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GENERALISED SOLUTIONS OF HESSIAN EQUATIONS

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We introduce a definition of generalised solutions of the Hessian equation $S_m(D^2u) = f$ in a convex set $\Omega \subset \mathbb{R}^n$, where $S_m(D^2u)$ denotes the m-th symmetric function of the eigenvalues of D^2u, $f \in L^p(\Omega)$, $p \geqslant 1$, and $m \in \{1, \dots, n\}$. Such a definition is given in the class of semi-convex functions, and it extends the definition of convex generalised solutions for the Monge-Ampère equation. We prove that semiconvex weak solutions are solutions in the sense of the present paper.

0. INTRODUCTION

In this note we deal with the so-called *Hessian equations*:

$$S_m(D^2u) = f > 0 \quad \text{in } \Omega. \tag{0.1}$$

Here $S_m(D^2u)$ denotes the m-th symmetric function of the eigenvalues of the Hessian matrix of u, $m \in \{1, \dots, n\}$, and Ω is an open bounded subset of $\mathbb{R}^n$.

The aim of the present paper is introducing a definition of *generalised solution* of equation (0.1). To do this, we restrict ourselves to the class of semiconvex functions defined over a convex set Ω and we prove that if u belongs to such class, $n+1$ Borel real measures $\sigma_0(u;\cdot), \dots, \sigma_n(u;\cdot)$, can be defined, which generalise the integrals of the functions $S_m(D^2u)$. Namely, if $u \in C^2(\Omega)$, then

$$\binom{n}{k}\sigma_k(u;\eta) = \int_\eta S_k(D^2u), \quad k = 0, \dots, n,$$

for every Borel subset η of Ω. Then we say that u is a generalised solution of (0.1) if

$$\binom{n}{m}\sigma_m(u;\eta) = \int_\eta f,$$

for every Borel subset η of Ω (see Definition 4.1).

Notice that, if u is convex, then $\sigma_n(u;\cdot)$ is the measure of the subgradient map of u (see Section 2 for details). Hence for $m = n$, that is, when (0.1) is the Monge-Ampère

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equation, the above definition coincides with the usual notion of convex generalised solution of the Monge-Ampère equation, introduced by Aleksandrov (see for instance [**2, 12**] and references therein).

The present paper originated from a private communication between the authors and professor Neil Trudinger, during a conference on Elliptic PDE held in Cortona (Italy) in May 1996. Trudinger [**11**] establishes existence and uniqueness results for weak solutions of certain Dirichlet problems involving equations (0.1); such solutions are limit of solutions of smooth approximating problems. In Theorem 4.2 we prove that a semiconvex weak solution of (0.1) is also a generalised solution.

In Section 1 we give some preliminaries on semiconvex functions, while in Section 2 and Section 3 we state and then prove the existence of the measures $\sigma_k(u;\cdot)$. Finally, in Section 4, we give the definition of the generalised solutions and we prove Theorem 4.2.

1. Semiconvex functions

In this section we recall briefly the notion of a semiconvex function. This class of functions was studied by several authors: see for instance [**3**] and [**7**].

Throughout, Ω is an open convex and bounded subset of $\mathbb{R}^n$ and $\|\cdot\|$ denotes the Euclidean norm. $\mathcal{L}^n$ stands for the n-dimensional Lebesgue measure and $\mathfrak{B}(A)$ is the family of Borel subset of a measurable set $A \subset \mathbb{R}^n$.

Definition: A real-valued function u, defined in Ω, is *semiconvex* if there exists $c \geqslant 0$ such that the function $u(x) + c\|x\|^2/2$ is convex in Ω.

If u is semiconvex in Ω, we call the real number

$$sc(u,\Omega) = \inf\left\{c \geqslant 0 \ : \ u + \frac{c\|x\|^2}{2} \text{ is convex in } \Omega\right\} \tag{1.1}$$

the *semiconvexity modulus* of u in Ω.

