

ON THE MT^* - AND λ -CONJUGATES OF \mathfrak{Q}^λ SPACES

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1. Introduction. Marston Morse and William Transue (9, 10), motivated by their theory of bilinear functions, introduced and studied vector function spaces called MT -spaces for which each element of the dual is represented by an integral with respect to a suitable (C) measure. In this paper the definition of real MT -spaces is generalized to give spaces, called MT^* -spaces, for which part but not all of the dual is of integral type and this part is called the MT^* -conjugate of the space. In the theory of \mathfrak{Q}^λ spaces (6) a conjugate space is also defined. It will be called the λ -conjugate below. An \mathfrak{Q}^λ space is an MT^* -space if and only if it contains \mathfrak{R} , the space of all continuous functions with compact support. The purpose of this paper is to compare the MT^* - and λ -conjugates of \mathfrak{Q}^λ spaces that are MT^* -spaces.

When E is countable at infinity the MT^* -conjugates and λ -conjugates coincide. Conditions ensuring that the MT^* -conjugate contains the λ -conjugate are given in Theorem 3.1, that the λ -conjugate includes the MT^* -conjugate in Theorems 4.1 and 4.2. Examples are given of \mathfrak{Q}^λ spaces, including the space $\bar{\mathfrak{Q}}^1$ (4, p. 13), for each of which the λ -conjugate strictly contains the MT^* -conjugate. Theorem 4.2 shows that, for a class of spaces E more general than the spaces E that are countable at infinity, the λ -conjugate always contains the MT^* -conjugate. Whether or not this is true for all E is not known. This paper makes essential use of many of the results of references (3, 4) and (8). The author wishes to express his thanks to Professor Morse for making available pre-publication copies of (8) and (10).

2. MT^* - and \mathfrak{Q}^λ -spaces and their conjugates. Let E be an arbitrary locally compact space and let \mathbf{R}^E denote the space of real valued functions on E .

Definition 2.1. We call A a (real) MT^* -space if: (i) A is a vector subspace of \mathbf{R}^E , (ii) A contains \mathfrak{R} , (iii) A contains $|x|$ if it contains x and (iv) there is a non-trivial, monotone semi-norm \mathfrak{N}^A defined on A .

We note that a real MT^* -space satisfies condition (i) in the definition of real MT -spaces apart from the requirement that \mathfrak{R} be dense in A (9, p. 168). If A' denotes the topological dual of A then, as in (9, Corollary 10.1), for every $\Phi \in A'$, the restriction of Φ to \mathfrak{R} defines a Radon measure ϕ on E such that for every $f \in \mathfrak{R}$,

$$(2.1) \quad \Phi(f) = \int f d\phi.$$

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The set of elements Φ in A' for which (2.1) holds for every $f \in A$ will be called the MT^* -conjugate of A and denoted by A^* . The mapping $\Phi \rightarrow \phi$ of A' into the vector space of all Radon measures on E is an isomorphism defining the MT^* -measure conjugate \mathfrak{A}^* of A (cf. 9, p.169). If $\phi \in \mathfrak{A}^*$, every $f \in A$ is integrable (ϕ) and to $\Phi(f)$, $\Phi \in A^*$. The definition

$$\|\phi\|_{\mathfrak{A}^*} = \sup_{x \in A, \mathfrak{N}^A(x) > 0} \frac{|\int x d\phi|}{\mathfrak{N}^A(x)},$$

for all $\phi \in \mathfrak{A}^*$, gives $\|\phi\|_{\mathfrak{A}^*} = \|\Phi\|_{A'}$.

A real MT -space is an MT^* -space for which $A' = A^*$. Conversely, if A is an MT^* -space for which $A' = A^*$, it is a real MT -space. Condition (ii) for MT -spaces is then satisfied by hypothesis. If \mathfrak{R} is assumed to be non-dense in A then (1, Lemme, p. 57) implies the existence of $\Phi \in A'$, not the zero element, vanishing in \mathfrak{R} . The corresponding ϕ is then the zero measure so that for some $f \in A$, $0 = \int f d\phi \neq \Phi(f)$ contradicting the fact that $A' = A^*$. Examples 3.1 and 3.2 of (10) show that there are MT^* -spaces in which \mathfrak{R} is dense that are not real MT -spaces.

Definition 2.2 (11, p. 53). Let $\mathfrak{M}, \mathfrak{M}'$ be families of subsets of E closed under the formation of countable unions and complements where \mathfrak{M}' is a proper sub-family of \mathfrak{M} and has the additional property that $M \in \mathfrak{M}', A \in E, A \subset M$ imply that $A \in \mathfrak{M}'$. For an \mathfrak{M} -measurable real valued function $f(P)$ on E , let $\|f\|_{\infty}$, the ess sup of $|f(P)|$, be the infimum of the set of numbers α such that $E_{\alpha} = \{P: |f(P)| < \alpha\}$ is in \mathfrak{M}' , if this set is non-void. Otherwise let $\|f\|_{\infty} = \infty$. Let $\mathfrak{L}_{\infty} = \mathfrak{L}_{\infty}(E, \mathfrak{M}, \mathfrak{M}')$ denote the subspace of \mathbf{R}^E for which $\|f\|_{\infty}$ is defined and finite.

