the projections of the path and of the sections on the xy plane; and the last of these curves which the path meets, will determine the ultimate value of z. Now, since all the projections of the sections touch the y-axis at the origin, if OP, the projection of the path of the moving particle, does not touch the y-axis at the origin, it must, as it approaches the origin, cut the projections of all the sections in the neighbourhood of the origin, until it meets at the origin, the line OA, which is the projection of the section for which $z=1$. Hence, if OP does not touch the y-axis at the origin, or if the ultimate value of $y:x$ {or of $\phi(t):f(t)$} is not $\pm\infty$, then the ultimate value of z is unity. This quite agrees with what was proved in the earlier part of this paper.

Dr Thomas Muir has furnisht me with the following references to papers which I have not myself seen :—

Johnson, W. W. On the expression 0^0. *Analyst* III, pp. 118-121 (1876).

Franklin, F. Note on indeterminate exponential forms. *American Journal of Mathematics* I, pp. 368-9 (1878).

The so-called Simson line.

By John Alison, M.A.

This paper is an attempt to collect and arrange some of the propositions regarding the so-called Simson line, contained in various Mathematical Treatises and Journals. Proofs have been altered, or new ones substituted to suit the arrangement.

§ 1. *Figure* 20. If from any point on the circumference of a circle perpendiculars be drawn to the sides of an inscribed triangle, the feet of these perpendiculars are collinear. [The line on which they lie is known as the Simson line; but it is doubtful if Simson knew it. See *Nature*, Vol. XXX. p. 635.]

§ 2. There are two theorems of which the preceding may be regarded as a particular case.

(a). If lines be drawn from any point on the circumference of a circle, making equal angles in the same direction with the three sides of an inscribed triangle, the points of intersection of these lines with

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the respective sides are collinear. Poncelet, *Propriétés Projectives*, § 468.

(*b*). *Figure* 21. If from any point P in a plane perpendiculars be drawn to the three sides of a triangle ABC; and if the feet of these perpendiculars be joined, the locus of P so that the area of the pedal triangle may be constant is a circle concentric with the circle ABC.

As P moves from the centre of the circle ABC to its circumference, the area of the pedal triangle diminishes from $\frac{1}{4}$ (triangle ABC) to zero when the triangle becomes the Simson line.

Let QRS, the pedal triangle, be denoted by Δ', and the triangle formed by joining the middle points of the sides of the triangle ABC by Δ; the radius of the circle ABC, by R, and the distance between P and the centre of the circle ABC, by R′. Let O be the centre of the circumscribed circle, and AOK a diameter. Join CP and produce it to meet the circumference of the circle in D and draw BD. Join OP and CK, and draw the perpendiculars AH, BI, PE, RF to the lines BC, CA, BD, QS respectively, and produce QP to meet BA in G.

Then since each of the quadrilaterals BSPQ, PQCR is inscriptible in a circle, we have $\angle DBA = \angle DCA = \angle PQR$, and $\angle ABP = \angle PQS$; hence $\angle EBP = \angle FQR$, and the triangles EBP, FQR are similar; therefore $\frac{PE}{RF} = \frac{PB}{RQ}$ or $PE \cdot RQ = PB \cdot RF$.

From the similar triangles BPG and GSQ we get

$$\frac{QS}{BP} = \frac{GS}{GP} = \frac{GQ}{GB} = \frac{AH}{AB} = \frac{AC}{AK}; \text{ therefore } \frac{QS}{BP} = \frac{AC}{AK} \text{ or } \frac{QS \cdot RF}{BP \cdot RF} = \frac{AC}{AK}.$$

Similarly $\frac{RQ}{CP} = \frac{AB}{AK}$ ... ... (1).

Again, the angles EDP, BAC being equal, we have

$$\frac{PE}{PD} = \frac{BI}{AB} \quad \dots \quad \dots \quad \dots \quad \dots \quad (2).$$

By (1) and (2) $\frac{RQ \cdot PE}{CP \cdot PD} = \frac{AB \cdot BI}{AB \cdot AK} = \frac{BI}{AK}$.

But it has been shown that $RQ \cdot PE = PB \cdot RF$; $\therefore \frac{PB \cdot RF}{CP \cdot PD} = \frac{BI}{AK}$;

and since $\frac{QS \cdot RF}{PB \cdot RF} = \frac{AC}{AK}$, therefore $\frac{QS \cdot RF}{CP \cdot PD} = \frac{AC \cdot BI}{AK^2}$.

Now $QS \cdot RF = 2\Delta'$, $CP \cdot PD = DO^2 - OP^2 = R^2 - R'^2$, $AC \cdot BI = 8\Delta$, and $AK^2 = 4R^2$; consequently we get finally $\frac{\Delta'}{\Delta} = \frac{R^2 - R'^2}{R^2}$.

