ON THE GROUP RING

IAN G. CONNELL

Let R be the discrete group ring of the group G over the ring A. In this paper we attempt to find necessary and sufficient conditions on G and A so that R will have some standard ring-theoretic property; among the properties considered are those of being artinian, regular, self-injective, and semi-prime.

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1. Notation. Throughout this paper *ring* will mean associative ring with $1 \neq 0$, and a *subring* S of the ring T will always contain the unit element of T. All modules will be unitary and will be assumed to be right modules unless otherwise stated.

If A is a ring and G a group, $R = R(G, A)$ denotes the group ring. Thus a typical element of R is a finite formal sum

$$r = \sum r(g)g, \qquad r(g) \in A, g \in G.$$

Letting the elements of A commute with those of G, addition and multiplication are defined in R in the obvious way, making R a ring; A is a subring of R under the identification $a = a \cdot 1$. A, G, and R will have these fixed meanings throughout; a, g, and r will always denote elements of A, G, and R respectively.

Z denotes the ring of rational integers and C the centre of G. Δ, δ, ω, Ω are defined in Section 2; $o(G)$ and $\nu(G)$ in Section 4.

2. Preliminaries. In this section we collect some simple facts which will be useful later. A number of these have been observed by various authors, for example by Deskins **(6)**.

It is easily verified that $R(G, A)$ is a functor covariant in both variables.

$\mathfrak{L}(G)$ denotes the lattice of subgroups of G, and $\mathfrak{L}_r(S)$ the lattice of right ideals of the ring S. We define

$$\omega: \mathfrak{L}(G) \to \mathfrak{L}_r(R)$$

by letting ωH be the right ideal generated by the set $\{1 - g: g \in H\}$, and we define

$$\Omega: \mathfrak{L}_r(R) \to \mathfrak{L}(G)$$

by

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$$\Omega J = \{g : 1 - g \in J\}.$$

(We prove below that $\Omega J \in \mathfrak{L}(G)$.)

The left annihilator of the subset X of the ring S is denoted by

$$X^l = \{s \in S : sX = 0\}.$$

The right annihilator X^r is defined similarly. X^l (resp. X^r) is a left (right) ideal.

$\omega G = \Delta$ is called the *fundamental* (or *augmentation*) *ideal of* R. The *norm* of

$$r = \sum r(g) g$$

is defined to be

$$\delta(r) = \sum r(g) \in A.$$

The *trace* of r is $r(1)$.

PROPOSITION 1. *$1 - g \in \omega H$ if and only if $g \in H$.*

If the set $\{g_i\}$ generates the subgroup H, then the right ideal generated by $\{1 - g_i\}$ is ωH.

$(\omega H)^l \neq 0$ if and only if H is finite. ω is faithful, isotone, and preserves joins.

$$(1) \qquad \omega(H \cap K) \subseteq \omega H \cap \omega K.$$

ωH is an ideal if and only if H is a normal subgroup, and then

$$(2) \qquad R(G/H, A) \approx R(G, A)/\omega H,$$

where $\approx$ denotes canonical ring isomorphism.

If $J \in \mathfrak{L}_r(R)$, then $\Omega J \in \mathfrak{L}(G)$ and is normal if J is an ideal. Ω is onto, isotone, and preserves meets.

$$(3) \qquad \Omega(I \cup J) \supseteq \Omega I \cup \Omega J;$$

$$(4) \qquad \Omega\omega H = H,$$

$$(5) \qquad \omega\Omega J \subseteq \Delta \cap J,$$

$$(6) \qquad \Omega J = \Omega(\Delta \cap J),$$

$$(7) \qquad R/\Delta \approx A.$$

$\{1 - g : g \in G\}$ is an A-module basis for Δ. $r \in \Delta$ if and only if $\delta(r) = 0$.

Remark. Inequality may occur in (1), (3), and (5). For (1) and (5) take G to be the four-group and A the two-element field; for (3) take G to be the two-element group and A the integers (mod 4).

Proof. Whether H is normal or not, let $R(G/H, A)$ denote the free left A-module generated by the right cosets of H. Thus a typical element of $R(G/H, A)$ is a finite sum of the form

$$\sum r(g) Hg, \qquad r(g) \in A.$$

(If H is normal, then we can define a multiplication on $R(G/H, A)$ so that it becomes the group ring of G/H.) Define

$$\phi\colon R(G, A) \to R(G/H, A)$$

by

$$\phi \sum r(g) g = \sum r(g) Hg.$$

Clearly ϕ is an A-homomorphism; we wish to show that $\operatorname{Ker} \phi = \omega H$. Because of the linearity of ϕ to show that $\omega H \subseteq \operatorname{Ker} \phi$ it is enough to show that $\phi((1 - h)g) = 0$, where $h \in H$ and $g \in G$; but this is clear, for

$$\phi((1 - h)g) = Hg - Hhg = Hg - Hg = 0.$$

Conversely, $\phi r = 0$ if and only if

$$\sum r(g) Hg = 0,$$

i.e.,

$$\sum_{g \in K} \left\{ \sum_{h \in H} r(hg) \right\} Hg = 0,$$

where K denotes a class of coset representatives, i.e.,

$$\sum_{h \in H} r(hg) = 0, \qquad \text{for all } g \in K.$$

Thus $r \in \operatorname{Ker} \phi$ implies

$$\begin{aligned} r &= \sum_{g \in K} \sum_{h \in H} r(hg) hg \\ &= \sum_{g \in K} \sum_{h \in H} \{ r(hg) hg - r(hg) g \} \\ &= \sum_{g \in K} \sum_{h \in H} (1 - h)(-r(hg) g) \in \omega H, \end{aligned}$$

and therefore $\operatorname{Ker} \phi = \omega H$.