The space \mathfrak{L}_{∞} is an MT^* -space with semi-norm $\|\cdot\|_{\infty}$ if every continuous function with compact support is \mathfrak{M} -measurable and in particular if \mathfrak{M} contains all the relatively compact Borel sets on E . The elements of A' to which correspond finitely additive measures that are not countably additive are not in A^* . If $E = (0, 1)$, if \mathfrak{M} denotes the Borel sets on E and \mathfrak{M}' the Borel sets of the first category on E , \mathfrak{L}_{∞} is an MT^* -space for which A^* reduces to the zero element of A' (2, Corollary 1, p. 186).

If A is an MT^* -space, \mathfrak{N}^A the semi-norm on A , \mathfrak{N}^A will be called reflexive if

$$(2.2) \quad \mathfrak{N}^A(x) = \sup \left(\left| \int x d\phi \right| ; \|\phi\|_{\mathfrak{A}^*} \leq 1, \phi \in \mathfrak{A}^* \right).$$

The left side of (2.2) is never less than the right side. When \mathfrak{N}^A is reflexive and the supremum in (2.2) can be replaced by $\sup \int |x| d|\phi|$, for all $\phi \in \mathfrak{A}^*$ with $\|\phi\|_{\mathfrak{A}^*} \leq 1$, \mathfrak{N}^A has a natural extension to all of \mathbf{R}^E . If A is an MT -space \mathfrak{N}^A is always reflexive and has such an extension (9, § 11). For the example above with $A^* = 0$, \mathfrak{N}^A is not reflexive.

Length functions and the corresponding function spaces were introduced in terms of non topological spaces E and general measures in (6). In this

paper we suppose given a positive Radon measure μ on an arbitrary locally compact space E . If $\lambda(f)$ is defined and $0 \leq \lambda(f) \leq \infty$ for every μ -measurable function $f(P)$ with $0 \leq f(P) \leq \infty$ almost everywhere, λ is called a length function if:

- (L 1) $\lambda(f) = 0$ whenever f is μ -negligible,
- (L 2) $\lambda(f) \leq \lambda(g)$ whenever $f(P) \leq g(P)$ for all $P \in E$,
- (L 3) $\lambda(f + g) \leq \lambda(f) + \lambda(g)$,
- (L 4) $\lambda(kf) = k\lambda(f)$ for all $k > 0$,
- (L 5) $f_1(P) \leq f_2(P) \leq \dots$ for all P implies that $\lambda(\sup f_n) = \sup \lambda(f_n)$.

A length function λ will be called continuous at infinity if, for every f ,

$$(L 6) \quad \lambda(f) = \sup_K \lambda(f\chi_K),$$

for all compact sets $K \in E$, where χ_K denotes the characteristic function of K . If $|f(P)|$ is μ -measurable, $\lambda(f)$ will mean $\lambda(|f|)$. A function f (set B) will be called λ -negligible if $\lambda(f) = 0$ ($\lambda(\chi_B) = 0$). $\mathfrak{L}^\lambda = \mathfrak{L}^\lambda(E, \mu)$ will denote the vector subspace of \mathbf{R}^E consisting of all μ -measurable $f \in \mathbf{R}^E$ with $\lambda(f) < \infty$. \mathfrak{L}^λ will be an MT^* -space if and only if it contains \mathfrak{R} . $L^\lambda = L^\lambda(E, \mu)$ will denote the normed space associated with \mathfrak{L}^λ . Every space L^λ is a Banach space (6, Theorem 3.1).

For every length function λ a λ -conjugate length function λ^* is defined by

$$(2.3) \quad \lambda^*(g) = \sup \int f(P)g(P)d\mu \leq \infty,$$

the supremum being taken for all $f \in \mathfrak{L}^\lambda$ with $\lambda(f) \leq 1$. The space \mathfrak{L}^{λ^*} will be called the λ -conjugate of \mathfrak{L}^λ and L^λ . A length function is reflexive if $\lambda(f) = \lambda^{**}(f)$ for every non-negative μ -measurable f . Necessary and sufficient conditions for the reflexivity of λ are given in (7, (4.1)) (for a general measure space). It can be shown that when λ is reflexive and \mathfrak{L}^λ is an MT^* -space for which the MT^* -conjugate contains the λ -conjugate then λ is also reflexive as a semi-norm on \mathfrak{L}^λ , $|\phi| \in \mathfrak{A}^*$ if $\phi \in \mathfrak{R}^*$ and

$$\lambda(f) = \sup \left(\int |f| |d|\phi|; \|\phi\|_{\mathfrak{A}^*} \leq 1, \phi \in \mathfrak{A}^* \right)$$

permitting a natural extension of λ to all of $\bar{\mathbf{R}}_+^E$:

Let μ denote a positive Radon measure on E . For $1 \leq p < \infty$, $\mathfrak{N}_p(f) = (\int f^p d\mu)^{1/p}$ is defined and non-negative for every μ -measurable f that is defined and non-negative almost everywhere. \mathfrak{N}_p then defines a length function. (L 1) follows from (3, Théorème 1, p. 119), (L 2) from (3, Proposition 10, p. 109) and (L 5) from (3, Théorème 3, p. 110) all applied to f^p , and (L 3) and (L 4) from (3, Proposition 2, p. 127). The μ -negligible and \mathfrak{N}_p -negligible sets coincide. If E is countable at infinity λ will be continuous at infinity. If E is arbitrary the length function \mathfrak{N}_p will not be continuous at infinity if E contains a locally negligible set that is not μ -negligible. By (3, Théorème 5, p. 194) the spaces \mathfrak{L}^p and \mathfrak{L}^λ with $\lambda = \mathfrak{N}_p$ coincide.

