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EMBEDDING AND REPRESENTATION THEOREMS FOR CLONES AND VARIETIES

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To Bernard Neumann on the occasion of his eightieth birthday

We use the theory of clones to prove that a countably presented variety of algebras can be embedded in a variety of groupoids.

0. INTRODUCTION

In Section 1 we show that any countable collection of functions $f_i\colon S^{n_i} \to S$, $n_i \geqslant 1$, $i = 1,2,3,\ldots$ on a countably infinite set S can be generated, under composition, by a single function $f\colon S^2 \to S$. In Section 3 we prove that any countable clone can be represented as a clone of functions and then, in Section 4, we deduce that any finitely or countably presented variety of algebras can be "embedded" in a variety of groupoids.

1. GENERATING FUNCTIONS

Let $S = \{1,2,3,\ldots\}$ and let $f_i\colon S^2 \to S$, $i = 1,2,3,\ldots$ be a countable collection of functions of two variables on S. Partition S into subsets $S_1, S_2, S_3, \ldots$ where

$$S_i = \{2^{2-1}(2x-1)\colon x \in S\},\ i = 1,2,3,\ldots$$

and let $f\colon S^2 \to S$ be defined by

(i) $f(x,x) = 2x$, $s \in S$;

(ii) $f(x,2x) = 2x-1$, $x \in S$;

(iii) $f(x,y) = f_i\big((x+2^i)/2^{i+1}(y+1)/2\big)$, $x \in S_{i+1}$, $y \in S_1$, $i = 1,2,3,\ldots$;

(iv) $f(x,y)$ is arbitrary for all other values of x and y.

From (i), (ii) $f(x,\,f(x,x)) = 2x-1$, $x \in S$. Also, from (iii),

$$f_i(x,y) = f\big(2^i(2x-1),\, 2y-1\big),\ x,y \in S. \tag{1.1}$$

Now if we put $g_1(x) = f(x,x)$, $g_{i+1}(x) = f(g_i(x),\, g_i(x))$, $i = 1,2,3\ldots$, then $g(x) = 2^i x$, for all i. Hence

$$g_i(f(x,\,f(x,x))) = 2^i(2x-1).$$

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From (1.1) we now have

$$f_i(x,y) = f(g_i(f(x,\ f(x,x))),\ f(y,\ f(y,y))). \tag{1.2}$$

That is, each f_i can be obtained by repeated composition from f and the projection function $p_1(x,y) = x$, $p_2(x,y) = y$.

Now any function of one variable f can be replaced by a function of two variables g, where $g(x,y) = f(x)$ for all x, y. Furthermore, (see Sierpinski [6]) any function of n variables can be written as a composition of functions of two variables. For example, take any bijection from S^2 to S, say $h(x_1,x_2) = 2^{x_1-1}(2x_2-1)$, and define iterates h_i by $h_1 = h$ and

$$h_{i+1}(x_1,x_2,\ldots,x_{i+2}) = h(x_1,\ h_i(x_2,x_3,\ldots,x_{i+2})),\ i = 1,2,3,\ldots. \tag{1.3}$$

Then for any $f\colon S^n \to S$, $n \geqslant 3$, there is a function $g\colon S^2 \to S$ such that

$$f(x_1,x_2,\ldots,x_n) = g(x_1,\ h_{i-2}(x_2,x_3,\ldots,x_n)). \tag{1.4}$$

Combining the above remarks, we have the following:

THEOREM 1. *Let $f_i\colon S^{n_i} \to S$, $i = 1,2,3,\ldots$ be a countable collection of functions on a countable set S. Then there is a function $f\colon S^2 \to S$ such that f generates each f_i.*

Remark. For S finite, a similar result follows from the existence of Sheffer stroke functions on any finite set (see, for example, Evans and Hardy [2]).

