Canad. Math. Bull. Vol. **33** (1), 1990

RING DERIVATIONS ON FUNCTION ALGEBRAS

BY

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ABSTRACT. In this paper we show that a ring derivation on a function algebra is trivial provided that the Choquet boundary of the algebra contains a dense sequentially non-isolated set.

1. **Introduction.** An operator D on a commutative algebra A is called a *ring derivation* if for all x, y in A, $D(x+y) = D(x) + D(y)$ and $D(xy) = xD(y) + yD(x)$. Throughout this paper, A denotes a function algebra and $Ch(A)$ denotes the Choquet boundary of A. A subset J of $Ch(A)$ is called a *dense sequentially non-isolated set* if J is dense in $Ch(A)$ and every point in J is the limit of a non-trivial sequence of points of $Ch(A)$. Throughout we follow the definitions and notations as given in [2].

In [6], Singer and Wermer show that the range of a continuous derivation on a commutative Banach algebra is contained in the radical, and recently, Thomas [8] has extended this result to all, not necessarily continuous, linear derivations. In [4], Johnson shows that a linear derivation on a semi-simple commutative Banach algebra A is necessarily continuous and hence a linear derivation is trivial on A. It is well known that there exists a non-trivial ring derivation (see [3]) on the algebra of complex numbers. In [1], Becker and in [5], Nandakumar have shown that all ring derivations are continuous on the algebra of analytic functions on an open region equipped with the topology of uniform convergence on compact subsets of the region. In Section 2 we give some basic lemmas and a theorem. In Section 3 we prove the main theorem. In Section 4 we give three examples.

Now we state the main theorem.

THEOREM. *Let A be a function algebra and let D be a ring derivation on A. If there exists a dense sequentially non-isolated set J in $Ch(A)$, then $D = 0$.*

2. Lemmas.

LEMMA 1. *If D is a ring derivation on a commutative algebra with identity, then $D(\alpha) = 0$ whenever α is rational.*

We omit the proof of the above lemma since it follows easily from the definition of the ring derivation.

Received by the editors May 9, 1988 and, in revised form, August 16, 1988.

AMS 1980 Subject Classification: 13B10, 46J10, 46J20

https://doi.org/10.4153/CMB-1990-012-1 Published online by Cambridge University Press

Now we state the following theorem which is a consequence of Theorems 1.6.3, and 2.3.4 of [2].

THEOREM 2. *Let A be a function algebra and let x_0 be an element of $Ch(A)$. If f is in A, then there exist two functions g and h in A such that $g(x_0) = h(x_0) = 0$ and $f - f(x_0) = gh$.*

LEMMA 3. *Let A be a function algebra and let D be a ring derivation on A. If $D(\alpha) = 0$ for every complex number α, then $D = 0$.*

PROOF. Let x_0 be in $Ch(A)$ and let f be in A. Then by Theorem 2 there exist g and h in A such that $g(x_0) = h(x_0) = 0$ and $f - f(x_0) = gh$. Hence we get

$$D(f) = D(f) - D(f(x_0)) = D(f - f(x_0)) = D(gh) = gD(h) + hD(g).$$

Evaluating the above at x_0, we obtain

$$D(f)(x_0) = 0, \text{ for all } x_0 \in Ch(A).$$

Since $Ch(A)$ is a boundary, we have $D(f) = 0$ and f being arbitrary the result follows.

LEMMA 4. *Let $K = \{x_n\}_{n=1}^{\infty} \cup \{x_0\}$ be a subset of $Ch(A)$ such that $\lim_{n\to\infty} x_n = x_0$. If g is a non-negative continuous function on K then there exists a function f in A such that $f|_K = g$.*

PROOF. Without loss of generality we assume that $g(x_n) \leqq 1$ and $\lim_{n\to\infty} g(x_n) = 0$. Otherwise consider g_1 defined by $g_1(x_n) = \|g\|_K^{-1}(g(x_n) - g(x_0))$. Since the maximal ideal space of A is Hausdorff, there exists a sequence of pairwise disjoint neighbourhoods $\{U_n\}$ such that x_n is in U_n for each n. For a given $\epsilon > 0$ let U_0 be a neighbourhood of x_0 such that $g(x_n) < \epsilon$ whenever x_n is in U_0. Let N be the largest index such that x_N is not in U_0. By Theorem 2.3.4 of [2] there exists a finite sequence of functions $\{f_n\}_{n=1}^{N}$ in A such that for each $n \leqq N$, $\|f_n\| \leqq 1, f_n(x_n) > 1 - (\epsilon/N)$, and $|f_n(y)| < \epsilon/N$ for y not in U_n. Let $f = \sum_{i=1}^{N} g(x_i) f_i$. We claim that $\|f - g\|_K < 2\epsilon$. For if $n \leqq N$ then