We denote the class of semiconvex functions in Ω by $W(\Omega)$ and set $W(\Omega, c) = \{u \in W(\Omega) \ : \ sc(u,\Omega) \leqslant c\}$ for every $c \geqslant 0$.

For a convex function v let $\partial v(x)$ be the subdifferential of v at x. If u is semiconvex we denote by $\partial u(x)$ the set $\{w - cx \ : \ w \in \partial\big(u + \big(c\|x\|^2\big)/2\big)(x)\}$, which coincides with the *Clarke generalised gradient* of u at x (see [**7**] for references). By well-known properties of the subdifferential of a convex function, $\partial u(x)$ is a nonempty closed and convex set for every $x \in \Omega$.

2. Generalised symmetric functions of the Hessian

We recall that, for real numbers $\beta_1, \dots, \beta_n$, and $1 \leqslant m \leqslant n$, the m-th symmetric function of $\beta_1, \dots, \beta_n$ is defined by

$$S_m(\beta_1, \dots, \beta_n) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant n} \beta_{i_1} \beta_{i_2} \dots \beta_{i_m} .$$

Furthermore, we set $S_0(\beta_1, \dots, \beta_n) = 1$.

If u is of class C^2, we denote by $S_m(D^2 u)$ the m–th symmetric function of the eigenvalues of the Hessian matrix $D^2 u$ of u. For a Borel subset η of Ω, and for $\rho \geqslant 0$, consider the set $P_\rho(u;\eta) = \{x + \rho \nabla u(x) \, : \, x \in \eta\}$. If $u \in C^2(\overline{\Omega})$, then by the area formula, for sufficiently small ρ, we get

$$\mathcal{L}^n(P_\rho(u;\eta)) = \int_\eta \det\left(I + \rho D^2 u\right) = \sum_{i=0}^{n} \left(\int_\eta S_i(D^2 u) \right) \rho^i ;$$

here I stands for the $n \times n$ identity matrix.

Now let $u \in W(\Omega)$; for any nonnegative ρ and for any subset $\eta \subseteq \Omega$, we set

$$P_\rho(u;\eta) = \{z \in \mathbb{R}^n \, : \, z = x + \rho v, \, x \in \eta, \, v \in \partial u(x)\} .$$

Clearly if $u \in C^1(\Omega) \cap W(\Omega)$ this definition coincides with the one given above.

The following result generalises [**4**, Theorem 1.1] (see also [**8**, Proposition 3.1]).

Theorem 2.1. *Let Ω be an open bounded convex set in $\mathbb{R}^n$, let $u \in W(\Omega, c)$ for some $c \geqslant 0$, and let u be Lipschitz. Then, for every Borel subset $\eta \subseteq \Omega$ and for every $\rho \in [0, 1/c)$, the set $P_\rho(u;\eta)$ is Lebesgue measurable. Moreover, there exist $n+1$ real-valued Borel measures $\sigma_i(u;\cdot)$, $i = 0, \dots, n$, such that:*

$$\mathcal{L}^n(P_\rho(u;\eta)) = \sum_{j=0}^{n} \binom{n}{i} \sigma_j(u;\eta) \rho^j , \tag{2.1}$$

for every $\rho \in [0, 1/c]$ and for every Borel subset η of Ω.

If u is convex, then, as proved in [**4**, Theorem 3.1] $\sigma_n(u;\eta) = \mathcal{L}^n(\{v \in \mathbb{R}^n : v \in \partial u(x), \, x \in \eta\})$, for every $\eta \in \mathfrak{B}(\Omega)$. Thus $\sigma_n(u;\cdot)$ is the measure of the subgradient map of u.

Remark. In [**6**] Federer established a well-known Steiner formula for *sets with positive reach*. Formula (2.1) can be seen as a counterpart of such formulas in the context of semiconvex functions. Note that, as proved by Fu [**7**], sets of positive reach can be characterised as sublevel sets of semiconvex functions.