In a similar manner when the point P is without the circumscribed circle it may be proved that $\frac{\Delta'}{\Delta} = \frac{R'^2 - R^2}{R^2}$; consequently $\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{R^2}$ according as the point is *within* or *without* the circumscribed circle. When P is on the circumscribed circle $R = R'$; hence $\Delta' = 0$, that is, the triangle QRS becomes a straight line.

The Mathematician, (1847) Vol. II. p. 37 (Davis). An analytical solution is given in Vol. I. (1843).

§ 3. If the projections (orthogonal or oblique) of a point on the three sides of a triangle are collinear, the locus of the point is the circumscribed circle of the triangle.

§ 4. If from any point on a circle three chords be drawn, and if on the same side of these chords three similar segments of circles be described, the three circles so constructed will intersect two by two on a straight line.

Conversely, if three circles have one common point, and intersect again two by two on a straight line; and if chords be drawn from the common point cutting off similar segments from the three circles in the same direction, the extremities of these three chords shall be concyclic with the common point.

§ 5. *Figure* 20. If three circles pass through one point which is concyclic with their centres, they shall intersect two by two on a straight line.

Let P be the common point, X, Y, Z the centres, draw diameters through P to A, B, C respectively.

P is the external centre of similitude of the circles round XYZ and ABC; and it is on the circle XYZ; therefore it is on the circle ABC The theorem then becomes a case of § 4.

The points of intersection lie on the Simson line P(ABC).

Cor. The Simson line P(ABC) makes, with the sides of the triangle ABC taken by twos, three triangles, the centres of whose circumscribed circles are concyclic with P on a circle of half the radius of the circle ABC. *Nouvelles Annales de Mathématiques*, 2nd Series, Vol. X., p. 206.

§ 6. *Figure* 22. For every complete quadrilateral there can be found a point such that the feet of the perpendiculars drawn from it on the four sides are collinear.

For the locus of the point, the feet of whose perpendiculars on AB, BC, CD are collinear, is the circle round CBE (§ 3); and the locus of the point, the feet of whose perpendiculars on BC, CD, DA are collinear, is the circle round CDF.

Therefore the feet of the perpendiculars from P, the second point of intersection of the two circles on the four lines AB, BC, CD, DA are collinear.

Cor. 1. The circumscribed circles of the four component triangles of any complete quadrilateral pass through one point.

Cor. 2. For every complete quadrilateral there is a point P such that if it be joined to the six vertices, and if circles be described on these lines as diameters, the six circles intersect three by three in four points which are collinear.

The line of collinearity is the Simson line of each of the component triangles with respect to P, and may be called the Simson line of the complete quadrilateral. Catalan's *Théorèmes et Problèmes*, 6th ed., p. 34.

§ 7. *Figure* 20. If a triangle ABC inscribed in a circle have its vertex A fixed; and if the sides AB, AC be such that the rectangle under their distances from a fixed point P on the circle is constant; the Simson line P(ABC) meets BC on a fixed circle.

Draw a tangent at A, and on it drop a perpendicular PD′.

P,B′,A,C′,D′ are concyclic, and therefore ∠ PD′B′ = ∠ PC′B′, and ∠ D′B′P = ∠ D′AP = ∠ ACP, the angle in the alternate segment, = ∠ B′A′P, the angle in the same segment of the circle PB′A′C.

Since ∠ PD′B′ = ∠ PC′A′ and ∠ PB′D′ = PA′C′, therefore the triangles PB′D′ and PC′A′ are similar and PD′·PA′ = PB′·PC′.

Since A and P are fixed points on the circle, therefore PD′ is constant; and if PB′·PC′ be constant PA′ is constant; that is, A′ lies on a circle whose centre is P. PA′ the radius $= \dfrac{PB'\cdot PC'}{PD'}$. Since PA′B is a right angle BC has this circle for its envelope.

Cor. If DEF be the triangle whose sides touch the circle at A, B, and C, and if PD′, PE′, PF′ be the perpendiculars from P on EF, FD, and DE respectively; PA′·PB′·PC′ = PD′·PE′·PF′.

For $\quad PD' \cdot PA' = PB' \cdot PC'$,
$PE' \cdot PB' = PC' \cdot PA'$,
and $PF' \cdot PC' = PA' \cdot PB'$.
Hence $PD' \cdot PE' \cdot PF' = PA' \cdot PB' \cdot PC'$.

§ 8. *Figure* 23. The Simson line D(ABC) is parallel to AP, BQ, and CR; P, Q, R being the points where the perpendiculars from D on BC, CA, AB meet the circle again.

$\angle BAP = \angle BDP$ = complement of $\angle DBC$,
= complement of $\angle DAB'$,
= complement of $\angle DC'B'$,
= $\angle AC'B'$.

Therefore AP and C'A' are parallel; hence also BQ, CR are parallel to D(ABC). Catalan's *Théorèmes et Problèmes*, p. 36.

Name the othocentres of the four triangles ABC, BCD, CDA, DAB, O_4, O_1, O_2, O_3 respectively.