$$\begin{aligned} |f(x_n) - g(x_n)| &\leq \sum_{\substack{i=1 \\ i\neq n}}^{N} |g(x_i)f_i(x_n)| + |g(x_n)|\,|1 - f_n(x_n)| \\ &\leq (N-1)(\epsilon/N) + |g(x_n)|(\epsilon/N) \\ &\leq \epsilon \leqq 2\epsilon \end{aligned}$$

since $g(x_n) \leqq 1$. If $n > N$ or $n = 0$ then

$$\begin{aligned} |f(x_n) - g(x_n)| &\leqq |f(x_n)| + |g(x_n)| \\ &\leqq \sum_{i=1}^{N} |g(x_i)f_i(x_n)| + |g(x_n)| \\ &< N(\epsilon/N) + \epsilon \\ &= 2\epsilon. \end{aligned}$$

https://doi.org/10.4153/CMB-1990-012-1 Published online by Cambridge University Press

Hence g is in the closure of $A\mid_K$. Since K is a peak set in the weak sense (see pp 113, [2]), $A\mid_K$ is closed by Corollary 2.4.3 of [2] and the result follows.

3. Proof of the main theorem.

PROOF.Let x_0 be in J and let $\{x_n\}$ be a sequence of points in $Ch(A)$ such that $x_n \longrightarrow x_0$. For a given real number α choose a sequence of positive real numbers $\{s_n\}$ such that αs_n is rational for each n and $s_n \longrightarrow 1$. The function g defined by $g(x_0) = 1$, $g(x_n) = 1$ for n odd, and $g(x_n) = s_n$ for n even is a continuous function on $K = \{x_n\}_{n=1}^{\infty} \cup \{x_0\}$. By Lemma 4 there exists a function f in A such that $f\mid_K = g$. By Theorem 2 there exist two sequences of functions $\{h_n\}$ and $\{g_n\}$ in A with $g_n(x_n) = h_n(x_n) = 0$ for each n and such that

$$f - 1 = f - f(x_n) = g_n h_n \text{ for } n \text{ odd}$$

and

$$\alpha f - \alpha s_n = \alpha f - \alpha f(x_n) = g_n h_n \text{ for } n \text{ even .}$$

Operating with D on both sides and evaluating at x_n we obtain by Lemma 1 that $D(f)(x_n) = 0$ for n odd and $D(\alpha f)(x_n) = 0$ for n even. Since $D(f)$ and $D(\alpha f)$ are continuous functions in A we have $D(f)(x_0)$ and $D(\alpha f)(x_0) = 0$. Since $f(x_0) = 1$ we have

$$\begin{aligned} D(\alpha)(x_0) &= f(x_0)D(\alpha)(x_0) + \alpha D(f)(x_0) \\ &= D(\alpha f)(x_0) \\ &= 0 \text{ for all } x_0 \in J. \end{aligned}$$

Since J is dense in $Ch(A)$, $D(\alpha) = 0$ on $Ch(A)$. This implies $D(\alpha) = 0$. It is easy to see that $D(\alpha) = 0$ for any complex number α. Now by Lemma 3, the result follows.

4. Examples.

EXAMPLE 1. Let A be the disc algebra of continuous functions on the closed unit disc which are analytic in the interior. The ring derivations on this algebra are trivial since the Choquet boundary of A is the unit circle which is obviously a dense sequentially non-isolated set.

EXAMPLE 2. Consider the long line L (see page 71, [7]) constructed from the ordinal space $[0, \Omega]$ (where Ω is the least uncountable ordinal) by placing between each ordinal α and its successor $\alpha + 1$ a copy of the unit interval $I = (0, 1)$. Then L is a compact connected Hausdorff space when equipped with the order topology under the obvious ordering. The Choquet boundary of the function algebra A of complex valued continuous functions on L is L itself. Every point except Ω is the limit of a sequence of points of L. Hence $J = L \setminus \{\Omega\}$ is a dense sequentially non-isolated set. Thus every ring derivation on A is trivial.

In Example 1, the entire Choquet boundary can be taken as a dense sequentially non-isolated set. But in Example 2, only a proper subset of $Ch(A)$ has that property.

EXAMPLE 3. Let A be the algebra of all continuous complex valued functions on $X = [0, 1] \cup \{2\}$. The Choquet boundary of A is X. Since X has an isolated point there are no dense sequentially non-isolated sets in X. Since there exists a non-trivial ring derivation on the algebra of complex numbers, it is easy to see that there exists a non-trivial ring derivation on A.

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