3. Proof of Theorem 2.1

In order to prove Theorem 2.1 we need some preliminary results.

Lemma 3.1. *Let $u \in W(\Omega, c)$, $c \geqslant 0$, and let $0 \leqslant \rho c < 1$. There exists a Lipschitz map π_u from $P_\rho(u;\Omega)$ to Ω, such that for every $z \in P_\rho(u;\Omega)$, $z = \pi_u(z) + \rho v$, where $v \in \partial u(\pi_u(z))$.*

Proof: Let $z, z' \in P_\rho(u;\Omega)$ and let $x, x' \in \Omega$ be such that $z = x + \rho v$, $v \in \partial u(x)$, and $z' = x' + \rho v'$, $v' \in \partial u(x')$. We prove that there exists a constant $L > 0$ such that $\|x - x'\| \leqslant L\|z - z'\|$. We choose a coordinate system such that $x = (0, 0, \dots, 0)$ and $x' = (t', 0, \dots, 0)$, $t' \geqslant 0$. Let $u^*(t) = u(t, 0, \dots, 0)$: such a function is defined and semiconvex, with semiconvexity modulus not greater than c, on an open interval $(-\varepsilon, t' + \varepsilon)$, for some $\varepsilon > 0$. Moreover, if $v = (v_1, v_2, \dots, v_n)$ and $v' = (v_1', v_2', \dots, v_n')$, then $v_1 \in \partial u^*(0)$ and $v_1' \in \partial u^*(t')$. By the definition of subgradient, this implies $v_1' - v_1 \geqslant -ct'$, and, if $c\rho < 1$, then

$$\begin{aligned}\|z - z'\| \geqslant |\langle z - z', e\rangle| &= |\langle x + \rho v, e\rangle - \langle x' + \rho v', e\rangle| \\ &= |\rho(v_1' - v_1) + t'| \geqslant t'(1 - c\rho) = \|x - x'\|\,(1 - c\rho),\end{aligned}$$

where $e = (1, 0, \dots, 0)$. Thus π_u is well defined and $\|x - x'\| = \|\pi_u(z) - \pi_u(z')\| \leqslant L\|z - z'\|$, with $L = 1/(1 - c\rho)$. ∎

Remark. The continuity of π_u implies that $P_\rho(u;\eta) = \pi_u^{-1}(\eta)$ is measurable for any Borel subset η of Ω.

For a positive R, let $B(R)$ be the open ball centred at the origin with radius R.

Lemma 3.2. *Let $u \in W(\Omega, c)$, $c \geqslant 0$, and let u be Lipschitz. Then there exists a semiconvex function $w \in W(\mathbb{R}^n, c)$ which extends u to $\mathbb{R}^n$. Furthermore w is radially symmetric and C^∞ in the complement of $B(R)$, for some $R > 0$.*

Proof: We consider the function $k(x) = u(x) + \left(c\|x\|^2\right)/2$, which is convex and Lipschitz in Ω. [**4**, Lemma 2.3] ensures that k can be extended by a convex and Lipschitz function k^*, defined in $\mathbb{R}^n$, which is radially symmetric and C^∞ in the complement of $B(R)$, for a suitable R. Consequently $w(x) = k^*(x) - \left(c\|x\|^2\right)/2$ provides the required extension of u. ∎

Lemma 3.3. *Let w be a semiconvex and Lipschitz function defined in $\mathbb{R}^n$, such that w is radially symmetric and C^∞ in the complement set of $B(R)$, $R > 0$. Then, for every ρ such that $1/\rho > sc(w, \mathbb{R}^n)$, $P_\rho(w; B(2R)) = B(2R + \rho L)$, where $L = \|Dw(x)\|$ for $\|x\| = 2R$.*

