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Recurrence without uniform recurrence

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Abstract We construct a minimal homeomorphism of a compact space such that a sequence of its iterates converges pointwise to the identity but no sequence of its iterates converges uniformly

1 *Introduction*

The object of this paper is to construct the system described below

THEOREM 1 1 *There exists a compact, complete, metric space* (X, d) *and a homeomorphism* $T\ X \to X$ *such that*

(i) *the set* $\{T^n x\ \ n \geq 0\}$ *is dense in* X *for each* $x \in X$,

(ii) $\sup_{x \in X} d(T^m x, x) \geq 1$ *for each* $m \geq 1$, *yet*

(iii) *there exists a sequence* $n(1) < n(2) <$ *such that* $d(T^{n(j)}x, x) \to 0$ *as* $j \to \infty$ *for each* $x \in X$

I should like to thank B Weiss for suggesting this problem and for pointing out a flaw in my original attack Since, in my opinion, the interest of this paper lies more in the method of construction than in the result itself I shall begin by indicating why we might expect such a construction to be fairly complicated The following result was pointed out to me, again by B Weiss

LEMMA 1 2 *Suppose that* (X, d) *is a compact metric space and* $T\ X \to X$ *is a homeomorphism such that* $\{T^n x\ \ n \geq 0\}$ *is dense in* X *for each* $x \in X$ *Suppose further that we can find a sequence* $n(1) < n(2) <$ *of integers and a sequence* $\varepsilon(1) > \varepsilon(2) >$ *of positive real numbers such that* $\varepsilon(j) \to 0$ *as* $j \to \infty$ *and such that for each* $x \in X$ *there exists an integer* $j_0(x)$ *with*

$$d(T^{n(j)}x, x) \leq \varepsilon(j) \qquad \text{for all } j \geq j_0(x)$$

Then $\sup_{x \in X} d(T^{n(j)}x, x) \to 0$ *as* $j \to \infty$

Proof Let

$$E(k) = \{x \in X\ \ d(T^{n(j)}x, x) \leq \varepsilon(j) \text{ for all } j \geq k\}$$

Then each $E(k)$ is closed and, by hypothesis, $\bigcup_{k=1}^{\infty} E(k) = X$ By the Baire category theorem we can find a k_0 and a non-empty open set U such that $U \subseteq E(k_0)$ Since $\{T^n x\ \ n \geq 0\}$ is dense in X, the sets $T^{-n}U$ $[n \geq 0]$ form an open cover of X and so, by compactness, we can find an M such that $\bigcup_{m=0}^{M} T^{-m}U = X$, i e such that for each $x \in X$ there exists an $0 \leq m \leq N$ with $T^m x \in U$

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Now let $\omega(\varepsilon)$ be the common modulus of continuity of $I, T^{-1}, T^{-2}, \quad, T^{-M}$, i e let

$$\omega(\varepsilon)=\sup_{0\le m\le N}\ \sup\{d(T^{-m}u, T^{-m}v) \ \ d(u,v)\le\varepsilon\}$$

We observe that $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$ and that we can now conclude the proof For if $x\in X$ then $T^m x\in U$ for some $0\le m\le M$ and so, if $k\ge k_0$

$$d(T^{n(k)}T^m x, T^m x)\le\varepsilon(k),$$

whence

$$d(T^{n(k)}x, x)=d(T^{-m}(T^{n(k)}T^m x), T^{-m}(T^m x))\le\varepsilon(k)$$

Thus $\sup_{x\in X} d(T^{n(k)}x, x)\le\omega(\varepsilon(k))$ for all $k\ge k_0$ and the lemma follows □

2 *Reduction to a semi-combinatorial problem*

Let Λ be a compact subset of $\mathbb{R}$ and let $\Lambda^{\mathbb{Z}}$ be the space of two-sided sequences $x\ \ \mathbb{Z}\to\Lambda^{\mathbb{Z}}$ Let d be the distance defined on $\Lambda^{\mathbb{Z}}$ by

$$d(x,y)=\sup_{k\in\mathbb{Z}} 2^{-|k|}|x_k-y_k|$$

and let T be the shift map $T\ \ \Lambda^{\mathbb{Z}}\to\Lambda^{\mathbb{Z}}$ given by