2. Clones

Let S be a nonempty set and let C be a collection of functions $f\colon S^n \to S$, on some fixed positive integer, such that

(i) C contains the projections $p_i(x_1,x_2,\ldots,x_n) = x_i$, $i = 1,2,\ldots,n$;
(ii) C is closed under the $(n+1)$-ary composition operation Σ where $\Sigma f g_1 g_2 \ldots g_n$ is the function $S^n \to S$ defined by

$$\Sigma f g_1 g_2 \ldots g_n\colon (x_1,x_2,\ldots,x_n) \to f(g_1(\mathrm{x}),\ g_2(\mathrm{x}),\ldots,g_n(\mathrm{x})) \quad \text{for all } \mathrm{x} = (x_1,x_2,\ldots,x_n) \text{ in } S^n. \tag{2.1}$$

C is called *n-clone* of functions on S.

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Note that Σ satisfies the generalised associative law for composition

(2.2) $\Sigma\Sigma f g_1 g_2 \dots g_n h_1 h_2 \dots h_n = \Sigma f \Sigma g_1 h_1 h_2 \dots h_n \dots \Sigma g_n h_1 h_2 \dots h_n$ for all f, g_i, h_j in C.

An *abstract n-clone* may be defined as an algebra on a set C with n constants $p_1, p_2, \dots, p_n$ and an $(n+1)$-ary operation $\Sigma\colon C^{n+1} \to C$ such that

(2.3)
(i) $\Sigma x p_1 p_2 \dots p_n$ for all $x \in C$
(ii) $\Sigma p_i y_1 y_2 \dots y_n = y_i$ for all $y_1, y_2, \dots, y_n$ in C, $i = 1, 2, 3, \dots, n$
(iii) $\Sigma\Sigma x y_1 y_2 \dots y_n z_1 z_2 \dots z_n = \Sigma x \,\Sigma y_1 \mathbf{z} \Sigma y_2 \mathbf{z} \dots \Sigma y_n \mathbf{z}$ for all $x \in C$, $\mathbf{y}$, $\mathbf{z} \in C^n$.

Examples of n-clones are:

1. C is the set of derived n-ary operations of an algebra A;
2. C is a free algebra on n generators $g_1, g_2, \dots, g_n$ and

$$\Sigma u v_1 v_2 \dots v_n = u\alpha \text{ for all } u,\ v_i \text{ in } C;$$

where α is the endomorphism mapping $g_i \to v_i$, $i = 1, 2, \dots n$;
3. C is the set of homomorphisms $A^n \to A$ of some algebra A.

Other examples may be found in Evans [**3**].

We may generalise the notion of n-clone to that of *heterogeneous clone* (or simply *clone*). Here, in the function case, S is a non-empty set, and collections $C^{(n)}$ of functions $f\colon S^n \to S$, $n = 1, 2, 3, \dots$ such that $C^{(n)}$ contains the projections p_i, $i = 1, 2, \dots n$, and the set $C = C^{(1)} \cup C^{(2)} \cup C^{(3)} \cup \dots$ is closed under the composition operations Σ^n_m where

(2.4) $$\Sigma^n_m : C^{(m)} \times \left(C^{(m)}\right)^n \to C^{(n)}$$

and $\Sigma^n_m f g_1 g_2 \dots g_m$, $f \in C^{(m)}$, $g_i \in C^{(n)}$ is the function $S^n \to S$ given by

(2.5) $\Sigma^n_m f g_1 g_2 \dots g_m : (x_1, x_2, \dots, x_n) \to f(g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_m(\mathbf{x}))$ for all $\mathbf{x} = (x_1, x_2, \dots, x_n)$ in S^n.