Proof: By the continuity of π_w we get

$$\begin{aligned}\overline{P_\rho(w; B(2R))} = \overline{\pi_w^{-1}(B(2R))} &= \pi_w^{-1}\left(\overline{B(2R)}\right) \\ &= \pi_w^{-1}(B(2R) \cup \partial B(2R)) = \pi_w^{-1}(B(2R)) \cup \pi_w^{-1}(\partial B(2R)).\end{aligned}$$

On the other hand $P_\rho(w; B(2R)) = \pi_w^{-1}(B(2R))$ is open, hence $\partial P_\rho(w; B(2R)) \subseteq \pi_w^{-1}(\partial B(2R)) = \partial B(2R + \rho L)$. Since the only open sets whose boundary is contained in $\partial B(2R+\rho L)$ are $B(2R+\rho L)$ and its complement, and since the Lipschitz continuity of w entails that $P_\rho(w;\eta)$ is bounded for every bounded η, the assertion follows. $\square$

Let $u_\varepsilon = \phi_\varepsilon * u$ be the standard mollification of u. As usual $\phi_\varepsilon(x) = (1/\varepsilon^n)\phi(x/\varepsilon)$, $\varepsilon > 0$, where $\phi \in C_0^\infty(\mathbb{R}^n)$ is a radially symmetric function supported in the unit ball, such that $0 \leqslant \phi \leqslant 1$ and $\int_{\mathbb{R}^n} \phi = 1$.

It is easily seen that if $u \in W(\mathbb{R}^n, c)$, then $u_\varepsilon \in W(\mathbb{R}^n, c)$ for every $\varepsilon \geqslant 0$, and u_ε converges uniformly to u on compact sets as $\varepsilon \to 0$. For brevity, we set $u_i = u_{1/i}$, $i \in \mathbb{N}$.

Proof of Theorem 2.1: Let w be a function which extends u to $\mathbb{R}^n$, radially symmetric and C^∞ outside a ball $B(R)$; such a function exists by Lemma 3.2. Consider the sequence w_i, $i \in \mathbb{N}$. It is easily seen that, for sufficiently large i, w_i is radially symmetric outside the ball $B(2R)$; let $B = B(4R)$ throughout.

For a fixed $\rho \in [0, 1/c)$ and for every Borel subset $\eta \subset \Omega$, the sets $P_\rho(w;\eta)$ and $P_\rho(w_i;\eta)$ are measurable for every $i \in \mathbb{N}$. (See the remark following Lemma 3.1.)

Let $\Theta(\rho,\eta) = \mathcal{L}^n(P_\rho(w;\eta))$ and $\Theta_i(\rho,\eta) = \mathcal{L}^n(P_\rho(w_i;\eta))$, $\forall\, \eta \in \mathfrak{B}(B)$, $\forall\, i \in \mathbb{N}$.

We first prove that the sequence of measures Θ_i, converges weakly to Θ in B. (We refer to [**1**] for the notion of weak convergence of measures and related properties.) By [**1**, Theorem 4.5.1] it suffices to prove that $\lim_{i\to\infty} \Theta_i(\rho, B) = \Theta(\rho, B)$, and that for every closed $\eta \subset B$ we have $\limsup_{i\to\infty} \Theta_i(\rho,\eta) \leqslant \Theta(\rho,\eta)$.

By Lemma 3.3 we get $\Theta(\rho, B) = \mathcal{L}^n(B(4R+\rho L))$ and $\Theta_i(\rho, B) = \mathcal{L}^n(B(4R+\rho L_i))$, where $L_i = \|Dw_i(x)\|$, for $x \in \partial B$. By the uniform convergence of the sequence w_i to w on compact sets, we have $\lim_{i\to\infty} L_i = L$ where $L = \|Dw(x)\|$ for $x \in \partial B$. Hence it follows that $\lim_{i\to\infty} \Theta_i(\rho, B) = \Theta(\rho, B)$.

