact support such that $K \subset \mathrm{supp}(\phi)$ and $\phi|_K \equiv 1$. Then $\nu_\xi(\phi) = 1$ and hence $\xi \notin \mathrm{GTr}(J)$. Hence, $\mathrm{GTr}(J) = \emptyset$.

Proposition 6.4. *Let $\xi, \zeta \in \tilde{\Omega}_c$, let $z = [\xi - \zeta]_2$ and let $J \lhd \bar{\mathbb{K}}$ be an ideal.*

(i) *If $J \lhd \bar{\mathbb{K}}$ is an ideal such that $z \in J$, then $\mathcal{G}_{\xi,J}(\Omega) = \mathcal{G}_{\zeta,J}(\Omega)$.*

(ii) *If $J \lhd \bar{\mathbb{K}}$ is a prime ideal, then $z \in J$ if and only $\mathcal{G}_{\zeta,J}(\Omega) = \mathcal{G}_{\xi,J}(\Omega)$.*

(iii) *If $J \lhd \mathcal{G}(\Omega)$ is an ideal, then $\mathrm{Tr}(J) = \mathrm{Tr}(\bar{J})$ and $\mathrm{GTr}(J) = \mathrm{GTr}(\bar{J})$.*

(iv) *If $x_0 \in \mathrm{GTr}(J)$ and $L = \nu_{x_0}(J)$, then $\tilde{\Omega}_c \cap \{x_0 + x \mid x \in L^n\} \subset \mathrm{GTr}(J)$.*

(v) *If $J_1 \neq J_2 \lhd \bar{\mathbb{K}}$ are ideals, then $\mathcal{G}_{\xi,J_1}(\Omega) \neq \mathcal{G}_{\xi,J_2}(\Omega)$.*

Proof. Choose first an open connected relatively compact subset $A \subset \Omega$ which contains the supports of ξ and ζ. Let $f \in \mathcal{G}_{\xi,J}(\Omega)$. Proposition 4.4 of [**4**] gives that $f(\xi) - f(\zeta) = \langle \nabla f(c) \mid \xi - \zeta \rangle$ for some $c \in \tilde{\Omega}_c$. The generalized Cauchy–Schwarz inequality tells us that $|f(\xi) - f(\zeta)| \leqslant [\nabla f(c)]_2 z \in J$. By convexity of ideals it follows that $f(\zeta) \in J$.

Let $f \in \mathcal{G}(\Omega)$ be such that $\hat{f}(\varepsilon, x) := \langle x - \xi_\varepsilon \mid x - \xi_\varepsilon \rangle$ is one of its representatives. Then $f(\xi) = 0$ and so $f \in \mathcal{G}_{\xi,J}(\Omega)$. Hence, $f(\zeta) = z^2 \in J$. Since the latter is a prime ideal, the result follows.

Let $x_0 \in \mathrm{GTr}(J)$ and suppose that $x_0 \notin \mathrm{GTr}(\bar{J})$. There then exists an element $f \in \bar{J}$ such that $f(x_0) = 1$. Now choose a sequence (f_n) in J converging to f. Then $f_n(x_0) \to f(x_0) = 1$. Hence, there exists an n_0 such that $f_{n_0}(x_0)$ is a unit and hence x_0 does not belong to $\mathrm{GTr}(J)$, which is a contradiction.

Suppose that $x = \xi + h$ with $h_i \in L$ for all $i = 1, \dots, n$. Then, by [**4**, Proposition 4.4], we have that there exists $c \in \tilde{\Omega}_c$ such that $f(x) - f(\xi) = \langle \nabla f(c), h \rangle \in L$. Hence, if $f \in J$, then $f(x) \in L$ and so $x \in \mathrm{GTr}(J)$.