If $g \in H$, then $1 - g \in \omega H$ by definition. Conversely, if $g \notin H$, then $\phi(1 - g) = H - Hg \neq 0$ so that $1 - g \notin \operatorname{Ker} \phi = \omega H$. Thus if $H_1 \neq H_2$, then $\omega H_1 \neq \omega H_2$, that is, ω is faithful. Clearly ω is isotone.

If $\{g_i\}$ generates H, let J denote the right ideal generated by $\{1 - g_i\}$; clearly $J \subseteq \omega H$. Conversely it is sufficient to show that $1 - h \in J$ for each $h \in H$. Now h is a group word in the g_i and we proceed by induction on the length t of the word. First it is true for words of length 1: $1 - g_i \in J$ and

$$1 - g_i^{-1} = (1 - g_i)(-g_i^{-1}) \in J.$$

Now suppose it is true for words h of length t; any word of length $t + 1$ is of the form hg_i or hg_i^{-1}, and

$$1 - hg_i^{\pm 1} = (1 - h) g_i^{\pm 1} + (1 - g_i^{\pm 1}) \in J$$

since $1 - h$ and $1 - g_i^{\pm 1}$ are in J.

Next, ω preserves joins. For if $\{h_i\}$ generates H and $\{k_j\}$ generates K, then $\{h_i\} \cup \{k_j\}$ generates $H \cup K$. Thus $\omega(H \cup K)$ is generated by $\{1 - h_i\} \cup \{1 - k_j\}$ and therefore coincides with $\omega H \cup \omega K$.

If $H = \{h_1, \ldots, h_n\}$, then

$$(h_1 + \ldots + h_n)(1 - h_i) = 0$$

for each i and therefore

$$h_1 + \ldots + h_n \in (\omega H)^l.$$

Conversely let

$$a_1 g_1 + \ldots + a_k g_k \in (\omega H)^l,$$

and suppose H infinite. Then we can choose h so that $g_1 h$ is distinct from all the g_i and then

$$(a_1 g_1 + \ldots + a_k g_k)(1 - h) \neq 0,$$

a contradiction; hence H must be finite.

Inequality (1) is obvious.

Now suppose H is a normal subgroup. We wish to prove that ωH is also a left ideal. First we observe the simple fact:

LEMMA. *To verify that a subset J of R is a left ideal it is sufficient to show that J is closed under* (i) *addition, and* (ii) *multiplication on the left by any a and by any g* (with the obvious modifications for right and two-sided ideals).

The present case clearly follows from $a(1 - h) = (1 - h)a \in J$ and

$$g(1 - h) = (1 - ghg^{-1})g = (1 - h')g \in J.$$

Conversely if ωH is an ideal and if $h \in H$, then $1 - h^{-1} \in \omega H$ and

$$g^{-1}h(1 - h^{-1})(-g) = 1 - g^{-1}hg \in \omega H.$$

Hence $g^{-1}hg \in H$ and H is normal.

If H is normal, then $R(G/H, A)$ is the group ring of G/H and the mapping ϕ defined above, shown to be an A-module homomorphism with kernel ωH, is now actually a ring homomorphism as is easily verified (where $\phi a = a$ for all a; in other words, ϕ is an A-algebra homomorphism). Since ϕ is clearly onto, we have the isomorphism (2).

Now ΩJ is a subgroup, for $1 \in \Omega J$ and

$$\begin{aligned} g, g' \in \Omega J &\Rightarrow 1 - g, 1 - g' \in J \\ &\Rightarrow 1 - g^{-1}g' = (1 - g)(-g^{-1}g') + (1 - g') \in J \\ &\Rightarrow g^{-1}g' \in \Omega J. \end{aligned}$$

If J is an ideal, then ΩJ is normal; for if $h \in \Omega J$ and $g \in G$, then $h = 1 - j$, $j \in J$ so that $g^{-1}hg = 1 - g^{-1}jg = 1 - j'$, $j' \in J$, and therefore $g^{-1}hg \in \Omega J$.

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Now

$$g \in \Omega\omega H \Leftrightarrow 1 - g \in \omega H \Leftrightarrow g \in H,$$

hence $\Omega\omega H = H$. Thus Ω is onto; clearly it is isotone and preserves meets.

$\Omega I \cup \Omega J$ is the smallest subgroup containing all g for which $1 - g$ is in either I or J; but $\Omega(I \cup J)$ is a subgroup containing all such g, and inequality (3) follows.