$$(Tx)=x_{k+1} \qquad (k\in\mathbb{Z})$$

Then $(\Lambda^{\mathbb{Z}}, d)$ is a complete, compact, metric space and T a homeomorphism Furstenberg and his collaborators Katznelson and Weiss have brilliantly exploited the analytic structure of $(\Lambda^{\mathbb{Z}}, d, T)$ to obtain results on the combinatorial structure of $\Lambda^{\mathbb{Z}}$ (see [**1**]) We shall reverse the process by using a combinatorial construction in $\Lambda^{\mathbb{Z}}$ to obtain the analytic Theorem 1 1 Examples of such constructions are discussed in [**1**, Chapter 1, §§ 3, 5] The examples given there take Λ to be a finite set but we shall use $\Lambda=[-1,1]$ The fact that Λ is then connected is essential for our construction

From now on T and d will have the meanings assigned to them in the previous paragraph We intend to deduce Theorem 1 1 from the following lemma which is not yet completely combinatorial since it mentions metric closure

LEMMA 2 1 *There exists a subset X of $[-1,1]^{\mathbb{Z}}$ and a sequence $n(1)<n(2)<$ with the following properties*

(i) *If $x, y\in X$ then given $k\ge 1$ and $\varepsilon>0$ we can find an $m\ge 1$ such that $|y_{m+r}-x_r|\le\varepsilon$ for all $|r|\le k$*

(ii) *For each $m\ne 0$ there exists an $x\in X$ such that $|x_m-x_0|\ge 1$*

(iii) *If $x\in X$ and $p\ge k+1$ then, for each $i\in\mathbb{Z}$,*

$$|x_i-x_{i+n(p)}|>2^{2-k}\Rightarrow|x_i-x_{i+n(l)}|\le 2^{2-l}$$

for all $p-1\ge l\ge k$

(iv) *If $x\in X$ then $Tx, T^{-1}x\in X$*

(v) *X is closed in $([-1,1]^{\mathbb{Z}}, d)$*

Proof of Theorem 1 1 *from Lemma* 2 1 Condition (v) tells us that (X, d) is complete and compact and condition (iv) shows us that T (restricted to X) is a homeomorphism Conditions (i), (ii) and (iii) yield the corresponding conditions of the theorem Thus if $x, y\in X$ and $\varepsilon>0$ then, choosing $k\ge 1$ with $2^k>\varepsilon^{-1}$, condition (i) tells us that we can find an $m \geq 1$ such that $|y_{m+r} - x_r| \leq \varepsilon$ for all $|r| \leq k$ and so

$$d(T^m y, x) = \sup_{r \in \mathbb{Z}} 2^{-|r|} |y_{m+r} - x_r| \leq \varepsilon$$

Thus the orbit of each point y is dense and (i) holds Similarly condition (ii) tells us that for each m there exists an $x \in X$ with $|x_m - x_0| \geq 1$ and so $d(T^m x, x) \geq 1$ Finally we observe that if $x \in X$ condition (iii) implies that $|x_i - x_{i+n(p)}| \geq 2^{2-r}$ for at most one value of $p \geq r+1$ and so $x_i - x_{i+n(p)} \to 0$ as $p \to \infty$ for each $i \in \mathbb{Z}$ Thus $d(T^{n(p)} x, x) \to 0$ as $p \to \infty$ for each $x \in X$ as required □

The reader should observe that, although condition (iii) of the lemma forces $d(T^{n(p)} x, x) \to 0$, the convergence can have hiccups These hiccups enable us to evade the conclusion of Lemma 1 2 From a more combinatorial point of view our problem has been to find a condition which is weak enough to be compatible with condition (i) and (ii) and yet strong enough to force pointwise convergence

3 *Reduction to a purely combinatorial problem*

The next step is in an obvious direction We show how Lemma 2 1 can be deduced from a purely combinatorial lemma

LEMMA 3 1 *There exists a sequence of integers $m(j) \geq 5$ and a collection of subsets $U(j) \subseteq [-1, 1]^{n(j)}$, where $n(j) = m(1)m(2) \quad m(j)$, with the following properties (we adopt the convention that if $u \in U(j)$ then $u_{ln(j)+r} = u_r$ for all $l \in \mathbb{Z}$, $1 \leq r \leq n(j)$)*

