A ONE-PARAMETER SUBSEMIGROUP WHICH MEETS MANY REGULAR $\mathscr{D}$-CLASSES

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We give an example which answers affirmatively the following problem posed by Hofmann and Mostert (**1**, p. 200):

P. 10. Let S be a compact semigroup with identity 1, zero 0, and totally ordered $\mathscr{D}$-class space $S/\mathscr{D}$. Suppose that there is a one-parameter semigroup containing 0 and 1. Can S have any regular $\mathscr{D}$-classes aside from $D(0)$ and $D(1)$?

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1. We first give a method of embedding a given semigroup S in a semigroup R so that in the commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & R \\ \downarrow & & \downarrow \\ S/\mathscr{D} & \longrightarrow & R/\mathscr{D} \end{array}$$

the bottom map is an isomorphism of partially ordered spaces and some non-regular $\mathscr{D}$-class of S is embedded into a regular $\mathscr{D}$-class of R.

We generalize the notion of a Rees product as follows: Let X and Y be sets and let S' be a partial semigroup (i.e., a set on which a partially defined associative multiplication is given) containing a subsemigroup S. Suppose that SS' and $S'S$ are defined and $SS'S \subset S$. If $[\ ,\ ]: Y \times X \rightarrow S'$ is a function, then the set $X \times S \times Y$ becomes a semigroup $[X, S, Y]$ with the multiplication $(x, s, y)(a, b, c) = (x, s[y, a]b, c)$. If X, Y, and S' are topological spaces, $[\ ,\ ]$ is a map, and the partial multiplication on S' is continuous, then $[X, S, Y]$ is a topological semigroup.

LEMMA. *Let $[X, S, Y]$ be as above and suppose that*

(i) *S' contains a subsemigroup A with identity 1 such that S is an ideal of A,*

(ii) *there exist $a \in X$ and $b \in Y$ with $[b, a] = 1$,*

(iii) *$SS' \cup S'S \subset A$,*

(iv) *for each $x \in X$ ($y \in Y$) there exists $y \in Y$ ($x \in X$) and $s \in S$ such that $s[y, x]$ ($[y, x]s$) is a unit of A,*

(v) *$J_S(s) = J_A(s)$ for $s \in S$.*

Then the $\mathscr{J}$-classes of $[X, S, Y]$ are exactly the sets $X \times J_S(s) \times Y$, for $s \in S$.

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Proof. Assume that s and t are $\mathscr{J}$-equivalent in S. Then $s \in (S \cup 1)t(S \cup 1)$. Let $u \in X$ and $v \in Y$. By hypothesis, $S[Y \times u]$ is a subset of A and contains a unit of A. Since S is an ideal of A, $S[Y \times u]$ contains the identity of A. Thus,

$$(S \cup 1)t(S \cup 1) \subset (S \cup 1)(S[Y \times u])t([v \times X]S)(S \cup 1) \subset (S \cup 1)[Y \times u]t[v \times X](S \cup 1).$$

Hence, $(x, s, y) \in [X, S, Y](u, t, v)[X, S, Y]$ for all $(x, y) \in X \times Y$. Similarly,

$$(u, t, v) \in [X, S, Y](x, s, y)[X, S, Y];$$

therefore, (u, t, v) is $\mathscr{J}$-equivalent to (x, s, y) in $[X, S, Y]$.

Now suppose that (x, s, y) and (u, t, v) are $\mathscr{J}$-equivalent in $[X, S, Y]$. Let (x_1, s_1, y_1) and (x_2, s_2, y_2) be in $[X, S, Y]$. If

$$(x, s, y) = (x_1, s_1, y_1)(u, t, v)(x_2, s_2, y_2),$$

then

$$(x, s, y) = (x_1, s_1[y_1, u]t[v, x_2]s_2, y_2).$$

Thus

$$s \in SS'tS'S \subset AtA.$$

If $(x, s, y) = (x_1, s_1, y_1)(u, t, v)$, we can similarly show that $s \in At \subset AtA$. Dually, $t \in AsA$; thus, t and s are $\mathscr{J}$-equivalent in A. By (v), t and s are $\mathscr{J}$-equivalent in S.

2. Under the conditions of (1), an element $(x, s, y) \in [X, S, Y]$ is idempotent if and only if $s[y, x]s = s$. If $s[y, x]s = s$, then s is a regular element of the partial semigroup S'. However, s need not be a regular element of the subsemigroup S.

We use this fact to embed a non-regular $\mathscr{J}$-class $J_S(s)$ of S into a regular $\mathscr{J}$-class $X \times J_S(s) \times Y$ of $[X, S, Y]$.