For case of reading Σ^n_m will be written simply as Σ when this causes no ambiguity, and vector notation will also be used for the same purpose, for sequences of functions as well as for sequences of elements of S. Thus, $\Sigma^n_m f g_1 g_2 \dots g_m$ may be written as $\Sigma f \mathbf{g}$ and (2.2) as

(2.6) $$\Sigma\Sigma f \mathbf{g}\mathbf{h} = \Sigma f \Sigma g_1 \mathbf{h} \Sigma g_2 \mathbf{h} \dots \Sigma g_n \mathbf{h}.$$

The abstract version of a general clone of functions is defined as follows. We have a set $\mathcal{C}$ which is the disjoint union of sets $\mathcal{C}^{(1)}, \mathcal{C}^{(2)}, \mathcal{C}^{(3)}, \ldots$. For each n, $\mathcal{C}^{(n)}$ contains elements $p^{(n)}, p_2^{(2)}, \ldots, p_n^{(n)}$ (we omit the superscripts whenever possible) and there is a partial operation Σ on $\mathcal{C}$ which is the union of the operations $\Sigma_m^n \colon \mathcal{C}^{(m)} \times (\mathcal{C}^n)^m \to \mathcal{C}^{(n)}$, $m, n \geqslant 1$, such that the following axioms are satisfied:

(2.7)

(i) $\Sigma x p_1 p_2 \ldots p_m = x$, for any x in $\mathcal{C}^{(m)}$;

(ii) $\Sigma p_i y_1 y_2 \ldots y_m = y_i$, for any projection p_i in $\mathcal{C}^{(m)}$ and $y_1, y_2, \ldots y_m$ in $\mathcal{C}^{(n)}$;

(iii) $\Sigma\, \Sigma x y_1 y_2 \ldots y_m z_1 z_2 \ldots z_n = \Sigma\, x\, \Sigma y_1 \mathbf{z} \Sigma y_2 \mathbf{z} \ldots \Sigma y_m \mathbf{z}$

for any x in $\mathcal{C}^{(m)}, y_1, y_2, \ldots, y_m$ in $\mathcal{C}^{(n)}$ and $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ in $\left(\mathcal{C}^{(t)}\right)^n$.

The examples of n-clones given earlier can be extended. The set of all derived operations of an algebra A is a clone. Similarly, the set of all homomorphisms $A^n \to A$, $n = 1, 2, 3, \ldots$, is a clone. In both cases the clone operation is composition. For the third examples, we take $\mathcal{C}^{(n)}$ to be the free algebra $F_n(\mathcal{V})$ in the variety $\mathcal{V}$ and the value of the clone operation $\Sigma u v_1 v_2 \ldots v_m$ for $u \in F_m(\mathcal{V})$, $v_i \in F_n(\mathcal{V})$ is defined to be the image of u under the homomorphism which maps the generating set of $F_m(\mathcal{V})$ onto $v_1, v_2, \ldots, v_m$ in $F_n(\mathcal{V})$.

3. Representing clones as clones of functions

Let $\mathcal{C}$ be an n-clone, that is, a set $\mathcal{C}$, an $(n+1)$-ary operation Σ and projection elements $p_1, p_2, \ldots, p_n$, satisfing (2.3). It is a simple matter to extend to n-clones the Cayley representation theorem for groups and semigroups. To each $c \in \mathcal{C}$, we assign the function $f_c \colon \mathcal{C}^n \to \mathcal{C}$ where

$$f_c(x_1, x_2, \ldots, x_n) = \Sigma c x_1 x_2 \ldots x_n, \ \mathrm{x} \in \mathcal{C}_n$$

and it is easily checked that $c \to f_c$ is an isomorphism from $\mathcal{C}$ to a clone of functions. A corresponding theorem holds for general heterogeneous clones but is more complicated to prove.

Let $\mathcal{C}$ be a heterogeneous clone with elements

$$\mathcal{C} = \mathcal{C}^{(1)} \cup \mathcal{C}^{(2)} \cup \mathcal{C}^{(3)} \cup \ldots$$

where $\mathcal{C}^{(m)}$ is the set of elements of arity m. Let 0 be an element not in $\mathcal{C}$ and consider the set S of all sequences

$$(x_1, x_2, x_3, \ldots),\ x_i \in \mathcal{C}^{(i)} \cup \{0\} \text{ such that for some } i,\ x_i \neq 0, \text{ and}$$

(3.1) (i) if $x_i = 0$, then $x_j = 0$ for all $j < i$

(ii) if $x_i \neq 0$, then $x_j = \Sigma x_i p_1^{(j)} p_2^{(j)} \ldots p_i^{(j)}$ for all $j > i$.

