have $F'F = E'X$, $F'C = E'Y$ and $\angle F' < \angle E'$ and so, by the open mouth theorem, it follows that $BE > XY > CF$ as required.

Acknowledgement

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107.13 An interesting generator of Archimedean circles

A very simple but, we think, very hard to prove, Proposition 1 for Archimedean circles (see [1, 2, 3]) led us to an interesting generalisation, and unexpectedly not so hard to prove, Proposition 2.

Proposition 1: From a point P on a circle with centre O and diameter AB we drop the perpendicular PC to AB such that $AC = 2a$, $CB = 2b$. From P we draw the tangents PQ, PR, to these circles and the perpendicular from O meets the line CP at the point X. The symmetric circles relative to XO, namely $I(r)$ and $J(r)$ that are tangent to the line QR and internally tangent to the circles with diameters AB, XO are Archimedean circles i.e.

$$r = \frac{ab}{a + b}.$$

If S is the external centre of similitude (Figure 1) of the circles with diameters AC, CB then the inversion with pole S and power $SA \cdot SB = SC^2$ transforms the circles with diameters AC, CB and maps to itself the circle $P(PC)$ that passes through Q, R. Hence the inverse of Q lies on the circle $P(PC)$ and the circle with diameter CB and hence this point is R, which means that the line QR passes through S (Figure 1).

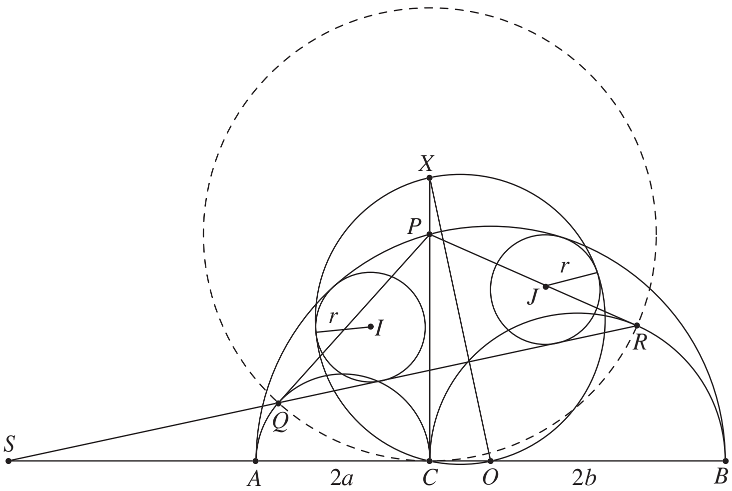

FIGURE 1

So in order to prove Proposition 1 it is sufficient to prove for the circle $J(r)$ the more general Proposition 2 below

Proposition 2: From a point P on a circle with centre O and diameter AB we drop the perpendicular PC to AB such that $AC = 2a$, $CB = 2b$. On the extension of CP we take an arbitrary point N and from the external centre of similitude of the circles with diameters AC, CB we draw a line L perpendicular to NO. The circle $X(r)$ that is tangent to the line L and internally tangent to the circles with diameters AB, NO is an Archimedean circle i.e.

$$r = \frac{ab}{a + b}.$$

Proof: If O is the centre of Cartesian coordinates and AB is the axis of abscissas then (Figure 2) we have the points $A(-a - b, 0)$, $B(a + b, 0)$, $C(a - b, 0)$ and the midpoints of AC, CB, $O_1(-b, 0)$, $O_2(a, 0)$. If $S(S_1, 0)$ is the external centre of similitude of the circles with diameters AC, CB and $a < b$ then from $\dfrac{O_1 - S_1}{O_2 - S_1} = \dfrac{a}{b}$ or $\dfrac{-b - S_1}{a - S_1} = \dfrac{a}{b}$ we get $S_1 = \dfrac{a^2 + b^2}{a - b}$. Let $Y(r_1)$ be the circle with diameter NO and $X(r)$ be the circle that is internally tangent to the circles $Y(r_1)$, $O(a + b)$ and to the line L.