Consider the standard embedding of $\bar{\mathbb{K}}$ in $\mathcal{G}(\Omega)$, i.e. the elements of $\bar{\mathbb{K}}$ are constant functions. Then it is clear that $J \subset \mathcal{G}_{\xi,J}(\Omega)$ and from this the result is obvious. □

A direct proof of Proposition 6.4 (iii) would be as follows: let $x_0 \in \mathrm{GTr}(J)$ and $f \in \bar{J}$. There exists a sequence (f_n) of J such that $f_n \to f$. It follows that $\nu_{x_0}(f_n) \to \nu_{x_0}(f)$ and hence $\nu_{x_0}(\bar{J}) \subset \overline{\nu_{x_0}(J)}$. Since ideals of $\bar{\mathbb{K}}$ are not dense, it follows that $x_0 \in \mathrm{GTr}(\bar{J})$.

Since $\bar{\mathbb{K}}$ is not an integral domain, it is clear that if J is a prime ideal of $\mathcal{G}(\Omega)$, then $J \cap \bar{\mathbb{K}} \neq (0)$. We shall use this fact in what follows.

Theorem 6.5.

(i) *For every $\xi \in \tilde{\Omega}_c$, the map ν_ξ is a continuous epimorphism.*

(ii) *If $J \subset \mathcal{G}(\Omega)$ is an ideal, then $\mathrm{GTr}(J)$ is closed.*

(iii) *Let $\mathcal{M} \lhd \mathcal{G}(\Omega)$ be a maximal ideal with non-empty generalized trace; choose $\xi \in \mathrm{GTr}(\mathcal{M})$ and set $\underset{\sim}{m} = \nu_\xi(\mathcal{M})$. Then*

  (a) *$\mathcal{M} \cap \bar{\mathbb{K}} = \underset{\sim}{m}$ is the subring of constant functions of $\mathcal{M}$,*

  (b) *$\nu_x(\mathcal{M}) = \underset{\sim}{m}$ and $\mathcal{M} = \mathcal{G}_{x,\underset{\sim}{m}}(\Omega)$ for all $x \in \mathrm{GTr}(\mathcal{M})$,*

  (c) *$\mathrm{GTr}(\mathcal{M}) = \{\xi + h \in \tilde{\Omega}_c \mid h \in \underset{\sim}{m}^n\}$.*

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Proof. To prove (i) we consider a sequence in $\mathcal{G}(\Omega)$ $f_n \to f_0$. Then $\kappa(f_n) \to \kappa(f_0)$ and so, for all $\xi \in \tilde{\Omega}_{\mathrm{c}}$, we have that $f_n(\xi) \to f_0(\xi)$ and hence $\nu_\xi(f_n) \to \nu_\xi(f_0)$.

Choose a sequence $\xi_n \to \xi_0$ in $\mathrm{GTr}(J)$. If $\xi_0 \notin \mathrm{GTr}(J)$, then there exists $f \in \mathcal{G}(\Omega)$ such that $f(\xi_o) = 1$. But $i(f)$ is continuous and so $\kappa(f)(\xi_n) \to \kappa(f)(\xi_0)$. Since $f_0(\xi)$ is a unit then so is $f_n(\xi)$ for sufficiently large n (see [**2**]) and thus ξ_n does not belong to $\mathrm{GTr}(J)$.

Consider $\bar{\mathbb{K}}$ embedded in $\mathcal{G}(\Omega)$ as the constant functions. Then it is clear that $\mathcal{M} \cap \bar{\mathbb{K}} \subset \underline{m}$. On the other hand, if there were $y \in \underline{m}$, $y \notin \mathcal{M} \cap \bar{\mathbb{K}}$, then $\mathcal{G}(\Omega) = \mathcal{M} + \mathcal{G}(\Omega)y$ and so $\bar{\mathbb{K}} = \nu_\xi(\mathcal{G}(\Omega)) = \nu_\xi(\mathcal{M} + \mathcal{G}(\Omega)y) \subset \underline{m} + \bar{\mathbb{K}}y = \underline{m}$, which is a contradiction. Hence, $\underline{m} = \mathcal{M} \cap \bar{\mathbb{K}}$, proving (a).