Note that if $i < j < k$, by (3.1) and the generalised associative law

$$x_k = \Sigma x_j p_1^{(k)} p_2 \dots p_j^{(k)} = \Sigma x_i p_1^{(k)} p_2^{(k)} \dots p_i^{(k)}. \tag{3.2}$$

Thus, a sequence in S begins with an initial segments of 0's (possibly empty) and after the first non-zero x_i, each term is determined uniquely by (3.1).

We now define two sequences $s = (x_1, x_2, x_3, \dots)$, $t = (y_1, y_2, y_3, \dots)$ in S to be *equivalent*, $s \equiv t$, if there is some non-zero term x_i such that $x_i = y_i$. Let $\overline{s}$ denote the equivalence class containing s and let $\overline{S}$ denote the set of equivalence classes. Note that in each $\overline{s}$ there will be a unique sequence with a shortest initial segment of 0's and the first non-zero term in this sequence determines $\overline{s}$.

For each $c \in C^{(m)}$, $m = 1, 2, 3, \dots$, in the clone C we define a function $f_c\colon \overline{S}^m \to \overline{S}$ as follows. Let $\overline{s}_1, \overline{s}_2, \overline{s}_3, \dots \overline{s}_m$ be elements of S and let i be such that for all j each sequence in $\overline{s}_j$ has its ith term non-zero. Denote this ith term for $\overline{s}_j$ by x_j, $j = 1, 2, \dots m$. Then we define

$$f_c(\overline{s}_1, \overline{s}_2, \dots, \overline{s}_m) = \overline{t} \tag{3.3}$$

where $\overline{t}$ contains all sequences of S having

$$\Sigma\, c x_1 x_2 \dots x_m \tag{3.4}$$

as ith term. Since $x_j \in C^{(i)}$, this element also belongs to $C^{(i)}$.

We claim that the f_c form a clone of functions on $\overline{S}$ and that $c \to f_c$ is a isomorphism from C to this clone of functions.

(i) f_c is well-defined in the sense that it is independent of the particular i we choose. This follows from (3.2).

(ii) $c \to f_c$ is one-one. For if $f_c = f_d$, then

$$f_c(\overline{p}_1, \overline{p}_2, \dots, \overline{p}_m) = f_c(\overline{p}_1, \overline{p}_2, \dots, \overline{p}_m),$$

where $p_1, p_2, \dots, p_m$ are the projection elements in $C^{(m)}$. Hence, $\Sigma\, c p_1 p_2 \dots p_m = \Sigma\, d p_1 p_2 \dots p_m$ and so by (2.7), $c = d$.

(iii) $c \to f_c$ is a homomorphism. Let $c \in C^{(m)}$ and $d_1, d_2, \dots, d_m \in C^{(n)}$. Then $d = \Sigma c d_1 d_2 \dots d_n$ belongs to $C^{(n)}$. We have to show that

$$f_d = \Sigma f_c f_{d_1} f_{d_2} \cdots f_{d_m}. \tag{3.5}$$

Let $s_j \in S$, $j = 1, 2, \ldots, n$ so that $(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) \in \bar{S}^n$ and let $f_{d_1}(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) = \bar{t}_l$, $1 \leqslant l \leqslant m$ where, by (3.3), $\bar{t}_l$ consists of all S-sequences having $\Sigma\, d_l x_1 x_2 \ldots x_n$ as ith term, for some i such that each x_j is a non-zero ith term of S_j. Then $f_c(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) = \bar{w}$ where $\bar{w}$ is the class of sequences having, as the ith term

$$\Sigma\, dx_1, x_2 \ldots x_n = \Sigma\, \Sigma\, cd_1 d_2 \ldots d_m x_1 x_2 \ldots x_n$$
$$\Sigma\, c\, \Sigma\, d_1 \mathbf{x} \Sigma d_2 \mathbf{x} \ldots \Sigma d_m \mathbf{x}.$$