§ 9. *Figure* 23. The Simson line D(ABC) bisects DO_4.

Since AP is parallel to C'A', therefore MAPA' is a parallelogram, and $AM = A'P = A'O_1$.

O_4, O_1 are the images of N and P in BC; therefore O_4O_1 and NP are antiparallels with respect to BC. But AD is also antiparallel to NP with respect to BC; therefore O_4O_1DA is a parallelogram, and $AO_4 = DO_1$; $AM + AO_4 = DO_1 + O_1A'$; $MO_4 = DA'$; therefore $MO_4A'D$ is a parallelogram, and DO_4 one of the diagonals is bisected by the other diagonal D(ABC). Catalan's *Théorèmes et Problèmes*, p. 37.

§ 10. *Figure* 22. The orthocentres of the four component triangles of a complete quadrilateral are collinear.

For if P the intersection of the four circumscribed circles be joined to the four orthocentres O_1, O_2, O_3, O_4, the lines PO_1, PO_2, PO_3, PO_4 are bisected by the Simson lines of the triangles with respect to P (§ 9). But these Simson lines are coincident (§ 6, Cor. 2); the points of bisection of PO_1, PO_2, PO_3, PO_4 are collinear; therefore O_1, O_2, O_3, O_4 lie on a line parallel to the Simson line of the complete quadrilateral.

§ 11. *Figure* 22. The Simson line of a complete quadrilateral is at right angles to the line passing through the middle points of the three diagonals.

O_1 is the orthocentre of ABF, X, Y, Z the points where O_1A, O_1B, O_1F cut the sides opposite A, B, F at right angles. AX, BY, FZ are pairs of points on the circles on AC, BD, FE as diameters; therefore the potencies of the three circles with respect to O_1 are $O_1A \cdot O_1X$, $O_1B \cdot OY$, $OF \cdot OZ$. But since AXBY and BYFZ are sets of concyclic points; $O_1A \cdot O_1X = O_1B \cdot O_1Y = O_1F \cdot O_1Z$; that is, O_1 is a point on the radical axis of the three circles on the diagonals as diameters.

Hence also O_2, O_3, O_4 are on the radical axis of the three circles, which is at right angles to the line through their centres, the middle points of the three diagonals. The Simson line is parallel to $O_1O_2O_3O_4$ (§ 10) and is therefore at right angles to the line joining the middle points of the diagonals.

§ 12. *Figure* 23. Given four points A, B, C, D on a circle, the Simson lines A(BCD), B(CDA), C(DAB), D(ABC) are concurrent.

AO_4O_1D is a parallelogram (§ 9) and T is the intersection of the diagonals; therefore T bisects AO_1 and DO_4. Hence also T bisects BO_2 and CO_3. But the lines AO_1, BO_2, CO_3, DO_4 are bisected by the Simson lines A(BCD), B(CDA), C(DAB), D(ABC) respectively; therefore the four Simson lines pass through T. *Educational Times Reprint*, Vol. I., Question 1431.

Cor. I. Since O_1, O_2, O_3, O_4 are the external centres of similitude of the circumscribed circle ABC and the nine-point circles of the triangles BCD, CDA, DAB, and ABC respectively, and the radius of the nine-point circle is half the radius of the circumscribed circle; hence the nine-point circles bisect AO_1, BO_2, CO_3, DO_4 respectively and so pass through T.

The nine-point circles of the four triangles formed by joining any four points in a plane are concurrent. See *Lady's and Gentleman's Diary*, 1864, p. 55.

Cor. 2. If A be a variable point on the circle, then the Simson lines D(ABC), B(ADC) intersect on the nine-point circle of DBC at a constant angle.

However A move the Simson lines of the four triangles are concurrent with their nine-point circles. But the triangle DBC is fixed; therefore its nine-point circle is the locus of the intersection of D(ABC) and B(ADC).

Since D(ABC), B(ADC) pass through the fixed points A′ and H respectively on the nine-point circle of DBC and intersect on the same circle, therefore their angle of intersection is constant.

The angle is equal to $\angle A'UH = 2\angle A'DC$,

$= $ twice the complement of $\angle BCD$.

§ 13. *Figure* 23. (*a*) $O_1O_2O_3O_4$ is a quadrilateral equal in all respects to ABCD.

For O_4O_1 has been shown to be parallel and equal to DA (§ 9); hence also O_1O_2 is equal and parallel to AB,

O_2O_3 „ „ BC,

and O_3O_4 „ „ CD.

Catalan's *Théorèmes et Problèmes*.

(*b*) T is the internal centre of similitude of the two quadrilaterals.

(*c*) If three points be taken from one of the sets of concyclic points (ABCD), ($O_1O_2O_3O_4$) and the non-corresponding point from the other set, and if any three of these four points be taken as vertices of a triangle, the fourth point is its orthocentre. For example, take the four points $ABCO_4$; O_4 is the orthocentre of ABC by construction, and by the same construction A is the orthocentre of BCO_4,

B „ CO_4A,

C „ O_4AB.