For every μ -measurable $f(P)$, with $0 \leq f(P) \leq \infty$ almost everywhere, write

$$\bar{\mathfrak{N}}_p(f) = \sup_K \mathfrak{N}_p(f\chi_K),$$

where K runs through the compact subsets of E . It is easily verified that $\bar{\mathfrak{N}}_p$ is a length function that is continuous at infinity and that the $\bar{\mathfrak{N}}_p$ -negligible sets are the locally negligible sets of E . We write $\bar{\mathfrak{Q}}^p$ and \bar{L}^p for the spaces $\mathfrak{Q}^\lambda, L^\lambda$ with $\lambda = \bar{\mathfrak{N}}_p$. From (4, Proposition 7, p. 13) it follows that $\bar{\mathfrak{Q}}^1$ is the space of essentially integrable functions for μ .

LEMMA 2.1. *If $f \in \bar{\mathfrak{Q}}^p$, the set $E(f) = (P:f(P) \neq 0)$ is the union of a countable sequence of compact sets and a locally negligible set E_0 .*

THEOREM 2.1. *If every locally null (μ) subset of E is μ -negligible, in particular if E is countable at infinity, \mathfrak{Q}^p and $\bar{\mathfrak{Q}}^p, 1 \leq p < \infty$ coincide. If E contains a locally null subset that is not μ -negligible, $\bar{\mathfrak{Q}}^p$ strictly contains $\mathfrak{Q}^p, \bar{L}^p$ and $L^p, 1 \leq p < \infty$, are equivalent.*

Lemma 2.1 is the analogue of (3, Lemme 2, p. 194). It is implied by (4, Corollaire, p. 13) for $p = 1$. Theorem 2.1 for $p = 1$ is a consequence of results in (4, § 2). In both cases the extension to all $p, 1 < p < \infty$ is not difficult.

When $p = \infty$ and \mathfrak{M} denotes the μ -measurable subsets of E , two length functions are obtained from Definition 2.2 by taking: (1) $\lambda = \mathfrak{N}_\infty = || \cdot ||_\infty$ with \mathfrak{M}' the μ -negligible subsets of E ; and (2) $\lambda = \bar{\mathfrak{N}}_\infty = || \cdot ||_\infty$ with \mathfrak{M}' the locally negligible subsets of E . $\bar{\mathfrak{N}}_\infty$ is continuous at infinity, \mathfrak{N}_∞ is not if E contains a locally negligible set that is not μ -negligible. The \mathfrak{Q}^λ spaces $\mathfrak{Q}_\infty(E, \mathfrak{M}, \mathfrak{M}')$ corresponding to (1) and (2) respectively will be denoted below by \mathfrak{Q}^∞ and $\bar{\mathfrak{Q}}^\infty$. These spaces are MT^* -spaces. Since $\bar{\mathfrak{N}}_\infty(f) \leq \mathfrak{N}_\infty(f), \mathfrak{Q}^\infty$ is always contained in $\bar{\mathfrak{Q}}^\infty$. In contrast to the case with $p < \infty, L^\infty$ and \bar{L}^∞ need not coincide. For example, if E contains a locally negligible set D with $\mu^*(D) = \infty, f_a(P) = a\chi_D$ is in \mathfrak{Q}^∞ for every finite a and the equivalence classes \hat{f}_a in L^∞ are different for different positive values of a . Each f_a is in $\bar{\mathfrak{Q}}^\infty$ but all belong to the equivalence class of $g(P) \equiv 0$. We note that the dual of \mathfrak{Q}^1 is $\bar{\mathfrak{Q}}^\infty$.

3. The λ -conjugate of $A = \mathfrak{Q}^\lambda$. In the sequel E will denote an arbitrary locally compact space, μ a positive Radon measure on E, K, K_i compact subsets of E , and λ will be an arbitrary length function for which \mathfrak{Q}^λ contains \mathfrak{R} . $\lambda(B)$ will be an abbreviation for $\lambda(\chi_B)$. A function g is locally integrable if it is μ -measurable and if $g\chi_K$ is integrable (μ) for every K . (Since E is locally compact this is equivalent to the definition in (8)).