But

$$\Sigma f_c f_{d_1} f_{d_2} \ldots f_{d_m}(\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_n) = f_c(\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_m)$$

where $\bar{z}$ is the class of sequences having

$$\Sigma\, c\, \Sigma d_1 \mathbf{x} \ldots \Sigma d_m \mathbf{x}$$

as ith term. Hence, $\bar{w} = \bar{z}$ and (3.5) is verified.

THEOREM 2. *Every countable clone is isomorphic to a clone of functions.*

4. EMBEDDING THEOREMS

The set of derived operations of an algebra $A = (A : F)$ is a clone $C(A)$ under composition. An algebra $B = (A : F')$ obtained by taking some set of derived operations of A as basic operations is called a *derived algebra* of A. Its clone $C(B)$ is a subclone of $C(A)$. Theorem 1 can be stated in the following equivalent forms

THEOREM 3.

(i) *Any countable algebra with countably many finitary operations is isomorphic to a derived algebra of some groupoid.*

(ii) *The clone of any algebra (countable with countably many finitary operations) is isomorphic to a subclone of the clone of some groupoid.*

To obtain a corresponding theorem for varieties, we first combine Theorems 1 and 2 in the following form.

THEOREM 4. *Any countable clone can be embedded in a clone which is generated by one element of arity two.*

Let $\mathcal{V}$ be a variety defined by a countable number of finitary operations. There are various ways of associating a clone $C(\mathcal{V})$ with the variety $\mathcal{V}$ (see, for example, W.D. Neumann [**6**]). We adopt a different approach. Regard the primitive operations of the variety $\mathcal{V}$ as generators of a clone $C(\mathcal{V})$ and translate the defining identities of $\mathcal{V}$ into defining relations for $C(\mathcal{V})$. For example, if $\mathcal{V}$ is given by two binary operations $+, \cdot$

and the defining identity $x(y+z) = xy + xz$, then $\mathcal{C}(\mathcal{V})$ is generated by elements m, a of arity two (corresponding to multiplication and addition) and satisfies the defining relation

$$\Sigma\, mp_1\, \Sigma ap_2p_3 = \Sigma\, a\, \Sigma\, mp_1p_2\, \Sigma\, mp_1p_3.$$

We omit the tedious description of the general case of this correspondence between $\mathcal{V}$ and $\mathcal{C}(\mathcal{V})$ and the verification (by induction on length and the rules of equational logic) that an identity holds in $\mathcal{V}$ if and only if the corresponding relation on the generators holds in $\mathcal{C}(\mathcal{V})$. Putting together the preceding theorems, we obtain the following result on the embedding of a variety in a variety of groupoids.

THEOREM 5. *Let $\mathcal{V}$ be a countably presented variety with finitary operations. Then there exists a variety of groupoids $\mathcal{W}$ such that to each n-ary operation of $\mathcal{V}$ there corresponds a groupoid word in n variables (a derived n-ary operation in $\mathcal{W}$) and an identity holds between the operations in $\mathcal{V}$ if and only if the corresponding identity holds between the derived operations of $\mathcal{W}$.*

5. Remarks

The origin of the above results is, of course, the original embedding theorem of Higman, Neumann, and Neumann [4] that any countable group can be embedded in a group generated by two elements. Many analogous theorems have been proved for other algebras, semigroups, quasigroups, rings, lattices, et cetera. For monoids, which are 1-clones, the result states that any countable monoid can be embedded in a monoid generated by two elements. A consequence of the results in this paper is that any n-clone $(n > 1)$ can be embedded in an n-clone generated by one element.

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