If three be chosen from the second set and one from the first the same is true, for ABCD and $O_1O_2O_3O_4$ are reciprocal figures; that is, the same construction which derived $O_1O_2O_3O_4$ from ABCD will derive ABCD from $O_1O_2O_3O_4$.

(*d*) Any pair of points from one of the sets (ABCD), ($O_1O_2O_3O_4$) with the non-corresponding pair from the other set, are four points on a circle whose radius is equal to that of ABCD or $O_1O_2O_3O_4$. For example, O_1BCO_4 are on an arc which is the image of BNC in BC, and so is part of a circle of equal radius with ABC.

§ 14. *Figure* 23. The Simson line D(ABC) is identical with the Simson line $O_1(BCO_4)$.

For D(ABC) passes through A′ the projection of D on BC, and through T the middle point of DO_4 (§ 9); and $O_1(BCO_4)$ passes through A′ the projection of O_1 on BC, and through T the middle point of O_1A.

Hence also D(ABC) coincides with $O_2(CO_4A)$ and $O_3(O_4AB)$.
Journal de Mathématiques Spéciales, 2nd Series, Vol. III. p. 30 (Weill).

Cor. 1. Every line in the quadrilateral ABCD is parallel to the corresponding line of $O_1O_2O_3O_4$, and if any line of one pass through T, the centre of similitude, so does the corresponding line of the other.

Hence $D(ABC)$ and $O_4(O_1O_2O_3)$ are parallel to each other, and both pass through T; therefore they coincide.

This gives a set of eight Simson lines which are coincident, viz.:—

(1) $D(ABC)$, $O_1(BCO_4)$, $O_2(CO_4A)$, $O_3(O_4AB)$,
$O_4(O_1O_2O_3)$ $A(O_2O_3D)$, $B(O_3DO_1)$, $C(DO_1O_2)$.

Of the points ABCD, $O_1O_2O_3O_4$ we may form eight sets of four concyclic points, and of the four points in a set we may form four triangles and construct the Simson line of each triangle with respect to the fourth concyclic point. Thus we get thirty two Simson lines which coincide as shown above in four sets of eight, all passing through T.

The other sets of coincident Simson lines are

(2) $A(BCD)$, $O_2(CDO_1)$, $O_3(DO_1B)$, $O_4(O_1BC)$,
$O_1(O_2O_3O_4)$, $B(O_3O_4A)$, $C(O_4AO_2)$, $D(AO_2O_3)$.

(3) $B(CDA)$, $O_3(DAO_2)$, $O_4(AO_2C)$, $O_1(O_2CD)$.
$O_2(O_3O_4O_1)$ $C(O_4O_1B)$, $D(O_1BO_3)$, $A(BO_3O_4)$.

(4) $C(DAB)$, $O_4(ABO_3)$, $O_1(BO_3D)$, $O_2(O_3DA)$,
$O_3(O_4O_1O_2)$ $D(O_1O_2C)$, $A(O_2CO_4)$, $B(CO_4O_1)$.

Cor. 2. The nine-point circles of the thirty-two triangles named all pass through T; and they coincide in eight sets of four circles; for the nine-point circle of ABC is also the nine-point circle of

BCO_4, CO_4A, and O_4AB.

§ 15. *Figure* 23. Any line through T bisecting one of the sides of the quadrilateral ABCD is at right angles to the opposite side; and any line through T bisecting one diagonal is perpendicular to the other diagonal.

Since AO_4O_1D is a parallelogram, T the intersection of diagonals and X the middle point of one side; therefore XT is parallel to AO_4 and so perpendicular to BC.

Y is the middle point of DB, and T the middle point of DO_4; therefore YT is parallel to BO_4 and so perpendicular to AC.

Cor. If S be the centre of the circumscribed circle and W the middle point of BC; XW and ST bisect each other.

For XT and SW are both perpendicular to BC; and SX and TW are both perpendicular to AD. Therefore SWTX is a parallelogram, and ST, WX, the diagonals bisect each other. *Lady's and Gentleman's Diary* for 1865, p. 60.

§ 16. *Figure* 23. The point T is such that

$$TA^2+TB^2+TC^2+TD^2=[\text{Diameter of } ABC]^2.$$

Draw CJ at right angles to BC. BJ is a diameter. $CJ=AO_4$.

Because T is the middle point of DO_4; therefore

$$2(TA^2+TD^2)=DA^2+AO_4^2=DA^2+CJ^2$$
$$=DA^2+BJ^2-BC^2,$$

also $$2(TB^2+TA^2)=AB^2+BJ^2-CD^2,$$
$$2(TC^2+TB^2)=BC^2+BJ^2-DA^2,$$

and $$2(TD^2+TC^2)=CD^2+BJ^2-AB^2.$$

Hence $$TA^2+TB^2+TC^2+TD^2=BJ^2.$$

Educational Times Reprint, Vol. I. Question 1431.