(i)$_{j+1}$ *If $u \in U(j)$ and $v \in U(j+1)$ then we can find an l, $1 \leq l \leq m(j+1)$, such that*

$$|u_r - v_{ln(j)+r}| \leq 2^{-j-1} \text{ for all } 1 \leq r \leq n(j)$$

(ii)$_j$ *If $1 \leq |m| \leq n(j) - 1$ we can find a $u \in U(j)$ and $1 \leq r$, $s \leq 2n(j)$ such that $s - r = m$ and $|u_r - u_s| \geq 1$*

(iii)$_j$ *If $u \in U(j)$ then, for each $j - 1 \geq p \geq k+1$ and each $i \in \mathbb{Z}$*

$$|u_i - u_{i+n(p)}| > 2^{2-k} \quad \Rightarrow \quad |u_i - u_{i+n(l)}| \leq 2^{2-l},$$

for all $p - 1 \geq l \geq k$

(iv)$_{j+1}$ *If $v \in U(j+1)$ and $1 \leq l \leq m(j)$ then writing $u_r = v_{ln(j)+r} (1 \leq r \leq n(j))$ we have $u \in U(j)$*

(v)$_{j+1}$ *There exists an $e(j) \in U(j)$ such that, if $v \in U(j+1)$ then $v_r = v_{n(j+1)-n(j)+r} = e(j)_r$ for all $1 \leq r \leq n(j)$*

Thus the 'sentences' of $U(j+2)$ are composed of 'words' from $U(j+1)$ and these words in turn are composed of 'letters' from $U(j)$ Notice that each 'sentence' in $U(j+2)$ begins and ends with the same 'buffer word' $e(j+1)$ Thus if we study a short sequence of 'letters' in some 'paragraph' or 'chapter' we know that the sequence either lies well within a single sentence or falls within two successive copies of the same 'buffer word'

Using Lemma 3 1 it is very easy to prove a lemma from which Lemma 2 1 follows almost immediately

LEMMA 3 2 *There exists a sequence $n(1) < n(2) < \quad$ and a collection of subsets $X(j)$ of $[-1, 1]^{\mathbb{Z}}$ with the following properties*

(i) *If $x \in X(j+1)$ and $y \in X(j+2)$ then given $1 \leq k \leq j$ we can find an $m \geq 1$ such that $|x_r - y_{r+m}| \leq 2^{-j-2}$ for all $|r| \leq j$*

(ii) *If* $1 \le |m| \le n(j)-1$ *we can find an* $x \in X(j)$ *such that* $|x_m - x_0| \ge 1$

(iii) *If* $x \in X(j)$ *then, for each* $j-1 \ge p \ge k+1$ *and each* $i \in \mathbb{Z}$

$$|x_i - x_{i+n(p)}| > 2^{2-k} \quad \Rightarrow \quad |x_i - x_{i+n(l)}| \le 2^{2-l}$$

for all $p-1 \ge l \ge k$.

(iv) *If* $x \in X(j)$, *then* Tx, $T^{-1}x \in X(j)$

(v) $X(j)$ *is closed in* $([-1,1]^{\mathbb{Z}}, d)$

(vi) $X(j) \supseteq X(j+1)$

Proof of Lemma 3 2 *from Lemma* 3 1 If we replace the $U(j)$ of Lemma 3 1 by their closures (with respect to the usual topology on $[-1,1]^{n(j)}$) the conditions of that lemma still apply We may therefore take the $U(j)$ to be closed We then define $X(j)$ to be the collection of all infinite strings of words from $U(j)$ More precisely, let us say that $x \in X(q,j)$ if and only if whenever $l \in \mathbb{Z}$ and $u_r = x_{ln(j)+q+r}$ $(1 \le r \le n(j))$ it follows that $u \in U(j)$ We set $X(j) = \bigcup_{q=0}^{n(j)-1} X(q,j)$ We note that

(iv)′ $X(q,j) = X(q+n(j), j)$ for all $q \in \mathbb{Z}$, and

(vi)′ $X(q,j) \subseteq X(q, j+1)$ for all $q \in \mathbb{Z}$

Conditions (iv) and (vi) follow at once Since we have taken $U(j)$ closed in $[-1,1]^{n(j)}$ it follows easily that $X(q,j)$ is closed in $([-1,1]^{\mathbb{Z}}, d)$ and condition (v) follows