Next we prove that, if $\varepsilon > 0$, η is a closed subset of B and i is sufficiently large, then

$$(P_\rho(w;\eta))_\varepsilon \supset P_\rho(w_i;\eta) \tag{3.1}$$

where $A_\varepsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) \leqslant \varepsilon\}$ for a set $A \subset \mathbb{R}^n$. We argue by contradiction: assume for every i there exist $\lambda_i \geqslant i$ and $x_{\lambda_i} \in \eta$ such that

$$x_{\lambda_i} + \rho D w_{\lambda_i}(x_{\lambda_i}) \notin (P_\rho(w;\eta))_\varepsilon\,. \tag{3.2}$$

Since η is compact and the functions w_i are uniformly Lipschitz, the sequences x_{λ_i} and $Dw_{\lambda_i}(x_{\lambda_i})$, are both bounded. Thus there exists a subsequence x_{μ_i} of x_{λ_i}, such that

$x_{\mu_i} \to x \in \eta$ and $Dw_{\mu_i}(x_{\mu_i}) \to v \in \partial w(x)$. Hence $\lim_{i\to\infty} x_{\mu_i} + \rho Dw_{\mu_i}(x_{\mu_i}) = x + \rho v$ which contradicts (3.2) since $x + \rho v \in P_\rho(w;\eta)$. Thus (3.1) is proved.

As i tends to infinity in (3.1), we obtain

$$\limsup_{i\to\infty} \Theta_i(\rho,\eta) \leqslant \mathcal{L}^n\big((P_\rho(w;\eta))_\varepsilon\big)\,,$$

for an arbitrary ε. On the other hand, since $P_\rho(w;\eta)$ is closed,

$$\Theta(\rho,\eta) = \inf_{\varepsilon>0} \mathcal{L}^n\big((P_\rho(w;\eta))_\varepsilon\big)\,.$$

Finally we proved that

$$\limsup_{i\to\infty} \Theta_i(\rho,\eta) \leqslant \Theta(\rho,\eta)\,.$$

Hence we conclude that the sequence Θ_i, $i \in \mathbb{N}$, converges weakly to the measure Θ.

Formula (2.1) applies to w_i, for every i:

$$\Theta_i(\rho,\eta) = \sum_{j=0}^{n} \binom{n}{j} \sigma_j(w_i;\eta)\rho^j\,, \quad \forall\, \eta \in \mathfrak{B}(B)\,, \tag{3.3}$$

where $\sigma_j(w_i;\eta) = \binom{n}{j}^{-1} \int_\eta S_j(D^2 w_i)dx$, $j = 0,1,\ldots,n$, are real bounded measures.

For a fixed $m > 0$ such that $n/m < 1/c$, let $\rho_k = k/m$, $k = 0,\ldots,n$. Writing equality (3.3) for $\rho = \rho_0, \rho_1, \ldots, \rho_n$, for every i we get the linear system

$$\Theta_i(\rho_k,\eta) = \sum_{j=0}^{n} \binom{n}{j} \sigma_j(w_i;\eta)\rho_k^j\,, \quad \forall\, \eta \in \mathfrak{B}(B)\,, \quad k = 0,1,\ldots,n\,.$$

The square matrix $\left(\binom{n}{j}\rho_k^j\right)$ is invertible, indeed it can be written as the product of a diagonal invertible matrix times a matrix of Vandermonde type. If (a_{jk}) denotes its inverse matrix, we can write

$$\sigma_j(w_i;\eta) = \sum_{k=0}^{n} \Theta_i(\rho_k,\eta)a_{jk}\,, \quad \forall\, \eta \in \mathfrak{B}(B)\,, \quad \forall\, j = 0,1,\ldots,n\,.$$

Notice that the coefficients a_{jk} are independent of i and η. Consequently, the sequence of measures $\sigma_j(w_i;\ )$, $i \in \mathbb{N}$, converges weakly for every $j = 0,1,\ldots,n$, as $i \to \infty$. Denote by $\sigma_j(w;\cdot)$ the weak limit of $\sigma_j(w_i;\cdot)$: $\sigma_j(w;\cdot)$ is a real bounded measure.