§ 17. *Figure* 24. The angle between two Simson lines D′(ABC) and D″(ABC) is equal to half of ∠ D′SD″ where S is the centre of the circumscribed circle of ABC.

$$\angle T=\angle TA'B-\angle TA''B,$$
$$=\angle B'D'C-\angle B''D''C, \text{ (since } A'B'CD', A''B''CD''$$

are inscribable quadrilaterals),

$$=\angle D''CA-\angle D'CA,$$
$$=\angle D''CD'=\frac{\angle D''SD'}{2}.$$

Cor. 1. If BD′ and CD″ meet in M outside the circle; ∠T = ∠BAC − ∠M. If M be inside the circle, ∠T = ∠M − ∠A.

If M any point in the plane be joined to A,B,C, cutting the circle in D′, D″, D‴, and if the three Simson lines relative to the points D′, D″, D‴ be drawn; then,

$$\angle T=\pm(\angle A-\angle M),$$
$$\angle T'=\pm(\angle B-\angle M'),$$
$$\angle T''=\pm(\angle C-\angle M'');$$

$$\therefore\quad \angle T+\angle T'+\angle T''=\pm(\text{two right angles}-\overline{\angle M+\angle M'+\angle M''});$$
$$\therefore\quad \angle M+\angle M'+\angle M''\pm(\angle T+\angle T'+\angle T'')=\text{two right angles.}$$

Educational Times Reprint, Vol. IX. Question 2480 (Tucker).

Cor. 2. If E, F, G be the vertices of any triangle inscribed in the circle ABC, the Simson lines E(ABC), F(ABC), G(ABC) make a triangle similar to EFG.

Cor. 3. This theorem gives us a method of finding a point on the circumscribing circle whose Simson line with respect to the given triangle will be parallel to a given line.

§ 18. *Figure* 24. The Simson lines relative to two points diametrically opposite intersect at right angles on the nine-point circle.

If D′ and D″ be diametrically opposite then ∠ D′SD″ = a straight angle; and the angle between D′(ABC) and D″(ABC) = $\frac{\angle D'SD''}{2}$ (§ 17) = one right angle.

The Simson line D′(ABC) passes through P′ the middle point of D′O; D″(ABC) passes through P″.

Since O is a centre of similitude of the two circles, when D′D″ becomes a diameter of ABC, P′P″ becomes a diameter of the nine-point circle; and the two lines passing through P′ and P″ and containing a right angle must intersect on the nine-point circle.

Cor. Two Simson lines at right angles intercept on any side of the triangle ABC between their points of intersection with the side and the extremities of the side two equal segments. For the points of intersection are the projections of the ends of a diameter. *Journal de Mathématiques Spéciales*, 2nd Series, Vol. III. p. 16.

§ 19. If ABC, A′B′C′ be two triangles in a circle having the pairs of vertices AA′, BB′, CC′ diametrically opposite, the two Simson lines P(ABC), P(A′B′C′) are at right angles to each other.

Since the triangle A′B′C′ is derived from ABC by rotating it through two right angles, therefore AH and A′H′ or A(ABC), A′(A′B′C′) are parallel.

P(ABC) makes with A(ABC) the angle $\frac{PSA}{2}$. (§ 17)

P(A′B′C′) makes with A′(A′B′C′) the angle $\frac{PSA'}{2}$.

The angle between P(ABC) and P(A′B′C′) is therefore

$$\frac{\angle PSA' + \angle PSA}{2} = \frac{\angle ASA'}{2} = \text{a right angle.}$$

Cor. If P′ be the diametral point of P, then the four Simson lines P(ABC), P′(ABC), P(A′B′C′), P′(A′B′C′) form a rectangle, two of whose vertices are on the nine-point circles of ABC and A′B′C′. *Educational Times Reprint*, Vol. VI. Question 1939. (Tucker).

§ 20. *Figure* 24. Two Simson lines D′(ABC) and D″(ABC) intercept on the nine-point circle of the triangle ABC two arcs, one of which is double of the other.

Since O the orthocentre of the triangle ABC is the external centre of similitude of the circumscribed and the nine-point circles, the arc P′P″ is similar to the arc D′D″, and subtends an angle at Q″ equal to the angle in the segment D′BD″.

But ∠T between the Simson lines is equal to the angle in the segment D′BD″. Therefore ∠T is equal to ∠P′Q″P″; ∠Q′P′Q″ is double of ∠P′Q″P″; and the arc Q′Q″ is double of the arc P′P″. *Journal de Mathématiques Spéciales*, 2nd Series, Vol. III. p. 13.