To prove (i) we observe that $x \in X(q, j+1)$ and $y \in X(p, j+2)$ where (using (iv)′) we may suppose $|q| \le n(j+1)/2+1$ and $p \ge n(j+2)+1$ Two cases arise according as $|q| > j$ or $|q| \le j$ If $|q| > j$ then, writing $Q = q$ if $q < 0$ and $Q = q - n(j+1)$ if $q \ge 0$, we know that if $u_s = x_{Q+s}$ $[1 \le s \le n(j+1)]$ and $v_t = y_{p+t}$ $[1 \le t \le n(j+2)]$ then $u \in U(j+1)$ and $v \in V(j+2)$ By condition (i) of Lemma 3 1 we can find an $1 \le l \le m(j+2)$ such that $|u_s - v_{ln(j+1)+s}| \le 2^{-j-2}$ for $1 \le s \le n(j+1)$ Thus taking $m = p + ln(j+1) - Q$ we have $m \ge 1$ and

$$|y_{m+r} - x_r| = |v_{m+p+r} - u_{r-Q}| = |v_{ln(j+1)+r-Q} - u_{r-Q}| \le 2^{-j-2}$$

for $|r| \le j$

If $|q| \le j$ then, observing that $y \in X(j+2, p) \le X(j+1, p)$, and using condition (v) of Lemma 3 2, we know that, writing $u_s = x_{q+s}$, $v_s = y_{p+s}$ $[|s| \le 2j]$ we have $u_s = e(j)_s = v_s$ for $1 \le s \le 2j$ and $u_s = e(j)_{n(j)+s} = v_s$ for $-2j \le s \le 0$ Thus $x_r = u_{r-q} = v_{r-q} = y_{p-q-r}$ for $|r| \le j$ and, setting $m = p - q$ we have $m \ge 1$ and $|y_{m-r} - x_r| = 0 \le 2^{-j-2}$ for $|r| \le j$ Thus (i) holds

The proof of (iii) involves a similar splitting into cases Using (iv)′, we know that $x \in X(q,j)$ with $q+1 \le i \le q+n(j)$ If $i+n(p) \le q+n(j)$ we set $u_r = x_{q+r}$ $(1 \le r \le n(j))$ so that $u \in U(j)$ Then $|x_i - x_{i+n(p)}| > 2^{2-k}$ implies $|u_{i-q} - u_{i+n(p)-q}| > 2^{2-k}$ which by Lemma 3 1(iii) implies $|u_{i-q} - u_{i+n(l)-q}| \le 2^{2-l}$ and so $|x_i - x_{i+n(l)}| \le 2^{2-l}$ for all $p-1 \le l \le k$ If, on the other hand, $i+n(p) > q+n(j)$ we know from Lemma 3 1(v) that

$$x_{q+n(j)-n(j-1)+r} = e(j-1)_r \qquad \text{for } 1 \le r \le 2n(j-1)$$

(where, by convention $e(j-1)_{n(j-1)+s} = e(j-1)_s$ $(1 \le s \le n(j-1))$ It follows that

$$x_i = e(j-1)_{i-q-n(j)+n(j-1)} = e(j-1)_{i-q-n(j)+2n(j-1)} = x_{i+n(j-1)}$$

Thus if $|x_i - x_{i+n(p)}| > 2^{2-k}$ we have $j-2 \ge p$ and

$$|e(j-1)_{i-q-n(j)+n(j-1)} - e(j-1)_{i-q-n(j)+n(j-1)+n(p)}| > 2^{2-k},$$

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whence, by Lemma 3 1(iii),

$$|e(j-1)_{i-q-n(j)+n(j-1)} - e(j-1)_{i-q-n(j)+n(j-1)+n(l)}| \le 2^{2-l},$$

and so $|x_i - x_{i+n(l)}| \le 2^{2-l}$ for all $p-1 \ge l \ge k$. Thus (iii) holds

Finally, to prove (ii) observe that by condition (ii) of Lemma 3 1 we can find $u \in U(j)$ and $1 \le r, s \le 2n(j)$ such that $s - r = m$ and $|u_r - u_s| \ge 1$ Setting $x_{ln(j)+t-r} = u_t$ $(1 \le t \le n(j), l \in \mathbb{Z})$ we have $x \in X(j)$ and $|x_0 - x_m| \ge 1$ □