When \mathfrak{Q}^λ is an MT^* -space it contains the characteristic function of every relatively compact μ -measurable subset of E . \mathfrak{Q}^λ contains \mathfrak{R} and since, given K , there exists a continuous function $f \in \mathfrak{R}$ coinciding with χ_K in K , if $B \subset K$ is μ -measurable $\lambda(B) \leq \lambda(K) \leq \lambda(f) < \infty$ and $\chi_B \in L^\lambda$. It then follows from

(2.3) that every $g \in \mathfrak{Q}^{\lambda*}$ is locally integrable. Thus g defines a Radon measure $g \cdot \mu$ (**8**, § 1) with values

$$(3.1) \quad \int f d(g \cdot \mu) = \int fg d\mu = g(f), \quad g \in (L^\lambda)',$$

for all $f \in K$. If for every $g \in \mathfrak{Q}^{\lambda*}$, (3.1) extends to all $f \in \mathfrak{Q}^\lambda$, the λ -conjugate of \mathfrak{Q}^λ is contained in the MT^* -conjugate. Since the right side of (3.1) is finite for all $f \in \mathfrak{Q}^\lambda$ if $g \in \mathfrak{Q}^{\lambda*}$, (**8**, Theorem 1.1) shows that it is sufficient to show that the left side is also always finite. There is no loss of generality in assuming that g is positive (that is non-negative) so that $g \cdot \mu$ is a positive Radon measure on E .

LEMMA 3.1. *If $f \in \bar{\mathbf{R}}_+^E$ is measurable (μ) and $g \geq 0$ is locally integrable and if $X = \cup_1^\infty K_i$, then*

$$(3.2) \quad \int^* fg\chi_X d\mu = \int^* f\chi_X d(g \cdot \mu) \leq \infty,$$

and both are finite if $f \in \mathfrak{Q}^\lambda$ and $g \in \mathfrak{Q}^{\lambda*}$.

Proof. The equality (3.2) is a corollary of (**8**, Lemma 5.1). When $f \in \mathfrak{Q}^\lambda$ and $g \in \mathfrak{Q}^{\lambda*}$, $\int^* fg\chi_X d\mu \leq \int fg d\mu < \infty$.

LEMMA 3.2. *If $f \in \mathfrak{Q}^\lambda$ is a positive lower semi-continuous function and $g \in \mathfrak{Q}^{\lambda*}$ then (3.1) holds.*

Proof. There is no loss of generality in supposing that g is non-negative. Then, using (3.1),

$$\int f d(g \cdot \mu) = \sup_{\substack{h \in \mathfrak{R} \\ 0 < h \leq f}} \int h d(g \cdot \mu) = \sup_{\substack{h \in \mathfrak{R} \\ 0 < h \leq f}} \int hg d\mu \leq \int fg d\mu < \infty.$$

Note. Lemmas 3.1 and 3.2 are also a consequence of (**4**, (Propositions 2 and 3, p. 9, and Theorem 1, p. 43)). See also (**8**, Note, p. 478).

LEMMA 3.3. *If $g \in \bar{\mathbf{R}}_+^E$ is locally integrable every μ -negligible set is $g \cdot \mu$ -negligible.*

Proof. Suppose that B is μ -null. If $g \in \bar{\mathbf{R}}_+^E$, $\int g\chi_B d\mu = 0$ by (**3**, Théorème 1, p. 119). Given $\epsilon > 0$, there exists an open set $U \supset B$ with $\mu(U) < \frac{1}{2}\epsilon$ and a l.s.c. function $h \geq \chi_{Bg}$ with $\mu(h) < \frac{1}{2}\epsilon$. Then $\text{env. sup. } (\chi_U, h)$ is l.s.c., $\geq \chi_{Ug}$ and

$$\int^* \chi_{Ug} d\mu \leq \int^* h d\mu + \int^* \chi_U d\mu < \epsilon.$$

If g is locally integrable, the measure $g \cdot \mu$ is defined ≥ 0 and, by (**3**, Corollaire 4, p. 158), and Lemma 3.1,

$$\begin{aligned} (g \cdot \mu)^*(B) &\leq (g \cdot \mu)^*(U) = \sup_{K \subset U} g \cdot \mu(K) = \sup_{K \subset U} \int \chi_K d(g \cdot \mu) \\ &= \sup_{K \subset U} \int \chi_K g \, d\mu \leq \int \chi_U g \, d\mu < \epsilon. \end{aligned}$$

Since ϵ is arbitrary B is $g \cdot \mu$ -null.

THEOREM 3.1. *Let $g \in \mathcal{Q}^{\lambda*}$. Then (i) \hat{g} is in the λ -conjugate $\mathcal{Q}^{\lambda*}$ and is in the MT^* -conjugate if and only if $\int f d(g \cdot \mu) = 0$ for every $f \in \mathcal{Q}^{\lambda}$ for which fg vanishes in E and (ii) if the MT^* -conjugate does not contain the λ -conjugate, E contains a locally negligible set B with $\lambda(B) < \infty$, with $\lambda(U) = \infty$ for every open set $U \supset B$ and $\mathcal{Q}^{\lambda*}$ contains g with $g\chi_B = 0$ and with $(g \cdot \mu)^*(B) = \infty$.*