Example 1. Consider the following partial subsemigroups of the real line with addition as the operation: $S' = [-1, \infty[$, $A = [0, \infty[$ and $S = [1, \infty[$. Let $X = Y = [0, 1]$. Define $[\ ,\ ]\colon Y \times X \to S'$ by $[y, x] = \max(1, x + y) - 2$. Then $[X, S, Y]$ is a topological semigroup with the multiplication

$$(x, s, y)(u, t, v) = (x, s + t + [y, u], v).$$

It is easy to check that the hypotheses of the lemma are satisfied. Hence, the $\mathscr{J}$-classes of $[X, S, Y]$ are the sets $[0, 1] \times r \times [0, 1]$, $r \geqq 1$. The set of idempotents is

$$\{(x, r, y) \in [X, S, Y] | r + [y, x] = 0\} = \{(x, 1, y) \in [X, S, Y] | x + y \leqq 1\}.$$

Define $f\colon [1, \infty[\to [X, S, Y]$ by $f(r) = (1, r, 1)$; then f is an isomorphism of semigroups onto the subsemigroup $1 \times S \times 1$.

We can make $[X, S, Y]$ into a compact topological semigroup C by letting the element at infinity in the one-point compactification of $[X, S, Y]$ act as a zero. Then C is a compact semigroup with a totally ordered $\mathscr{J}$-class space in which the top $\mathscr{J}$-class is regular and C contains a piece $P = \overline{f([1, \infty[)}$ of a one-parameter semigroup as a cross section for the $\mathscr{J}$-classes. Note that P passes through the top $\mathscr{J}$-class without meeting it in an idempotent.

3. We now wish to extend the semigroup C of Example 1 to a semigroup Q with identity such that C will be an ideal of Q and P will be contained in a one-parameter semigroup from the identity to the zero of Q.

Suppose that T is a topological semigroup acting on the right and on the left of a topological semigroup R so that

$$\text{(a)} \qquad \begin{aligned} &(r\cdot t)r' = r(t\cdot r'), \quad (t\cdot r)\cdot t' = t\cdot(r\cdot t), \quad r\cdot(r'\cdot t) = rr'\cdot t,\\ &(t\cdot r)\cdot r' = t\cdot rr', \qquad (r\cdot t)\cdot t' = r\cdot tt', \qquad t\cdot(t'\cdot r) = tt'\cdot r \end{aligned}$$

for all $r, r' \in R$, $t, t' \in T$. Then $T \cup R$ is a topological semigroup with the multiplication which extends the given multiplications on T and R and satisfies $rt = r\cdot t$ and $tr = t\cdot r$ for $t \in T$, $r \in R$.

Let S be a closed ideal of T and let $f: S \rightarrow R$ be a continuous morphism of semigroups satisfying

$$\text{(b)} \qquad s\cdot r = f(s)r, \quad r\cdot s = rf(s), \quad f(ts) = t\cdot f(s), \quad f(st) = f(s)\cdot t$$

for $s \in S$, $t \in T$, $r \in R$.

Define an equivalence relation ρ on $T \cup R$ by $x\,\rho\,y$ if and only if $x = y$, $x = f(y)$, $y = f(x)$, or $f(x) = f(y)$. Then ρ is a closed congruence on $T \cup R$. We denote the quotient semigroup $(T \cup R)/\rho$ by $T\,\pi\,R$.

The semigroup $T\,\pi\,R$ has an identity if and only if T has an identity which acts on R on both sides as an identity. The Green classes of elements in $T\backslash S$ are the same whether they are taken relative to T or relative to $T\,\pi\,R$. If the $\mathscr{J}$-class spaces of both T and R are totally ordered and T meets the top $\mathscr{J}$-class of R, then the $\mathscr{J}$-class space of $T\,\pi\,R$ is totally ordered.

4. We now give an application of the extension method in § 3.

Let X, Y, S, S', and $[\ ,\ ]$ define a Rees product $[X, S, Y]$ as in § 1. Let A be a subsemigroup of S' such that S is an ideal of A and A has an identity 1. Suppose that there exists $(b, a) \in Y \times X$ with $[b, a] = 1$. Let T be a semigroup and I an ideal of T.

Suppose that T acts on X on the left and on Y on the right so that $I\cdot X = T\cdot a = a$ and $Y\cdot I = b\cdot T = b$. Suppose also that there exist functions

$$\phi: T \times X \rightarrow A \quad \text{and} \quad \psi: Y \times T \rightarrow A$$

satisfying the following:

(i) $\phi(tt', x) = \phi(t, t'\cdot x)\phi(t', x)$ and $\psi(y, tt') = \psi(y, t)\psi(y\cdot t, t')$ for $t, t' \in T$, $x \in X$, $y \in Y$;

(ii) $\psi(y, t)[y\cdot t, u] = [y, t\cdot u]\phi(t, u)$ for $u \in X$, $y \in Y$, $t \in T$;

(iii) $\phi(r, a)[b, x] = \phi(r, x)$ and $\psi(y, r) = [y, a]\psi(b, r)$ for $x \in X$, $y \in Y$, $r \in I$.

Let $R = [X, S, Y]$ and let T act on the right and on the left of R as follows:

$$t\cdot(x, s, y) = (t\cdot x, \phi(t, x)s, y),$$
$$(x, s, y)\cdot t = (x, s\psi(y, t), y\cdot t)$$

for $t \in T$ and $(x, s, y) \in R$.