Proof of Lemma 2 1 *from Lemma* 3 2 Set $X = \bigcup_{j=1}^{\infty} X(j)$ Then conditions (i), (iii), (iv) and (v) of Lemma 2 1 follow at once from conditions (i), (iii), (iv) and (v) of Lemma 3 2 To prove Lemma 2 1(ii) we observe that by Lemma 3 2(ii) we can find, for each $j \ge 1$, an $x(j) \in X(j)$ such that $|x(j)_m - x(j)_0| \ge 1$ Since $([-1, 1]^{\mathbb{Z}}, d)$ is compact the sequence of $x(j)$ must have a limit point x, say By Lemma 3 2(vi) $x(j) \in X(k)$ for all $j \ge k$ and so, since $X(k)$ is closed, $x \in X(k)$ for all $k \ge 1$ Thus $x \in X$ and, since $d(x(j), x) \to 0$, $|x_m - x_0| \ge 1$ □

Remark. In fact, our construction will be sufficiently explicit to allow us to write down a specific x without appealing to general results

4 *The induction*

Although Lemma 3 1 has an inductive form it does not, as it stands, lend itself to an inductive proof The key step in the paper consists in replacing it with a narrower, more specific, result which can be obtained by induction

LEMMA 4 1 *There exists a sequence of integers* $m(j) \ge 5$ *and three sequences of functions*

$$a(j,\ )\ [0, 1] \to [-1, 1]^{n(j)}$$
$$b(j,\ )\ [0, 1] \to [-1, 1]^{n(j)}$$
$$c(j,\ )\ [0, 1] \to [-1, 1]^{n(j)}$$

where $n(j) = m(1)m(2) \quad m(j)$ *with the following properties (we write* $U(j) = \{a(j, t)\ \ t \in [0, 1]\} \cup \{b(j, t)\ \ t \in [0, 1]\} \cup \{c(j, t)\ \ t \in [0, 1]\}$ *and adopt the convention that if* $u \in U(j)$ *then* $u_{ln(j)+r} = u_r$ *for all* $l \in \mathbb{Z}$, $1 \le r \le n(j)$*)*

$(i)_{j+1}$ *If* $u \in U(j)$ *and* $v \in U(j+1)$ *then we can find an* l, $1 \le l \le m(j+1)$, *such that*

$$|u_r - v_{ln(j)+r}| \le 2^{-j-1} \qquad \textit{for all } 1 \le r \le n(j)$$

$(ii)_j$ *If* $1 \le m \le n(j) - 1$ *we can find* $u \in \{a(j, 1), b(j, 1), c(j, 1)\}$, $1 \le r, s \le 2n(j)$ *such that* $s - r = m$ *and* $|u_r - u_s| \ge 1$

$(iii)_j$ *If* $u \in U(j)$ *then for each* $j - 1 \ge p \ge k + 1$ *and each* $i \in \mathbb{Z}$

$$|u_i - i_{i+n(p)}| > 2^{2-k} \quad \Rightarrow \quad |u_i - u_{i+n(l)}| \le 2^{2-l}$$

for all $p - 1 \ge l \ge k$

$(iv)_{j+1}$ *If* $v \in U(j+1)$ *and* $1 \le l \le m(j)$ *then writing* $u_r = v_{ln(j)+r}$ $(1 \le r \le n(j))$ *we have* $u \in U(j)$

$(v)_{j+1}$ *If* $v \in U(j+1)$ *then* $v_r = v_{n(j+1)-n(j)+r} = c(j, 0)_r$ *for all* $1 \le r \le n(j)$

$(vi)_j$ $a(j, 0) = b(j, 0) = c(j, 0)$

$(vii)_j$ *There exists an* $N(j)$, $1 \le N(j) \le n(j)$, *such that* $a(j, t)_{N(j)} = t$ *and* $b(j, t)_{N(j)} = -t$