Cor. *Figure* 25. If the arcs of the nine-point circle be measured in the positive direction we have

$$\begin{aligned} XY &= 2VH = 2XW = 2WY = 2KU, \\ YZ &= 2WK = 2YU = 2UZ = 2LV, \\ ZX &= 2UL = 2ZV = 2VX = 2HW, \\ YX &= 2UV = 2YL = 2LX = 2HK, \\ ZY &= 2VW = 2ZH = 2HY = 2KL, \\ XZ &= 2WU = 2XK = 2KZ = 2LH, \\ & \qquad XH = KY + LZ. \end{aligned}$$

For the sides and perpendiculars of a triangle are the Simson lines of the three vertices and the three diametrically opposite points respectively.

§ 21. *Figure* 23. The length of each segment of the line D(ABC) made by the sides of the triangle ABC is proportional to the distance of D from the vertex of the angle opposite the segment.

For triangles ACD and C′DA′ are similar;

therefore $\dfrac{DA'}{DC} = \dfrac{C'A'}{AC}$.

Triangles BDJ and DA′C are similar;

therefore $\dfrac{DA'}{DC} = \dfrac{BD}{BJ}$;

therefore $\dfrac{C'A'}{AC} = \dfrac{BD}{BJ}$; therefore $C'A' = \dfrac{AC}{2R} \cdot BD$.

Similarly $A'B' = \dfrac{BA}{2R} \cdot CD$.

$B'C' = \dfrac{CB}{2R} \cdot AD$.

[This theorem is a particular case of what is proved in § 2. (*b*), namely, $\dfrac{QS}{PB} = \dfrac{AC}{AK}$.]

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Cor. 1. If D coincide with A then C′A′ becomes the perpendicular from A on BC and $p = \frac{AC \cdot AB}{2R}$. Hence $2R \cdot p = AB \cdot AC$, a well known theorem.

Cor. 2. Since $A'B' + B'C' + C'A' = 0$; hence taking account of sign we get Ptolemy's theorem that

$$AC \cdot BD = AB\,CD + BC \cdot AD.$$

Cor. 3. If two points D and D′ be taken symmetrical with respect to the diameter through B one of the vertices of the triangle ABC; the segments opposite B of the Simson lines with respect to the two points are equal.

For the segments are $\frac{AC}{2R} \cdot BD$ and $\frac{AC}{2R} \cdot BD'$, and $BD = BD'$.

Cor 4. If ABCD be a quadrilateral in a circle, the segments of

	A(BCD) intercepted between			BC	and	CD,
	B(CDA)	,,	,,	CD	,,	DA,
	C(DAB)	,,	,,	DA	,,	AB,
and	D(ABC)	,,	,,	AB	,,	BC

are equal, for they are each equal to $\frac{AC \cdot BD}{2R}$.

A(BCD) and C(DAB) are equally inclined to BD,
B(CDA) and D(ABC) ,, ,, ,, ,, AC.

§ 22. If D and D′ be symmetrical points with respect to the centre then the sum of the squares of the segments of their Simson lines relative to one of the vertices of the triangle ABC is equal to the square of the side opposite that vertex.

$$\overline{D(ABC)}^2 + \overline{D'(ABC)}^2 = \left(\frac{AC \cdot BD}{2R}\right)^2 + \left(\frac{AC \cdot BD'}{2R}\right)^2$$
$$= \left(\frac{AC}{2R}\right)^2 (BD^2 + BD'^2) = \left(\frac{AC}{2R}\right)^2 \cdot (2R)^2$$
$$= AC^2.$$

Cor. The sum of the squares of the six segments of the Simson lines with regard to two centrally symmetrical points is equal to $AB^2 + BC^2 + CA^2$. §§ 21, 22, *Journal de Mathématiques Elémentaires* Vol. II. pp. 68-71 (Julliard).

§ 23. *Figure* 23. The projection of any side of a triangle on a Simson line is equal to the segment of the line intercepted by the other two sides.

Draw BK perpendicular to AP. AK is equal to the projection of AB on D(ABC).

The triangles ABK and DEC are similar. For the arc PC + arc BAD = arc BA + arc LP; take away the common parts, and arc AD = arc CL; add arc DL to each, and arc AL = arc CD; hence $\angle ABK = \angle DEC$.

But the triangles are right angled, and are therefore similar.

Therefore $$\frac{AK}{AB} = \frac{CD}{DE} = \frac{CD}{2R} = \frac{A'B'}{AB} \text{ (§ 21)};$$

and AK, the projection of AB on the Simson line, is equal to A′B′, the intercept between BC and CA.

§ 24. *Figure* 23. Since the perpendicular from B on D(ABC) cuts the circle in a point L so that LC = AD, we have the following theorem :—

If two points D, L be taken on the circle equidistant from A and C respectively, the line D(ABC) is perpendicular to LB and L(ABC) is perpendicular to DB.

Cor. When D is the middle point of AC, D(ABC) is at right angles to DB, the internal bisector of the angle ABC.

§ 25. *Figure* 26. ABCD is a square inscribed in a circle, and P a point on the circumference, the area of the quadrilateral formed by the Simson lines A(BCD), B(CDA), C(DAB), D(ABC) is equal to $\frac{1}{4}(PA \cdot PC + PB \cdot PD)$.