Note that, since $\mathcal{M}$ is a maximal ideal of $\mathcal{G}(\Omega)$, we have that $\underline{m}$ is a maximal ideal of $\bar{\mathbb{K}}$. In fact, if this were not the case, then there would exist a non-trivial ideal J of $\bar{\mathbb{K}}$ containing $\underline{m}$ properly. But then $\mathcal{M}$ would be properly contained in the non-trivial ideal $\mathcal{G}_{\xi,J}(\Omega)$, which is a contradiction. Now if $x \in \mathrm{GTr}(\mathcal{M})$, then $\underline{m} = \nu_x(\mathcal{M} \cap \bar{\mathbb{K}}) \subset \nu_x(\mathcal{M}) \neq \bar{\mathbb{K}}$. Hence, $\nu_x(\mathcal{M}) = \underline{m}$ and it also follows that $\mathcal{M} = \mathcal{G}_{x,\underline{m}}(\Omega)$, proving (b).

We now prove the last assertion. Let $x \in \mathrm{GTr}(\underline{m})$. We have already seen that, in this case, $\mathcal{M} = \mathcal{G}_{x,\underline{m}}(\Omega) = \mathcal{G}_{\xi,\underline{m}}(\Omega)$ and since $\mathcal{M}$ is a prime ideal it follows from Proposition 6.4 that $[\xi - x]_2 \in \underline{m}$. By lemma 6.1 and the convexity of ideals we have that $|\xi_i - x_i| \in \underline{m}$ for all $i = 1, 2, \dots, n$. So if we set $h_i = \xi_i - x_i$, then $h := (h_1, \dots, h_n) \in \underline{m}^n$ and $x = \xi + h$. Conversely, suppose that $x = \xi + h$ with $h_i \in \underline{m}$ for all $i = 1, n$. Then, by [**4**, Proposition 4.4], we have that there exists $c \in \tilde{\Omega}_{\mathrm{c}}$ such that $f(x) - f(\xi) = \langle \nabla f(c), h \rangle \in \underline{m}$. Hence, if $f \in \mathcal{M}$, then $f(x) \in \underline{m}$ and so $x \in \mathrm{GTr}(\mathcal{M})$. □

One should compare Theorem 6.5 (c) with the classical result which describes maximal ideals of the algebra of continuous functions of a compact topological space. In the present case, any function f of $\mathcal{M}$ induces a function $\bar{f} : (\bar{\mathbb{K}}/\underline{m})^n \to \bar{\mathbb{K}}/\underline{m}$ which belongs to the ideal determined by the image of ξ in $(\bar{\mathbb{K}}/\underline{m})^n$, i.e. $\bar{f}(\bar{\xi}) = 0$. In the case when $n = 1$, it is easy to see that if $f \in \mathcal{M}$, then some primitive F of f also belongs to $\mathcal{M}$.

Corollary 6.6. *Let $\mathcal{M} \lhd \mathcal{G}(\Omega)$ be a maximal ideal. Then*

(i) $\mathrm{Tr}(\mathcal{M})$ *has at most one element,*

(ii) $\mathrm{Tr}(\mathcal{M})$ *is non-empty if and only if $\underline{m} := \mathcal{M} \cap \bar{\mathbb{K}} \lhd \bar{\mathbb{K}}$ is a maximal ideal of $\bar{\mathbb{K}}$ and there exists a unique $\xi \in \Omega$ such that $\mathcal{M} = \mathcal{G}_{\xi,\underline{m}}(\Omega)$,*

(iii) $\mathrm{GTr}(\mathcal{M})$ *is non-empty if and only if $\underline{m} := \mathcal{M} \cap \bar{\mathbb{K}} \lhd \bar{\mathbb{K}}$ is a maximal ideal of $\bar{\mathbb{K}}$ and there exists $\xi \in \tilde{\Omega}_{\mathrm{c}}$ such that $\mathcal{M} = \mathcal{G}_{\xi,\underline{m}}(\Omega)$,*

(iv) *if $\mathcal{M}$ is dense,* $\mathrm{GTr}(\mathcal{M}) = \emptyset$.