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(viii)$_j$ *If* $1 \le k \le j-1$, $1 \le i \le n(j)$ *and* $t \in [0, 1]$ *then*

$$|a(j, t)_i - b(j, t)_i| > 2^{2-k} \quad \Rightarrow \quad |a(j, t)_{i+n(l)} - a(j, t)_i| \le 2^{2-l}$$

for all $j-1 \ge l \ge k$

Thus conditions (i)$_{j+1}$, (iii)$_j$ and (iv)$_j$ come over from Lemma 3 1 unchanged, conditions (ii)$_j$ and (v)$_{j+1}$ are strengthened whilst conditions (iv)$_j$, (vii)$_j$ and (viii)$_j$ are new A proof of Lemma 4 1 will thus give a proof of Lemma 3 1 The first step in the inductive proof of Lemma 4 1 is simple

LEMMA 4 2 *Let* $n(1) = m(1) = 5$, $N(1) = 3$ *and*

$$a(1, t) = (0, 0, t, 0, 0) \qquad (0 \le t \le 1)$$
$$b(1, t) = (0, 0, -t, 0, 0) \qquad (0 \le t \le 1)$$
$$c(1, t) = (0, 0, 0, 0, 0) \qquad (0 \le t \le 1)$$

Then conditions (ii)$_1$, (iii)$_1$, (vi)$_1$, (vii)$_1$ *and* (ix)$_1$ *of Lemma* 4 1 *are satisfied*

Proof Direct inspection (Condition (ix)$_1$ is vacuously satisfied) □

To complete the induction we use the following lemma (in which an attempt has been made to simplify the notation)

LEMMA 4 3 *Suppose* $n(j) \ge 5n(l)$ *for all* $j-1 \ge l \ge 1$ *and suppose that the three functions from* $[0, 1]$ *to* $[-1, 1]^{n(j)}$ *whose values at* $0 \le t \le 1$, *given by* $a(t) = a(j, t)$, $b(t) = b(j, t)$, $c(t) = c(j, t)$, *satisfy conditions* (ii)$_j$, (iii)$_j$, (v)$_j$, (vi)$_j$, (vii)$_j$ *and* (ix)$_j$ *of Lemma* 4 1 *Then we can find* $m(j+1) \ge 5$ *and three functions from* $[0, 1]$ *to* $[-1, 1]^{n(j+1)}$ (*where* $n(j+1) = n(j)m(j+1)$) *whose values at* $0 \le t \le 1$ *are given by* $A(t) = a(j+1, t)$, $B(t) = b(j+1, t)$, $C(t) = c(j+1, t)$ *and satisfy all the conditions* (i)$_{j+1}$ *to* (viii)$_{j+1}$ *of Lemma* 4 1

Proof Since a, b, c are continuous on $[0, 1]$ they are uniformly continuous We can therefore find an $\eta > 0$ such that η^{-1} is an integer and $|a(t)_r - a(\tau)_r|$, $|b(t)_r - b(\tau)_r|$, $|c(t)_r - c(\tau)_r| < 2^{-j-1}$ whenever $|t - \tau| \le \eta$ and $1 \le r \le n(j)$ We set $M = \eta^{-1} + 1$, $m(j+1) = 18M + n(j)$ and define A, B and C as follows

If $0 \le l \le 15M$, $1 \le r \le n(j)$ we set

$$A(t)_{ln(j)+r} = B(t)_{ln(j)+r} = C(t)_{ln(j)+r} = E_{ln(j)+r}$$

where

$$\begin{aligned} E_{ln(j)+r} &= a(l\eta)_r && (0 \le l \le M) \\ &= a(1)_r && (M+1 \le l \le 4M) \\ &= a((5M-l)\eta)_r && (4M+1 \le l \le 5M) \\ &= b((l-5M-1)\eta)_r && (5M+1 \le l \le 6M) \\ &= b(1)_r && (6M+1 \le l \le 9M) \\ &= b((10M-l)\eta)_r && (9M+1 \le l \le 10M) \\ &= c((l-10M-1)\eta)_r && (10M+1 \le l \le 11M) \\ &= c(1) && (11M+1 \le l \le 14M) \\ &= c((15M-l)\eta)_r && (14M+1 \le l \le 15M) \end{aligned}$$

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If $15M+1\le l\le 16M$, $1\le r\le n(j)$ we set

$$A(t)_{ln(j)+r}=C(t)_{ln(j)+r}=a((l-15M-1)\eta t)_r$$
$$B(t)_{ln(j)+r}=b((l-15M-1)\eta t)_r,$$

whilst if $16M+1\le l\le 17M$, $1\le r\le n(j)$ we set

$$A(t)_{ln(j)+r}=a((17M-l)\eta t)_r$$
$$B(t)_{ln(j)+r}=C(t)_{ln(j)+r}=b((17M-l)\eta t)_r$$