Proof. (i) Suppose that $g \in \mathcal{Q}^{\lambda*}$ is positive. If $f \in \mathcal{Q}^{\lambda}$, $\int fg \, d\mu < \infty$ and, for fixed f , the set where $fg \neq 0$ is the union of X , the union of a sequence of compact sets, and a μ -negligible set E' . If $E_0 = (P:fg(P) = 0)$, $E = E_0 \cup E' \cup X$ and E_0 is measurable (μ). Suppose that f is non-negative. Using Lemma 3.1,

$$\int fg \, d\mu = \int fg \chi_X d\mu = \int f \chi_X d(g \cdot \mu) \leq \int^* f d(g \cdot \mu)$$

and, using Lemma 3.3, we have

$$\begin{aligned} \int^* f d(g \cdot \mu) &\leq \int^* f \chi_{E'} d(g \cdot \mu) + \int^* f \chi_{E_0} d(g \cdot \mu) + \int^* f \chi_X d(g \cdot \mu) \\ &= \int^* f \chi_{E_0} d(g \cdot \mu) + \int fg \, d\mu. \end{aligned}$$

Equality then holds if

$$\int^* f \chi_{E_0} d(g \cdot \mu) = 0.$$

Conversely, if $g \in \mathcal{Q}^{\lambda*}$ and if \hat{g} is in the MT^* -conjugate, then, whenever $f \in \mathcal{Q}^{\lambda}$ and fg vanishes in E , $\int f d(g \cdot \mu) = \hat{g}(f) = \int fg \, d\mu = 0$. When one or both of f, g is not positive the extension is trivial.

(ii) Suppose that $\mathcal{Q}^{\lambda*}$ contains g with \hat{g} not in the MT^* -conjugate A^* . Since g^+ and g^- , the positive and negative parts of g are also in $\mathcal{Q}^{\lambda*}$ and \hat{g}^+ and \hat{g}^- in A^* would imply that \hat{g} was in A^* , there is no loss of generality in assuming that g is positive. There then exists a positive $f \in \mathcal{Q}^{\lambda}$ with $fg(P) = 0$ in E but with $\int^* f d(g \cdot \mu) > 0$ and, since $\int fg \, d\mu = 0$ (8, Lemma 2.1) implies that $\int^* f d(g \cdot \mu) = \infty$,

Let $E_n = (P:fg(P) > 1/n)$. The μ -measurability of f implies that E_n is μ -measurable. Since $\chi_{E_n} \leq n f(P)$, $\lambda(E_n) \leq n\lambda(f)$ for each n . Furthermore

$$\begin{aligned} \int \chi_{E_n} g \, d\mu &\leq n \int fg \, d\mu = 0, & n = 1, 2, \dots, \\ \sup_{K \subset E} \int \chi_{E_n \cap K} d(g \cdot \mu) &= \sup_{K \subset E} \int \chi_{E_n \cap K} g \, d\mu = 0, \end{aligned}$$

so that each E_n is locally negligible.

Let $f_n(P) = \min. (n, f(P))$. Then f_n increases to f and by (3, Théorème 3, p. 110),

$$\sup_n \int^* f_n \chi_{E_n} d(g \cdot \mu) = \int^* f d(g \cdot \mu) = \infty.$$

Thus, for all sufficiently large n ,

$$\begin{aligned} \int^* f_n \chi_{E_n} d(g \cdot \mu) &> 0, \\ \int^* \chi_{E_n} d(g \cdot \mu) &\geq n^{-1} \int^* f_n \chi_{E_n} d(g \cdot \mu) > 0, \end{aligned}$$

and (8, Lemma 2.1) implies that

$$(3.3) \quad \int^* \chi_{E_n} d(g \cdot \mu) = \infty.$$

Finally, for all n for which (3.3) holds, $\lambda(U) = \infty$ for every open set U containing E_n since otherwise Lemma 3.2 implies that

$$\int^* \chi_{E_n} d(g \cdot \mu) \leq \int \chi_U d(g \cdot \mu) = \int \chi_U g d\mu \leq \lambda(U) \lambda^*(g) < \infty,$$

giving a contradiction.

As a corollary we list some of the conditions that imply that the MT^* -conjugate A^* of $A = \mathfrak{Q}^\lambda$ contains the λ -conjugate \mathfrak{Q}^{λ^*} :

- (1) E is countable at infinity.
- (2) For every $g \in \mathfrak{Q}^{\lambda^*}$, $g \cdot \mu$ is bounded.
- (3) For every $g \in \mathfrak{Q}^{\lambda^*}$ and every μ -measurable B , $(g \cdot \mu)^*(B) = \infty$ implies that $\lambda(B) = \infty$.
- (4) For every locally integrable g , $(g \cdot \mu)^*(B) = \infty$, $\lambda(B) < \infty$, imply that $\lambda^*(g) = \infty$.
- (5) If B is μ -measurable and if $\lambda(U) = \infty$ for every open set U containing B then $\lambda(B) = \infty$.
- (6) $\lambda(E) < \infty$ or $\lambda(K)$ is bounded for all K in E .

If $A = \mathfrak{Q}^p$, $1 \leq p < \infty$, and B is μ -measurable then $\mathfrak{N}_p(B) = \mu^{*1/p}(B)$ which implies (5). To prove (6) suppose that $\lambda(E) = M < \infty$ or that $\lambda(K) \leq M < \infty$ for every $K \subset E$. The existence of B with $(g \cdot \mu)^*(B) = \infty$ implies that $(g \cdot \mu)^*(E) = \infty$ and (3, Corollaire 4, p. 158) implies that E contains compact subsets with arbitrarily large $(g \cdot \mu)$ -measure. Then

$$\lambda^*(g) \geq \sup_K \int \chi_K g d\mu / \lambda(K) \geq \sup_K g \cdot \mu(K) / M = \infty,$$

so that (4) applies. The spaces \mathfrak{Q}^∞ , $\bar{\mathfrak{Q}}^\infty$ satisfy (6).