Proof. Choose $\xi_1, \xi_2 \in \mathrm{Tr}(\mathcal{M})$. Let $\underline{m}_i := \nu_{\xi_i}(\mathcal{M})$, $i = 1, 2$. Then, by the previous theorem, $\underline{m}_1 = \underline{m}_2 =: \underline{m}$ is a maximal ideals of $\bar{\mathbb{K}}$ and $\mathcal{M} \subset \mathcal{G}_{\xi_i,\underline{m}}(\Omega)$. It follows that $\mathcal{G}_{\xi_1,\underline{m}}(\Omega) = \mathcal{G}_{\xi_2,\underline{m}}(\Omega)$. Invoking [**2**, Proposition 2.7], we conclude that $\xi_1 = \xi_2$. This proves (ii) and the first part of (i). The last part follows by Example 6.3.

Item (iii) is a consequence of the previous theorem and the definition of $\mathrm{GTr}(\mathcal{M})$, while (iv) is a consequence of the former and Proposition 6.4 (iii). □

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We give some examples which show that there exist maximal ideals whose generalized trace is empty. These ideals are dense and hence $\mathrm{Inv}(\mathcal{G}(\Omega))$ is not open. In the next section we shall prove the existence of a unique minimal dense ideal. We shall also give an example of a closed maximal ideal J such that $\mathrm{Tr}(J) = \emptyset$ but $\mathrm{GTr}(J) \neq \emptyset$.

Example 6.7. (a) From [**2**, Proposition 1.12 (b)] it is readily seen that $\mathcal{G}_{\mathrm{c}}(\Omega)$ is a dense ideal. Indeed, for any fixed $f \in \mathcal{G}(\Omega)$, by using a regularizing family for Ω (see [**1**, Notation 1.1]) it is easy to obtain a sequence $(\varphi_\nu)_{\nu\in\mathbb{N}}$ in $C_{\mathrm{c}}^\infty(\Omega)$ such that $\varphi_\nu f \to f$ if $\nu \to \infty$, in the sharp topology on $\mathcal{G}(\Omega)$. Since $\bar{\mathbb{K}} \cap \mathcal{G}_{\mathrm{c}}(\Omega) = 0$, from Corollary 6.6 (i), (ii) it follows that $\mathcal{G}_{\mathrm{c}}(\Omega)$ is not a maximal ideal and it must have empty trace. Denote by $\mathcal{I}$ a maximal ideal containing $\mathcal{G}_{\mathrm{c}}(\Omega)$. Then $\mathcal{I}$ is dense and $\mathrm{GTr}(\mathcal{I}) = \emptyset$.

(b) Let Ω be an non-void open subset of $\mathbb{R}^n$, let $J :=]0, 1[$ and let $o = (O_\lambda)_{\lambda\in J}$ be a family of open subsets of $\mathbb{R}^n$ verifying the following conditions:

(A1) if $A_\lambda := \Omega \cap O_\lambda$, then $\phi \neq A_\lambda \neq \Omega$ for all $\lambda \in J$;

(A2) $\lambda < \mu \Longrightarrow \emptyset \neq \bar{A}_\lambda \subsetneqq A_\mu$;

(A3) $\Omega = \bigcup_{\lambda\in J} A_\lambda$;

(A4) there exist $\zeta \in \partial\Omega$ and $\alpha \in J$ such that $\zeta \in O_\alpha$.

Now, for every $\lambda \in J$ define $\underline{a}_\lambda = \{f \in \mathcal{G}(\Omega) \mid \mathrm{supp}(f) \subset A_\lambda\}$. From (A2) it follows that $(\underline{a}_\lambda)_{\lambda\in J}$ is a strictly increasing family of ideals of $\mathcal{G}(\Omega)$; hence, $\underline{a}_o := \bigcup_{\lambda\in J} \underline{a}_\lambda$ is also an ideal of $\mathcal{G}(\Omega)$. It is easy to see that the following statements hold.