Let $t, t' \in T$ and let (x, s, y) and (u, v, w) be in R. It is straightforward to check that

$$(t\cdot(x, s, y))\cdot t' = t\cdot((x, s, y)\cdot t')$$

and that

$$t\cdot((x, s, y)(u, v, w)) = (t\cdot(x, s, y))(u, v, w).$$

Since T acts on X and (i) holds,

$$tt'\cdot(x, s, y) = t\cdot(t'\cdot(x, s, y)).$$

Since (ii) holds,

$$((x, s, y)\cdot t)(u, v, w) = (x, s, y)(t\cdot(u, v, w)).$$

Thus, the conditions (a) are satisfied.

Define $f: I \to R$ by $f(r) = (a, \phi(r, a), b)$ for each $r \in I$. Since (i) holds and $T\cdot a = a$, it follows that f is a homomorphism of semigroups.

Since (ii) holds,

$$\psi(b, r) = \psi(b, r)[b\cdot r, a] = [b, r\cdot a]\phi(r, a) = \phi(r, a).$$

For $r \in I$ and $(x, s, y) \in R$,

$$r\cdot(x, s, y) = (r\cdot x, \phi(r, x)s, y) = (a, \phi(r, a)[b, x]s, y) = f(r)(x, s, y)$$

since $I\cdot X = a$ and since (iii) holds. Furthermore,

$$(x, s, y)\cdot r = (x, s\psi(y, r), y\cdot r) = (x, s[y, a]\psi(b, r), b) = (x, s, y)f(r)$$

since $Y\cdot I = b$ and since (iii) holds. Similarly, $t\cdot f(r) = f(tr)$ and $f(r)\cdot t = f(rt)$ for $r \in I$ and $t \in T$. Thus, condition (b) is satisfied and we may form $T \pi R$.

If S, X, and Y are topological spaces, T is a topological semigroup, the partial multiplication on S' is continuous, I is a closed ideal of T, and all actions and functions are continuous, then $T \pi R$ is a topological semigroup.

Example 2. Let $R = [X, S, Y]$ be as in Example 1. Let $T = [0, \infty[$ with addition as the operation and let $I = [1, \infty[$. Let T act on the right and on the left of X and Y by $t\cdot x = x\cdot t = \min(1, x + t)$ for $t \in T$, $x \in [0, 1] = X = Y$. Define $\phi: T \times X \to A = [0, \infty[$ by $\phi(t, x) = \max(0, t + [1, x])$ and $\psi: Y \times T \to A$ by $\psi(y, t) = \max(0, t + [y, 1])$. Let $a = b = 1$.

It is tedious but not difficult to show that all the conditions of § 4 are satisfied. Thus, we can form $T \pi R$. Since I is a closed ideal of T, T and R are topological semigroups, and all actions and functions are continuous, $T \pi R$ is a topological semigroup.

We identify T with the semigroup $1 \times [0, \infty[\times 1$ with the multiplication $(1, x, 1)(1, y, 1) = (1, x + y, 1)$. The semigroup $T \pi R$ may be described as

$$(1 \times [0, \infty[\times 1) \cup ([0, 1] \times [1, \infty[\times [0, 1])$$

with the multiplication

$$(x, r, y)(u, s, v) = \begin{cases} (u + r, s, v) & \text{if } 1 \leqq s, r + u \leqq 1, \\ (x, r, y + s) & \text{if } 1 \leqq r, y + s \leqq 1, \\ (x, r + [y, u] + s, v) & \text{otherwise.} \end{cases}$$

Since the identity $(1, 0, 1)$ of T acts as an identity on R, $T \pi R$ has an identity. The $\mathscr{J}$-classes in $T \pi R$ are the same as the $\mathscr{D}$-classes. By the remarks made in § 3, $T \pi R$ has a totally ordered $\mathscr{D}$-class space. One can check that (x, s, y) and (u, t, v) are $\mathscr{D}$-equivalent in $T \pi R$ if and only if $s = t$. The only regular $\mathscr{D}$-classes of $T \pi R$ are $(1, 0, 1)$ and $[0, 1] \times 1 \times [0, 1]$.

We can embed $T \pi R$ in a compact semigroup Q by letting the element at infinity in the one-point compactification of $T \pi R$ act as a zero. The non-zero elements of Q have the same Green classes relative to Q as relative to $T \pi R$ and the $\mathscr{D}$-class space of Q is totally ordered.

Clearly, $(1 \times [0, \infty[\times 1)$ together with the zero of Q is a one-parameter semigroup running from the identity to the zero of Q. This one-parameter semigroup passes through the regular $\mathscr{D}$-class $D((1, 1, 1))$ without meeting it in an idempotent. Thus, this example answers (**1**, p. 200, P. 10) positively.

Reference

1. K. H. Hofmann and P. S. Mostert, *Elements of compact semigroups* (Charles E. Merrill Books, Columbus, Ohio, 1966).

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