THEOREM 3.2. *There exist MT^* -spaces \mathfrak{Q}^λ , with λ reflexive and continuous at infinity, for which the λ -conjugate strictly contains the MT^* -conjugate. In particular the spaces $\bar{\mathfrak{Q}}^p$, $1 \leq p < \infty$, are of this type for suitable μ , E .*

Proof. Let $E, D, T_n(y)$ and μ be defined as in the example of a locally compact space E that is not countable at infinity given in (3, Exercice 4, p. 116) and let $g = \chi_{E-D}$. Then $\overline{\mathfrak{M}}_\infty(g) = 1$ and $\hat{g} \in \tilde{L}^\infty = \mathfrak{Q}^{\lambda^*}, \lambda = \overline{\mathfrak{M}}_1$. As in (8, Exercice 4) $\chi_D \in \tilde{\mathfrak{Q}}^1$ but is not integrable ($g \cdot \mu$) so that \hat{g} does not belong to the MT^* -conjugate of $\tilde{\mathfrak{Q}}^1$

To give an example for $1 < p < \infty$ let E, D be as before but for δ fixed, $0 < \delta < 1, P_{ni}$ the point $(1 - n, i/n^2)$, define $\beta(P_{ni}) = n^{-2-\delta}$ and define $\beta(P) = 0$ for the points $(0, y)$ in E . Let μ' denote the measure determined by the masses $\beta(P)$. Define $g(P) = n^{\delta-1}$ for $P = P_{ni}$ ($i = 0, 1, \dots, n^2; n = 1, 2, \dots$); $g(P) = 0$ elsewhere in E . Actual computation shows that every compact subset of E has finite μ' -measure and that, for $\lambda = \overline{\mathfrak{M}}^p, \lambda^*(g) = \overline{\mathfrak{M}}^q(g) < \infty$ so that $\hat{g} \in \mathfrak{Q}^{\lambda^*}$. Suppose that B is a subset of $(0, y)$ dense (in the usual topology on \mathbf{R}) on some interval $(a, b), -1 < a < b < 1$ and let $U_{B,n} = \cup T_i(y)$ for all $i \geq n, y \in B$. Computation shows that $(g \cdot \mu')^*(U_{B,n}) = \infty$ for every n . As in the Bourbaki example every open set containing D contains some set $U_{B,n}$ and therefore $(g \cdot \mu')^*(D) = \infty$ although D is locally negligible ($g \cdot \mu'$). Since the dual of $\tilde{\mathfrak{Q}}^p$ coincides with the λ -conjugate, $1 < p < \infty$, the λ -conjugate strictly contains the MT^* -conjugate. The length functions $\overline{\mathfrak{M}}_p$ are continuous at infinity and reflexive.

Remark. The right side of (2.1) is $\mathfrak{N}_1(f, \phi)$. Replacing \mathfrak{N}_1 by $\overline{\mathfrak{M}}_1$ in the definition of the MT^* -conjugate gives a conjugate which always contains the λ -conjugate when $A = \mathfrak{Q}^\lambda$.

4. The MT^* -conjugate of $A = \mathfrak{Q}^\lambda$. In this section Φ denotes an arbitrary element of the MT^* -conjugate of $A = \mathfrak{Q}^\lambda, \phi$ the Radon measure corresponding to Φ determined by the restriction of Φ to K .

LEMMA 4.1. *If $\Phi \in A^*$ and $A = \mathfrak{Q}^\lambda(\mu)$ there exists a locally μ -integrable function g for which the measure $g \cdot \mu$ coincides with ϕ .*

Proof. There is no loss of generality in assuming $\Phi \geq 0$. By hypothesis every $f \in A$ is integrable (ϕ) with

$$(4.1) \quad \Phi(f) = \int f d\phi.$$

For each compact set K every μ -measurable subset e is in A and $\Phi(\chi_e) = \int \chi_e d\phi = \phi(e)$. If $\mu(e) = 0$,

$$|\phi(e)| = |\Phi(e)| \leq |\Phi|\mathfrak{N}^A(\chi_e) = |\Phi|\lambda(e).$$

Since $\lambda(e) = 0$ whenever $\mu(e) = 0$, every set that is locally μ -negligible is locally ϕ -negligible. The Lebesgue-Nikodym theorem (4, Théorème 2, p. 47) then implies that ϕ is a measure of base μ , that is, that there exists a locally integrable point function g with $\phi = g \cdot \mu$ (4, p. 42).

THEOREM 4.1. *Let $\Phi \in A^*$ and let g in \mathbf{R}^E be locally integrable with $g \cdot \mu \equiv \phi$. Then (i) $g \in \mathfrak{Q}^{\lambda^*}$ if and only if $\mu(g\chi_B) = 0$ for every ϕ -negligible set B that is*

the union of a sequence $\{B_i\}$ with each function χ_{B_i} in \mathfrak{Q}^λ , and (ii). If $g \notin \mathfrak{Q}^{\lambda*}$, E contains a set D , negligible ($g \cdot \mu$) and locally negligible (μ) with $\lambda(D) < \infty$ and $\mu^*(D) = \infty$.