(1°) For each $(X, \underline{b})$, where $\emptyset \neq X \subset \Omega$ and $\underline{b}$ is a proper ideal of $\bar{\mathbb{K}}$, we have

 (i) $\underline{a}_o \neq \mathcal{G}_{X,\underline{b}}(\Omega)$,

 (ii) $\underline{a}_\lambda \neq \mathcal{G}_{X,\underline{b}}(\Omega)$ for all $\lambda \in J$ such that $X \cap A_\lambda \neq \phi$.

(2°) $\underline{a}_o$ is a proper ideal of $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathrm{c}}(\Omega) \subset \underline{a}_o$.

These examples show, in another way, that there are non-maximal proper ideals of $\mathcal{G}(\Omega)$ which are different from $\mathcal{G}_{X,\underline{b}}(\Omega)$. This also gives a direct proof that $\mathcal{G}_{\mathrm{c}}(\Omega)$ is not a maximal ideal.

Proposition 6.8. $\mathrm{Rad}(\mathcal{G}(\Omega)) = 0$.

Proof. Suppose there exists a non-zero element $f \in \mathrm{Rad}(\mathcal{G}(\Omega))$. Choose $\xi \in \tilde{\Omega}_{\mathrm{c}}$ such that $y := f(\xi) \neq 0$. Since $\mathrm{Rad}(\bar{\mathbb{K}}) = 0$ (see [**2**]) there exists a maximal ideal $\underline{m} \lhd \bar{\mathbb{K}}$ such that $y \notin \underline{m}$. Then it follows that f does not belong to the maximal ideal $\mathcal{G}_{\xi,\underline{m}}(\Omega)$, which is a contradiction. □

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7. Supports of ideals

In this section we introduce the notion of the support of an ideal. This will be used to give examples of ideals whose generalized traces are not empty but whose traces are empty. We also give examples of maximal ideals whose supports coincide but are not equal. Furthermore, we prove the existence of a unique minimal dense ideal.

Let $V \subset U$ be open subsets of $\mathbb{R}^n$. We can consider the restriction map

$$r_V^U : \mathcal{G}(U) \to \mathcal{G}(V), \quad r_V^U(f) := f|_V.$$

It is clear that this is a $\bar{\mathbb{K}}$-algebra homomorphism which need not be surjective.

Definition 7.1. Let $\Omega \subset \mathbb{R}^n$ be open and let $J \lhd \mathcal{G}(\Omega)$ be an ideal. A point $x \in \Omega$ is *J-regular*, or J is *regular at* x, if it has an open relatively compact neighbourhood $V \subset \Omega$ such that $\mathcal{G}_c(V) \subset r_V^\Omega(J)$. The set of J-regular points is denoted by $[\Omega; J]$ and its complement in Ω is called the *support of* J and denoted by $\operatorname{supp}(J)$.

Lemma 7.2. *Let $J \lhd \mathcal{G}(\Omega)$ be an ideal and let $x \in \Omega$. The following are equivalent:*

(i) *x is J-regular;*

(ii) *there exists an open relatively compact neighbourhood V of x such that $1_V \in r_V^\Omega(J)$;*

(iii) *there exists an open relatively compact neighbourhood W of x such that $r_W^\Omega(J)$ contains a unit.*

Proof. It is easily seen that (i) $\implies$ (ii) $\implies$ (iii), so we only need to prove that (iii) $\implies$ (i). Let $g \in J$ be such that $f_0 := r_W^\Omega(g)$ is a unit. Choose $W_1 \subset W$ relatively compact and let ϕ be a Urysohn function whose support is contained in W with $\phi = 1$ in W_1. Since J is an ideal, it follows that $g_0 := \phi \bar{f}_0^1 g \in J$ and $r_{W_1}^\Omega(g_0) = 1_{W_1}$. From this, (i) now follows easily. □

Lemma 7.3. *Let $J \lhd \mathcal{G}(\Omega)$ be an ideal.*

(i) $\operatorname{Tr}(J) \subset \operatorname{supp}(J)$.