It is easy to check that conditions $(\mathrm{iv})_{j+1}$, $(\mathrm{v})_{j+1}$ and $(\mathrm{vi})_{j+1}$ are satisfied and to see that condition $(\mathrm{vii})_{j+1}$ is satisfied it suffices to take $N(j+1)=16Mn(j)+N(j)$ We check the remaining conditions $(\mathrm{i})_{j+1}$, $(\mathrm{ii})_{j+1}$, $(\mathrm{iii})_{j+1}$ and $(\mathrm{viii})_{j+1}$ one by one

Condition $(\mathrm{i})_{j+1}$ Suppose $u\in U(j)$ and $v\in U(j+1)$ Then $u\in\{a(t), b(t), c(t)\}$ for some fixed $0\le t\le 1$ Choose $1\le k\le M$ so that $|(k-1)\eta-t|<\eta$ By the choice of η

$$|E_{kn(j)+r}-a(t)_r|=|a((k-1)\eta)_r-a(t)_r|<2^{-j-1},$$
$$|E_{(5M+k)n(j)+r}-b(t)_r|=|b((k-1)\eta)_r-b(t)_r|<2^{-j-1},$$
$$|E_{(10M+k)n(j)+r}-c(t)_r|=|c((k-1)\eta)_r-c(t)_r|<2^{-j-1},$$

for all $1\le r\le n(j)$ It follows that

$$\min_{0\le p\le 2}\max_{1\le r\le n(j)}|v_{(5PM+k)n(j)+r}-u_r|<2^{-j-1}$$

and condition $(\mathrm{i})_{j+1}$ follows

Condition $(\mathrm{ii})_{j+1}$ We begin with two simplifying remarks Firstly since $u_{r+n(j+1)}=u_r$ the condition $1\le r$, $s\le 2n(j)$ may be ignored Secondly if $r-s=m$ then $n(j+1)+s-r=n(j+1)-m$ and so we need only consider $1\le m\le n(j+1)/2+1$ Even so, we shall distinguish 3 cases according as $1\le m\le Mn(j)$ and $m\not\equiv 0 \bmod n(j)$, $1\le m\le Mn(j)$ and $m\equiv 0 \bmod n(j)$, or $Mn(j)+1\le m$

If $1\le m\le Mn(j)$ and $m\not\equiv 0 \bmod n(j)$ then $m=kn(j)+\mu$ for some integers $1\le k\le M$ and $1\le\mu\le n(j)$ By condition $(\mathrm{ii})_j$ we know that there exist integers $1\le\rho$, $\sigma\le 2n(j)$ such that

$$\max(|a(1)_\rho-a(1)_\sigma|, |b(1)_\rho-b(1)_\sigma|, |c(1)_\rho-c(1)_\sigma|)\ge 1$$

It follows that

$$\max_{0\le p\le 2}(|C(1)_{((5p+1)M+k)n(j)+\mu}-C(1)_{(5p+1)Mn(j)+\sigma}|)\ge 1$$

We note that

$$(((5P+1)M+k)n(j)+\mu)-((5P+1)Mn(j)+\sigma)=m,$$

so our discussion of this case is complete

If $1\le m\le Mn(j)$ and $m\equiv 0 \bmod n(j)$ then $m=kn(j)$ for some $1\le k\le M$ Thus setting $r=(16M+k)n(j)+N(j)$ and $s=16Mn(j)+N(j)$ we have $r-s=m$ and, using condition $(\mathrm{vii})_j$,

$$|C(1)_r-C(1)_s|=16((M-k)\eta)_{N(j)}-a(1)_{N(j)}|$$

Finally, if $Mn(j)+1\le m\le n(j+1)/2+1$ then taking $r=16Mn(j)+N(j)$ and $s=r-m$ we have $1\le s\le(15M+1)n(j)$ and so $A(1)_s=C(1)_s$ whilst $A(1)_r=1$, $C(1)_r=-1$ Thus

$$|A(1)_r-A(1)_s|+|C(1)_r-C(1)_s|=|1-A(1)_s|+|A(1_s)+1|\ge 2$$

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and so $\max(|A(1)_r - A(1)_s|, |C(1)_r - C(1)_s|) \geq 1$ Combining the 3 cases discussed we obtain condition $(\mathrm{ii})_{j+1}$