The weak limits of the left and the right hand-sides of (3.3) must coincide, then

$$\mathcal{L}^n(P_\rho(w;\eta)) = \sum_{j=1}^{n} \binom{n}{j} \sigma_j(w;\eta)\rho^j\,, \quad \forall\, \eta \in \mathfrak{B}(B)\,, \quad \forall\, \rho \geqslant 0\,.$$

Finally, since $u = w$ in Ω, and Ω is open, for every $x \in \Omega$ we have $\partial u(x) = \partial w(x)$ and consequently $P_\rho(u;\eta) = P_\rho(w;\eta)$ for every Borel set η and for every $\rho \geqslant 0$. ∎

REMARK. Notice that the measures $\sigma_j(u;\cdot)$ are uniquely determined by virtue of the identity principle for polynomials.

4. Generalised Solutions of the Hessian equations

Consider the following Dirichlet problem involving the Hessian equations:

$$\text{(4.1)} \qquad \begin{cases} S_m(D^2u) = f > 0 & \text{in } \Omega\,, \\ u = g & \text{on } \partial\Omega\,. \end{cases}$$

Solvability of this problem in the classical sense is studied in [**5**] and [**10**]; while in [**11**] weak solutions are considered. The measures $\sigma_i(u;\cdot)$, introduced in the previous section, allow us to give a notion of generalised solution of this problem.

Definition 4.1: Let $\Omega \subset \mathbb{R}^n$ be a bounded convex open set, $f \in L^1_{loc}(\Omega)$ and $g \in C(\partial\Omega)$; a semiconvex function $u \in C(\overline{\Omega})$ is said to be a *generalised solution* of (4.1) if

$$\text{(4.2)} \qquad \begin{cases} \binom{n}{m}\sigma_m(u;\eta) = \int_\eta f(x)dx\,, & \forall \eta \in \mathfrak{B}(\Omega), \\ u = g & \text{on } \partial\Omega\,. \end{cases}$$

We prove that a semiconvex function which is limit of classical solutions, is a generalised solution in the sense of Definition 4.1. This implies in particular that if u is a weak solution of problem (4.1) in the sense of Trudinger [**11**], and u is semiconvex, then u is also a generalised solution.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex open set and let Ω_i be a sequence of smooth bounded convex open sets, converging to Ω in the Hausdorff metric. Moreover, let $u_i \in C^\infty(\Omega_i)$ and $f_i = S_m(D^2u_i)$ in Ω_i, $i \in \mathbb{N}$. If u_i converges uniformly on compact subsets of Ω to a semiconvex function u and f_i converges to f in $L^1(\Omega)$, then u is a generalised solution of the equation*

$$S_m(D^2u) = f \quad \text{in } \Omega\,.$$

Remarks. Here the functions f_i, $i \in \mathbb{N}$ are assumed to be extended as zero in $\mathbb{R}^n \setminus \Omega_i$.

For the notion of Hausdorff metric, see [**9**].

Proof: First consider the sequence of measures

$$\mu_i(\eta) = \int_\eta f_i(x)dx\,,$$

defined for every Borel subset $\eta \subset \Omega$. Since $f_i \to f$ in $L^1(\Omega)$, this sequence converges strongly to the measure

$$\mu(\eta) = \int_\eta f(x)dx\,;$$

thus $\lim_{n\to\infty} \mu_i(\eta) = \mu(\eta)$ for every $\eta \in \mathfrak{B}(\Omega)$.

On the other hand, from the uniform convergence of the sequence u_i to u, using the same argument as in the proof of Theorem 2.1, it follows that $\mu_i(\cdot)$ converges weakly to $\binom{n}{m}\sigma_m(u;\cdot)$ in Ω. By the uniqueness of the weak limit, this concludes the proof. ∎

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