(ii) *If $x \in [\Omega; J]$, then there exist $\phi \in J \cap \mathcal{G}_c(\Omega)$ and $\hat{\phi}$, a non-negative representative of ϕ, a neighbourhood $W \subset \Omega$ of x and $\eta \in \boldsymbol{I}$ such that $\hat{\phi}(\varepsilon, y) \geqslant \frac{1}{2}$ in $\boldsymbol{I}_\eta \times W$.*

(iii) *If $J_1 \subset J_2$ are ideals of $\mathcal{G}(\Omega)$, then* $\operatorname{supp}(J_2) \subset \operatorname{supp}(J_1)$.

(iv) $\operatorname{supp}(J) = \operatorname{supp}(\bar{J})$.

Proof. To see (i) we merely look at the definition of $\operatorname{Tr}(J)$: if $x_0 \in \operatorname{Tr}(J)$, then $f(x_0)$ is a non-unit for each $f \in J$ and hence the conclusion follows.

To prove (ii) choose an open relatively compact neighbourhood V of x and $f_0 \in J$ such that $f_0|V \equiv 1$. Choose now $W \subset V$ relatively compact and a Urysohn function ϕ_0 which is 1 on W and 0 outside V. Since J is an ideal, $\phi := \phi_0 f_0$ will give the conclusion.

We now prove (iii). Take $x_0 \in [\Omega; \bar{J}]$. There then exists $f \in J$ and a relatively compact neighbourhood V of x_0 such that $f|V \equiv 1$. On the other hand, there exists a sequence

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$f_n \in J$ converging to f. Now apply [**2**, Proposition 1.12] with V the first element of the exhaustion, i.e. $m = 0$, and $\nu = 1$ and $a = 2$ and Theorem 5.3 to obtain that, for some sufficiently large n_0, the restriction of f_{n_0} to V is a unit in $\mathcal{G}(V)$ and hence, by Lemma 7.3, we have that $x_0 \in [\Omega; J]$. Hence, $\operatorname{supp}(J) \subset \operatorname{supp}(\bar{J})$. □

We can now prove our main theorem of this section.

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^n$, $J \lhd \mathcal{G}(\Omega)$ an ideal. Then*

(i) *J is dense in $\mathcal{G}(\Omega)$ if and only if* $\operatorname{supp}(J)$ *is empty,*

(ii) *if J is prime, then* $\operatorname{card}(\operatorname{supp}(J)) \leqslant 1$,

(iii) *if J is maximal, then it is closed if and only if* $\operatorname{card}(\operatorname{supp}(J)) = 1$.

Proof. Suppose that J is dense. Then, by Lemma 7.3, we have that $\operatorname{supp}(J) = \operatorname{supp}(\mathcal{G}(\Omega)) = \emptyset$. Conversely, suppose that J has empty support. We shall prove that $\mathcal{G}_c(\Omega) \subset J$. In fact, let U be an open relatively compact subset of Ω. For every $x \in U$ there exists an open relatively compact subset $V_x \subset W_x$ and $\phi_x \in J$ such that $\phi_x|_{W_x} \equiv 1$. Lemma 7.3 tells us that we may choose ϕ_x to be non-negative and at least $\frac{1}{2}$ in V_x. The family $(V_x)_x \in U$ give an open covering of $\bar{U}$ and, hence, we may choose a finite number of them covering U. So $U \subset \bigcup_{1\leqslant i\leqslant n} V_{x_i}$. Define $\phi = \sum_{1\leqslant i\leqslant n} \phi_{x_i}$. Then $\phi \in J$ and is at least $\frac{1}{2}$ in U. Hence, $\phi|_U$ is a unit and so, by Lemma 7.2, we have that there exists $\psi \in J$ such that $\psi|_U \equiv 1$. Hence, any element of $\mathcal{G}(\Omega)$ whose support is contained in U belongs to J. Since U was arbitrary, our claim follows.