Condition $(\mathrm{iii})_{j+1}$ Observe first that this condition is vacuously satisfied if $j = 1$ We may therefore suppose $j \geq 2$ We distinguish 2 cases according as $p \leq j-1$ or $p = j$ If $p \leq j-1$ the argument is exactly the same as that used to establish condition (iii) in the proof of Lemma 3 2 from Lemma 3 1 If $p = j$ we argue as follows

By the choice of η and the definitions of A, B and C

$$|A(t)_i - A(t)_{i+n(j)}|, \ |B(t)_i - B(t)_{i+n(j)}| < 2^{-j-1}$$

for all $1 \leq i \leq n(j+1)$ whilst

$$|C(t)_i - C(t)_{i+n(j)}| < 2^{-j-1}$$

for all $1 \leq i \leq (16M-1)n(j)$ and all $16Mn(j)+1 \leq i \leq n(j+1)$ $(0 \leq t \leq 1)$ Further, by $(\mathrm{v})_j$,

$$C(t)_{16Mn(j)-n(j-1)+r} = c(j-1, 0)_r = C(t)_{16Mn(j)+n(j)-n(j-1)+r}$$

for all $1 \leq r \leq n(j-1)$ and so

$$|C(t)_i - C(t)_{i+n(j)}| = 0 < 2^{-j-1}$$

for all $16Mn(j) - n(j-1) + 1 \leq i \leq 16Mn(j)$ Thus if $1 \leq i \leq n(j+1)$ and $|u_i - u_{i+n(j)}| > 2^{2-k}$ we can conclude that $u = C(\tau)$ for some $0 \leq \tau \leq 1$ and $i = (16M-1)n(j) + q$ for some $1 \leq q \leq n(j) - n(j-1)$

It follows that $u_i = C(\tau)_i = a(\tau)_q$ and $u_{i+n(j)} = C(\tau)_{i+n(j)} = b(\tau)_q$ Since $|a(\tau)_q - b(\tau)_q| > 2^{2-k}$ we see from $(\mathrm{viii})_j$ that $|a(\tau)_{q+n(l)}| \leq 2^{2-l}$ and so, since $a(\tau)_{q+n(l)} = C(\tau)_{i+n(l)} = u_{i+n(l)}$, that $|u_i - u_{i+n(l)}| \leq 2^{2-l}$ for all $j-1 \geq l \geq k$ as required

Condition $(\mathrm{viii})_{j+1}$ As we observed above, $|A(t)_i - A(t)_{i+n(j)}| < 2^{-j-1}$ for all $1 \leq i \leq n(j+1)$ and all $0 \leq t \leq 1$ Thus we need only check condition $(\mathrm{viii})_{j+1}$ for $j-1 \geq l \geq k$ since the case $l = j$ is settled automatically Since when $j = 1$ there are no further values of l to consider we may suppose $j \geq 2$

By construction $A(t)_i = B(t)_i$ for all $1 \leq i \leq 15Mn(j)$ and, by condition $(\mathrm{vi})_j$, $A(t)_i = B(t)_i$ for all $Pn(j) - n(j-1) + 1 \leq i \leq Pn(j)$ Thus if $|A(t)_i - B(t)_i| > 2^{2-k}$ it follows that $i = kn(j) + r$ where $15M \leq k \leq 17M - 1$ and $1 \leq r \leq n(j) - n(j-1)$, and so $A(t)_i = a(\tau)_r$, $B(t)_i = b(\tau)_r$ for some $1 \geq \tau \geq 0$ We now have $|a(\tau)_r - b(\tau)_r| > 2^{2-k}$ whence, by condition $(\mathrm{viii})_j$, $|a(\tau)_{r+n(l)} - a(\tau)_r| \leq 2^{2-l}$ for all $j-1 \geq l \geq k$ Since $a(\tau)_{r+n(l)} = A(t)_{i+n(l)}$ this yields $|A(t)_{i+n(l)} - A(t)_i| \leq 2^{2-l}$ for all $j-1 \geq l \geq k$ as required □

Lemmas 4 2 and 4 3 together give Lemma 4 1 and so the proof is complete

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