ABC and ADC are diametral triangles, and therefore (§ 19) their Simson lines with respect to P cut one another at right angles. For the same reason the Simson lines of ABD and CBD intersect at right angles. By the construction of the figure the vertices of the quadrilateral lie on the sides and diagonals of the square.

$$\text{(§ 21) } A'B' = \frac{AB}{2R} \cdot PC, \quad A'D' = \frac{CD}{2R} \cdot PA,$$

$$B'C' = \frac{CD}{2R} \cdot PB, \quad C'D' = \frac{AB}{2R} \cdot PD.$$

But since the quadrilateral A′B′C′D′ is made up of two right-angled triangles, its area is equal to

$$\frac{A'B' \cdot A'D' + C'B' \cdot C'D'}{2} = \frac{AB^2}{8R^2}(PA \cdot PC + PB \cdot PD),$$

$$= \tfrac{1}{4}(PA \cdot PC + PB \cdot PD).$$

Cor. If ABCD be any rectangle in a circle, the area of the

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quadrilateral formed by the four Simson lines of P, if P be opposite AB or CD, is equal to

$$\tfrac{1}{4}(PA\cdot PC+PB\cdot PD).\frac{AD^2}{2R^2}.$$

Educational Times, Vol. XXXVI. p. 152. Question 7373. (Edwardes.)

§ 26. *Figure* 25. Let XYZ, HKL be the pedal and medial triangles respectively of the triangle ABC; U, V, W, the middle points of AO, BO, CO, where O is the orthocentre of ABC. (1) The Simson lines of the vertices of each of the triangles XYZ, HKL with respect to the other are concurrent at the centre of gravity of the perimeter of XYZ. (2) The Simson lines of the points U, V, W with respect to the triangle XYZ form a triangle, which has for its centre of similitude with ABC the centre of gravity of XYZ, and for its orthocentre the centre of gravity of the perimeter of XYZ.

(1) The Simson line K(XYZ) passes through the middle of XZ, and is parallel to BY, the internal bisector of XYZ. (§§ 18, 24, Cor.) But lines drawn through the middle points of the sides of a triangle parallel to the internal bisectors of the opposite angles are concurrent at the centre of gravity of the perimeter of the triangle.

Y(HKL) is the parallel drawn through D the middle of BY to the line BS, where S is the circumscribed centre of ABC, and orthocentre of HKL. (§ 9.) Let T be its point of intersection with PT drawn from the middle point of YZ parallel to AX. The triangles DTP and BYZ are respectively similar to ZOY and XYO; $\frac{ZY}{YO}=\frac{DP}{PT}=\frac{ZB}{2PT}$; $\frac{YZ}{YO}=\frac{ZB}{OX}$. Hence XO = 2PT; therefore T is the centre of gravity of the perimeter of XYZ.

(2) U(XYZ), V(XYZ), W(XYZ) (§ 24, Cor.) are the parallels drawn through P, Q, R respectively to the sides BC, AC, AB; let M be the intersection of the two last. The line XP cuts QR in N; the line MN is parallel to AP and equal to $\frac{AP}{2}$. Then if MA cut NP in G, $NG=\frac{GP}{2}$; therefore G the centre of homology is the centre of gravity of XYZ.

T is the orthocentre of the triangle formed by these three Simson lines; because TM is parallel to AX.

Mathesis, Vol. V. p. 58 (Van Aubel); and *Educational Times*, Vol. 37, p. 380 (Tucker).

§ 27. *Figure* 25. The triangle formed by the three Simson lines U(XYZ), V(XYZ), W(XYZ) has its sides respectively parallel to and half of BC, CA, AB; so has the triangle UVW.

U, V, and W are the middle points of the arcs YZ, ZX, and XY. UX, VY, WZ are the internal bisectors of the angles of the triangle XYZ; and VW, WU, UV are at right angles to them respectively.

Hence the proposition—If XYZ be any triangle in a circle, U, V, W the middle points of the arcs cut off by the sides; the Simson lines U(XYZ), V(XYZ), W(XYZ) form a triangle whose sides are equal and parallel to those of the triangle UVW. *Educational Times*, Vol. XXXV. p. 289, Question 7186 (Tucker).

§ 28. *Figure* 23. Parabolas may be described such that A, B, C, D are their respective foci, and the sides of their corresponding triangles, tangents. The tangents at the vertices of these parabolas are concurrent, and their directrices pass through O_1, O_2, O_3, O_4 respectively.

For if D be the focus, and AB, BC, CA tangents, then the feet of the perpendiculars from D on the tangents lie on the tangent at the vertex; that is, the Simson line A′B′C′ is the tangent at the vertex of the parabola. The four Simson lines are the four tangents at the vertices, and they pass through T. (§ 12.)