Proof. (i) There is no loss of generality in considering only positive f and Φ in the proof of (i) and (ii). If $f \in \mathfrak{Q}^\lambda$, $\Phi \in A^*$, then $f \in \mathfrak{Q}^1(\phi)$ and $E = E_0 \cup E' \cup X$, where $E_0 = (P: f(P) = 0)$, E' is ϕ -negligible and X is the union of a countable sequence of compact sets. Since E, E_0 and X are measurable (μ) so is E' . Since $\int f \chi_X d(g \cdot \mu) \leq \int f d(g \cdot \mu) = \int f d\phi < \infty$, (3.2) holds finitely. Since

$$fg\chi_{E_0}$$

vanishes,

$$\int fg\chi_{E_0} d\mu = 0.$$

Thus

$$\begin{aligned} (4.4) \quad \int f d\phi &= \int f d(g \cdot \mu) = \int fg\chi_X d\mu \leq \int^* fg d\mu \\ &\leq \int^* fg\chi_{E_0} d\mu + \int^* fg\chi_{E'} d\mu + \int fg\chi_X d\mu = \int^* fg\chi_{E'} d\mu + \int f d\phi. \end{aligned}$$

If $E'_i = (P \in E': f(P) > 1/i)$, $E' = \cup_1^\infty E'_i$, each E'_i is μ -measurable and $\lambda(E'_i) \leq i\lambda(f) < \infty$ so that each $\chi_{E'_i} \in \mathfrak{Q}^\lambda$. If the hypothesis of (i) is satisfied, $\int \chi_{E'} g d\mu = 0$ which implies that $\int \chi_{E'} fg d\mu = 0$. Then $\int fg d\mu = \int f d\phi = \Phi(f)$ for every $f \in \mathfrak{Q}^\lambda$ and $g \in \mathfrak{Q}^{\lambda*}$. Conversely, if $g \in \mathfrak{Q}^{\lambda*}$, $\int fg d\mu < \infty$ for every $f \in \mathfrak{Q}^\lambda$ and, by (8, Theorem 1.1),

$$(4.5) \quad \int fg d\mu = \int f d(g \cdot \mu) = \Phi(f)$$

for all $f \in \mathfrak{Q}^\lambda$. If B_i is ϕ -negligible with $\chi_{B_i} \in \mathfrak{Q}^\lambda$ (4.4) and (4.5) imply that

$$\int g\chi_{B_i} d\mu = 0$$

whence $\int g\chi_B d\mu = 0$ if $B = \cup_1^\infty B_i$.

(ii) If $g \notin \mathfrak{Q}^{\lambda*}$ (i) implies that there exists a ϕ -negligible set $B = \cup_1^\infty B_i$, where $\lambda(B_i) < \infty, i = 1, 2, \dots$, and such that $\mu^*(g\chi_{B_i}) > 0$. Then (8, Lemma 2.1) implies that $\mu^*(g\chi_B) = \infty$. Writing $B(n) = \cup_1^n B_i$, since $B(n) \uparrow B$, $\mu^*(g\chi_{B(n)})$ is positive and therefore infinite for all sufficiently large n , say $n > n_0$. We show that for a fixed $n > n_0$, $D = (P \in B(n) : g(P) > 0)$ satisfies all the conditions (ii). We note that $\lambda(D) \leq \lambda(B(n)) < \infty$ and that $\mu^*(g\chi_D) = \mu^*(g\chi_{B(n)}) = \infty$. Consider $g_m(P) = \min(m, g(P))$. It is locally integrable and defines a Radon measure $g_m \cdot \mu$ with $0 < g_m \cdot \mu \leq g \cdot \mu$ and $g_m \cdot \mu(D) \leq g \cdot \mu(D) \leq g \cdot \mu(B) = 0$. Now

$$\infty = \int^* g\chi_D d\mu = \sup_m \int^* g_m\chi_D d\mu,$$

and (8, Lemma 2.1) implies that $\mu^*(g_m \chi_D) = 0$ or ∞ for each m and therefore is infinite for all sufficiently large m so that

$$\int^* \chi_D d\mu \geq m^{-1} \int^* g_m \chi_D d\mu = \infty.$$

Finally, if $D_i = \{P \in D : g(P) > 1/i\}$, $D = \cup_1^\infty D_i$ and, for an arbitrary compact set K ,

$$\begin{aligned} \int \chi_{D \cap K} d\mu &= \sup_i \int \chi_{D_i \cap K} d\mu \leq \sup_i i \int g \chi_{D_i \cap K} d\mu \\ &\leq \sup_i i \int \chi_D d(g \cdot \mu) = 0. \end{aligned}$$

Thus D is locally μ -negligible.