Suppose that J is prime and x and y are distinct points of its support. We choose Urysohn functions ψ and ϕ such that $\phi(x) = 1 = \psi(y)$, $\phi(y) = 0 = \psi(x)$, and their supports are disjoint. Then $\phi \cdot \psi = 0$ and, since J is prime, $\psi \in J$, say. But then $x \in [\Omega; J]$, which is a contradiction.

If J is maximal and closed, then J is prime and not dense. Hence, by items (i) and (ii), $\operatorname{supp}(J)$ consists of a single point. Conversely if J is maximal and $\operatorname{supp}(J)$ consists of a single point, then $\operatorname{supp}(\bar{J})$ also consists of a single point. Hence, it is not dense and so $J = \bar{J}$. □

The above proof gives us the following somewhat surprising result.

Corollary 7.5. *Let $J \lhd \mathcal{G}(\Omega)$ be an ideal. Then J is dense if and only if $\mathcal{G}_c(\Omega) \subset J$. In particular, $\mathcal{G}_c(\Omega)$ is the intersection of the set of dense ideals.*

Corollary 7.6. *Let J be a non-dense prime ideal. Then* $\operatorname{card}(\operatorname{supp}(J)) = 1$.

Example 7.7. Here we construct two maximal ideals J_1, J_2 of $\mathcal{G}(\Omega)$ with equal support such that $\operatorname{GTr}(J_1) \neq \operatorname{GTr}(J_2)$ and such that $\operatorname{Tr}(J_i)$ is empty, $i = 1, 2$. We shall do this for $\Omega = \mathbb{R}^n$ but it is obvious that this can be done for any open subset of $\mathbb{R}^n$.

Let $\underset{\sim}{m} \lhd \bar{\mathbb{K}}$ be a maximal ideal. Let $\Omega = \mathbb{R}^n$, let $x_0 \in \Omega$ be any point and let $e_1 \in \mathbb{R}^n$ be any vector of norm 1. Let $\zeta = x_0 + 2\alpha_1 e_1$ and $\xi = x_0 + \alpha_1 e_1$ be points in $\tilde{\Omega}_c$. Then $[\zeta - \xi]_2 = \alpha_1$ is a unit and hence $\mathcal{G}_{\zeta,\underset{\sim}{m}}(\Omega) \neq \mathcal{G}_{\xi,\underset{\sim}{m}}(\Omega)$. Clearly, their generalized traces are different. Since $\operatorname{Tr}(J) \subset \operatorname{supp}(J)$, we just need to prove that their support equals

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$\{x_0\}$ and that x_0 does not belong to their trace. The last part is clear because $[\zeta - x_0]_2$ and $[\xi - x_0]_2$ are units. To calculate their support note that, since $\lim \zeta_\varepsilon = \lim \xi_\varepsilon = x_0$, we have that any neighbourhood of x_0 contains the support of ζ and ξ in the sense that, for sufficiently small ε, ζ_ε and ξ_ε belong to this neighbourhood. Working with a Urysohn function, it is now obvious that any element outside a neighbourhood of x_0 is a regular point for both ideals and, hence, by Theorem 7.4 their support equals $\{x_0\}$.

Example 7.8. Let J be an ideal of $\bar{\mathbb{K}}$ and let M be the ideal it generates in $\mathcal{G}(\Omega)$. Then it is easy to see that $\mathrm{supp}(M) = \Omega$ and $\mathrm{GTr}(M) = \tilde{\Omega}_{\mathrm{c}}$ and hence $\mathrm{Tr}(M) = \mathrm{supp}(M) = \Omega$.

This example shows that, contrary to what happens in $\bar{\mathbb{K}}$, maximal ideals are not generated by characteristic functions.

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