Because DO_4 is drawn from the focus D, and is bisected by the tangent A′B′C′ (§ 9); therefore O_4 is on the directrix.

§ 29. *Figure* 22. If a parabola touch four lines, its focus is the point of intersection of the circumscribed circles of the four triangles formed by the four lines, and the Simson line of the complete quadrilateral is the tangent at the vertex of the parabola.

The axis of the parabola is parallel to the line joining the middle points of the diagonals of the quadrilateral. The directrix is the line of the orthocentres of the component triangles.

§ 30. (*a*) Two Simson lines at right angles to one another are the asymptotes of an equilateral hyperbola passing through the vertices of the triangle. By § 18, Cor.

(*b*) The locus of the centres of equilateral hyperbolas circumscribed to a triangle is the nine-point circle.

For the asymptotes, the two Simson lines at right angles to each other, intersect on the nine-point circle. (§ 18.)

(*c*) An equilateral hyperbola circumscribed to a triangle passes through its orthocentre.

For, by § 14, the asymptotes of the equilateral hyperbola circumscribed to ABC are also the asymptotes of that round BCO_4; but a hyperbola is determined if its asymptotes and one point on the curve be given; hence the proposition.

§ 31. *Figure* 24. The point where D′(ABC) touches its envelop is found by producing Q′P′ to T so that P′T = Q′P′.

For as D′ and D″ approach indefinitely near, the angle T is indefinitely diminished. P′Q′ and P″Q″ in the limit are parallel, and P′P″ and Q′Q″ are antiparallel.

In the triangles TP′P″, TQ′Q″ we have in the limit

$$\frac{TP'}{TQ''} = \frac{\text{chord } P'P''}{\text{chord } Q'Q''} = \frac{\text{arc } P'P''}{\text{arc } Q'Q''} = \tfrac{1}{2}.$$

But $\dfrac{TP'}{TQ''}$ in the limit $= \dfrac{TP'}{TQ'} = \tfrac{1}{2}$;

therefore TP′ = P′Q′, and T is the point where D′(ABC) touches its envelop.

§ 32. *Figure* 27. The envelop of the Simson line is a three-cusped hypocycloid.

Let QPT be the Simson line, T the point of contact with its envelop. Draw a circle PTR through T so as to touch the nine-point circle PQS at P. Since QP = PT the circles are of equal radius, and OR = 3OP. As T traces out its curve it moves at every instant in the direction of the tangent TQ that is perpendicular to RT, and so that TP = PQ, and R moves round the circle ARB. These conditions will be satisfied if T be a fixed point on the circle PTR rolling within the circle ARB which is of triple radius; the curve thus traced is a hypocycloid of three cusps. §§ 30, 31, 32, *Journal de Mathématiques Spéciales,* 2nd Series, Vol. III. pp. 14, 15, 31 (Weill).

§ 33. *Figure* 28. The following are some of the properties of the three-cusped hypocycloid deduced by considering it as the envelop of the Simson line.

(1) If T_1T_2 be a tangent at T, meeting the curve again in T_1 and T_2; T_1T_2 is of constant length, and tangents at T_1 and T intersect at right angles on the inscribed circle.

For if Q_1T_1 and Q_1T_2 be constructed Simson lines at right angles they intersect at Q_1 on the nine-point circle of the triangle of reference (the inscribed circle of the hypocycloid); and since they meet their envelop at points such that $Q_1P_1 = P_1T_1$ and $Q_1P_2 = P_2T_2$, therefore T_1T_2 is double of P_1P_2, that is double of the diameter of the inscribed circle.

Since the sides and perpendiculars of a triangle are Simson lines with respect to that triangle, therefore T_1T_2 is a Simson line and touches the hypocycloid; and QQ_1, which is the perpendicular at Q, is also a tangent to the curve. (For the inscribed circle is the nine-point circle of the triangle $T_1T_2Q_1$.)

(2) P is the middle point of T_1T_2. Therefore any tangent to a hypocycloid meets the inscribed circle in two points, one of which bisects the tangent; and if a perpendicular be drawn to the tangent at the other point, the perpendicular is also a tangent to the hypocycloid.

(3) Since P is the middle point of TQ, therefore $QT_1 = TT_2$. That is, the part of the tangent intercepted between one of the branches of the curve and the inscribed circle is equal to the part between the other two branches.

(4) If SS_1 be drawn a tangent to the inscribed circle at Q_1, the angles QQ_1S, QQ_1S_1 are bisected by the tangents to the hypocycloid Q_1P_2 and Q_1P_1.

For since the triangle QQ_1T_2 is right angled, and P_2 is the middle point of the hypotenuse, therefore $\angle P_2QQ_1$ is equal to $\angle P_2Q_1Q$. But $\angle P_2QQ_1$ is the angle in the alternate segment, and is therefore equal to $\angle P_2Q_1S$. Since $\angle P_2Q_1P_1$ is a right angle, therefore $\angle QQ_1S_1$ is bisected by Q_1P_1.