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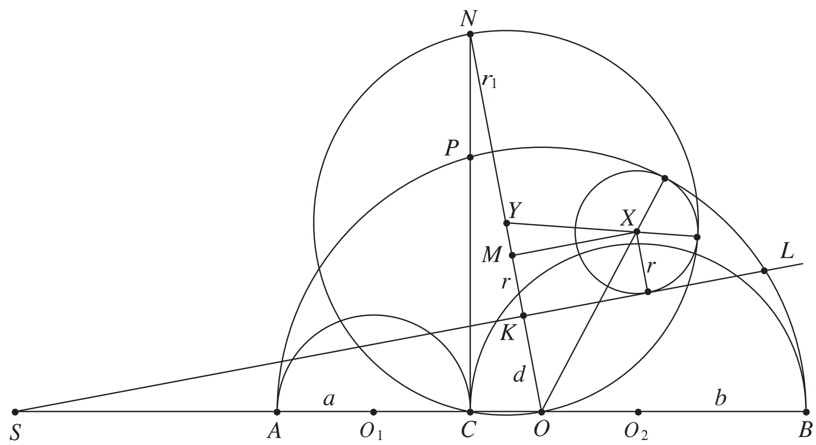

FIGURE 2

Let K be the intersection of L with ON and M the orthogonal projection of X on ON. It is obvious that $XY = r_1 - r$, $OX = a + b - r$ and if $OK = d$ that $OM = d + r$, $MY = r_1 - r - d$. The points K, C are on the circle with diameter SN, so the power of O gives $ON \cdot OK = SO \cdot OC$ or

$$2dr_1 = a^2 + b^2. \qquad (1)$$

The Pythagorean theorem gives

$$OX^2 - XY^2 = OM^2 - MY^2$$

or

$$(a + b - r)^2 - (r_1 - r)^2 = (d + r)^2 - (r_1 - r - d)^2$$

or

$$(a + b)^2 - 2r(a + b) = 2dr_1$$

or

$$a^2 + b^2 + 2ab - 2r(a + b) = a^2 + b^2$$

or

$$r = \frac{ab}{a + b}$$

and here ends the proof.

Note: Since $ON \geqslant OP = a + b$ the line L intersects the circles $O_1(a)$, $O_2(b)$ with diameters AC, CB or is tangent to them because the distance d_1 of O_1 from the line L is

$$d_1 = \frac{O_1S \cdot OK}{OS} = \left(-b + \frac{a^2 + b^2}{b - a}\right)\frac{b - a}{ON} = \frac{a(a + b)}{ON} \leqslant a.$$

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Special Cases

i) If $N \equiv P$ then the line L is tangent to the circles with diameters AC, CB.

ii) If $a = b$ then Proposition 1 and Proposition 2 cannot hold because the line L is parallel to AB, $C \equiv O$ and the perpendicular from O to L coincides with CP. In this case from (1) we conclude that, given the point N on the perpendicular bisector of AB not inside the circle, the line L must be parallel to AB at a distance $d = 2a^2 / ON$ and can be constructed as follows. The semicircle with diameter O_1B, where O_1 is the midpoint of AO, meets the segment ON at M, (Figure 3) and on the circle with diameter ON we take a point T such that $OT = OM$. The parallel from T to AB is the required line L.

So we have the following:

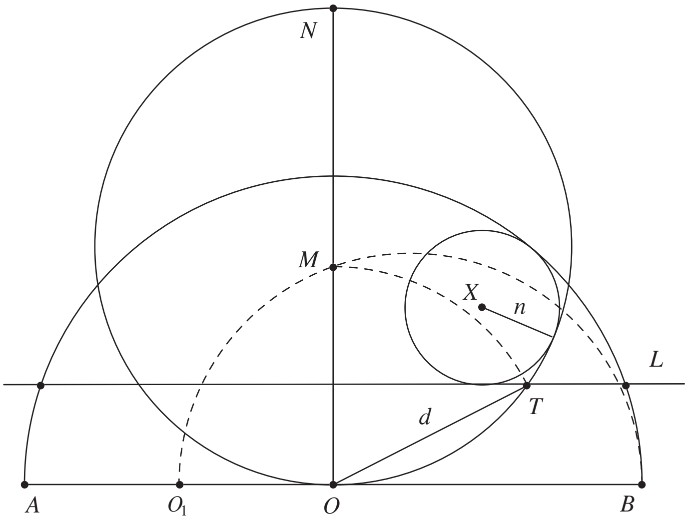

FIGURE 3

Proposition 3: From a point N on the perpendicular bisector of a diameter AB of the circle $O(2a)$, we draw a parallel line L to AB at a distance $d = 2a^2 / ON$, (constructed as above). Then the circle which is tangent to the line L and tangent internally to the circles with diameters AB, ON is an Archimedean circle with $r = \frac{1}{2}a$ relative to the congruent circles with diameters AO, OB.

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