A variety of conditions, sufficient to ensure that the λ -conjugate contains the MT^* -conjugate, follow from Theorem 4.1. We mention only: (i) E is countable at infinity, (ii) μ is bounded and (iii) $\mu^*(B) = \infty$ implies that $\lambda(B) = \infty$. Condition (iii) shows that if $A = \mathfrak{Q}^p$, $1 \leq p < \infty$, the λ -conjugate always contains the MT^* -conjugate so that these conjugates then coincide. Actually, each of (i)-(iii) implies that every locally integrable $g \in \mathbf{R}^E$ is in \mathfrak{Q}^{λ^*} whereas \mathfrak{Q}^{λ^*} will contain A^* if to each $\Phi \in A^*$ corresponds one $g \in \mathfrak{Q}^{\lambda^*}$ with $g \cdot \mu \equiv \Phi$. If g is locally integrable with $g \cdot \mu \equiv \Phi$, every g' that is locally equivalent to g is also locally integrable and $g' \cdot \mu \equiv \Phi$. Consider (8, Example 5.2) where $g = \chi_H$, $f = \chi_E$ and the length functions $\lambda = \overline{\mathfrak{M}}_p$, $1 \leq p < \infty$. Then $g \notin \mathfrak{Q}^{\lambda^*}$ but g is locally equivalent to the zero element of \mathfrak{Q}^{λ^*} . More generally, let E , D , and μ be defined as in Theorem 3.2 and let λ denote an arbitrary length function. Then D is locally negligible with $\mu^*(D) = \infty$ but every locally negligible subset of $E - D$ is μ -negligible. Replacing g by $g\chi_D$ gives a g' locally equivalent to g with $\int^* g' \chi_B d\mu = \int^* g \chi_{B \cap D} d\mu = 0$ for every locally negligible set B with $\lambda(B) < \infty$. Since

$$(g' \cdot \mu)^*(B) \leq \int^* g' \chi_B d\mu$$

by (4.4) this contradicts Theorem 4.1 (ii) if g' is not in \mathfrak{Q}^{λ^*} .

Edwards (5, p. 143) defines the μ -measure of a μ -measurable set B to be $\mu(B) = \sup_K \int \chi_{B \cap K} d\mu$, where K runs through the compact subsets of E .

THEOREM 4.2. *If E contains E^* with $\mu^*(A) < \infty$ for every locally negligible set contained in $E - E^*$ and if E^* is the union of a countable collection of sets of finite μ -measure, then for every MT^* -space \mathfrak{Q}^λ on E , the λ -conjugate contains the MT^* -conjugate.*

Proof. First suppose that $\mu(E^*) < \infty$. By the argument of (5, Theorem 7 (4)), $E^* = Q_1 \cup Q_2$ where Q_1 is the union of a countable collection of compact sets and Q_2 is locally null. If $A \subset E^*$ with $\mu(A) = 0$, $\mu^*(A) = \infty$, $\mu(Q_1 \cap A) = 0$. Set $g' = g\chi_X$, where $X = CE^* \cup Q_1$. If $K \subset CE$ or $K \subset Q_1$, $g(P) = g'(P)$ for $P \in K$. Suppose that $\mu(K \cap CE^*) > 0$ and $\mu(K \cap E^*) > 0$.

Then $K = N \cup_1^\infty K_n$ where N is μ -negligible and each K_n is compact and contained in one of E^* , CE^* (3, pp. 181-2). Since $\mu(K_n \cap Q_2) = 0, n = 1, 2, \dots$, $g = g'$ almost everywhere in each K and g' is locally equivalent to g . For each such $A \subset E^*$,

$$\mu^*(g'\chi_A) = \mu^*(g\chi_{Q_1 \cup A}) = 0$$

contradicting Theorem 4.2 (ii) if $g' \notin \mathfrak{L}^{\lambda*}$. If $E^* = \cup_1^\infty E_n$ with $\mu(E_n) < \infty, n = 1, 2, \dots$, each $E_n = Q_{1n} \cup Q_{2n}$ as above and the preceding argument holds when applied to $\cup_n Q_{1n}, \cup_n Q_{2n}$ in place of Q_1 and Q_2 respectively.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires* (Warsaw, 1932).
2. G. Birkhoff, *Lattice Theory*, Am. Math. Soc. Colloquium Publications, vol. XXV (New York, 1948).
3. N. Bourbaki, *Eléments de Mathématique*, Livre VI, Integration, chaps. I-IV (Paris, 1952).
4. ———, *Eléments de Mathématique*, Livre VI, Integration, chap. V (Paris, 1956).
5. R. E. Edwards, *A theory of Radon measures on locally compact spaces*, Acta. Math., 89 (1953), 133-164.
6. H. W. Ellis and I. Halperin, *Function spaces determined by a levelling length function*, Can. J. Math., 5 (1953), 576-592.
7. I. Halperin and W. A. J. Luxemburg, *Reflexivity of the length function*, Proc. Amer. Math. Soc., 8 (1957), 496-9.
8. M. Morse and W. Transue, *Products of a C-measure and a locally integrable mapping*, Can. J. Math., 9 (1957), 475-486.
9. ———, *Semi-normed vector spaces with duals of integral type*, J. d'Analyse Math., 4(1954-5), 149-186.
10. ———, *Vector subspaces A of $C^{\mathbb{B}}$ with duals of integral type*, J. Math. pures et appl., to appear.
11. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc., 72 (1952), 46-66.

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