Since $\omega\Omega J =$ right ideal generated by $\{1 - g : g \in \Omega J\}$ = right ideal generated by $\{1 - g : 1 - g \in J\} \subseteq J$, and since $\omega H \subseteq \Delta$ for any subgroup H, we have (5).

Statement (6) follows from the fact that $\{1 - g\} \subseteq \Delta$.

To prove (7) we observe that

$$R/\Delta = R(G, A)/\omega G \approx R(G/G, A) \approx A.$$

Notice that in the canonical *norm epimorphism* $\delta : R \to A$ so defined, an element of R is sent onto its norm.

Using $(1 - g)g' = (1 - gg') - (1 - g)$, one sees immediately that the A submodule generated by $\{1 - g\}$ is Δ. Finally, the element

$$\sum r(g)g = \delta(r) - \sum r(g)(1 - g)$$

is in Δ if and only if $\delta(r) = 0$.

We define

$$\omega_1 : \mathfrak{L}_r(A) \to \mathfrak{L}_r(R)$$

and

$$\Omega_1 : \mathfrak{L}_r(R) \to \mathfrak{L}_r(A)$$

by $\omega_1 J = JR$ (that is, $\omega_1 J$ is the right ideal generated by the subset J of R), and $\Omega_1 J = J \cap A$. Clearly $\Omega_1 J \in \mathfrak{L}_r(A)$.

We recall that an ideal J of a ring S is *prime* if $KL \subseteq J$, where K and L are ideals, implies $K \subseteq J$ or $L \subseteq J$. Equivalently, J is prime if $xSy \subseteq J$ implies $x \in J$ or $y \in J$. (In the commutative case the middle factor S may be omitted.)

PROPOSITION 2.

$$\omega_1 J = \{r \in R : \text{for all } g \in G, r(g) \in J\}. \tag{8}$$

ω_1 is a lattice monomorphism; moreover ω_1 preserves products. $\omega_1 J$ is an ideal if and only if J is an ideal and then

$$R(G, A/J) \approx R(G, A)/\omega_1 J. \tag{9}$$

Ω_1 is onto, isotone, and preserves (arbitrary) meets. Moreover if J is a prime ideal, then so is $\Omega_1 J$. We have

$$\Omega_1(J \cup K) \supseteq \Omega_1 J \cup \Omega_1 K, \tag{10}$$

$$\Omega_1(JK) \supseteq \Omega_1 J \cdot \Omega_1 K, \tag{11}$$

$$\Omega_1\omega_1 J = J, \tag{12}$$

$$\omega_1\Omega_1 J \subseteq J. \tag{13}$$

Proof. Most of the statements of the proposition are trivial, and we only indicate a few of the details of the proof.

It follows readily from (8) that ω_1 is faithful and preserves meets, joins, and products.

By (8) it is clear that if J is an ideal, then so is $\omega_1 J$; conversely if $\omega_1 J$ is an ideal, $a \in A$ and $j \in J$, we wish to show that $aj \in J$. But $aj \in \omega_1 J$ and, again by (8), $aj \in J$.

If $\phi: A \to A/J$ is the natural map for any ideal J in A, define $\phi': R(G, A) \to R(G, A/J)$ by

$$\phi'\sum r(g)g = \sum\phi(r(g))g. \tag{14}$$

Then ϕ' is a ring epimorphism and

$$\begin{aligned}\operatorname{Ker}\phi' &= \{r \in R(G, A): \phi(r(g)) = 0, \text{ for all } g\}\\ &= \{r: r(g) \in J, \text{ for all } g\} = \omega_1 J,\end{aligned}$$

and (9) follows.

Finally we prove that if J is a prime ideal in R, then $\Omega_1 J$ is a prime ideal in A, the proofs of all the remaining statements being easy. Suppose that H and K are ideals in A such that $HK \subseteq \Omega_1 J$. Now

$$\omega_1(HK) = \omega_1 H\cdot\omega_1 K \subseteq \omega_1\Omega_1 J \subseteq J.$$

Since $\omega_1 H$ and $\omega_1 K$ are ideals and J is prime, one is contained in J, say $\omega_1 H \subseteq J$. Applying Ω_1,

$$\Omega_1\omega_1 H = H \subseteq \Omega_1 J,$$

as required.

The preceding proposition can be generalized as follows. If H is a subgroup, then $R' = R(H, A)$ is a subring of $R = R(G, A)$ and we define

$$\begin{aligned}\omega_H&: \mathfrak{L}_r(R') \to \mathfrak{L}_r(R),\\ \Omega_H&: \mathfrak{L}_r(R) \to \mathfrak{L}_r(R'),\end{aligned}$$

by $\omega_H J = JR$ and $\Omega_H J = J \cap R'$. Again ω_H is a lattice monomorphism. If $H \subseteq C$, then ω_H preserves products, $\omega_H J$ is an ideal if and only if J is an ideal, and the analogues of (10)–(13) are true. We thus obtain the following proposition.

PROPOSITION 3. *If H is central and J is a prime ideal in $R(G, A)$, then $\Omega_H J$ is a prime ideal in $R(H, A)$.*

It is also easy to prove this by direct calculation.

The *support* of r is defined by

$$\mathrm{Supp}(r) = \{g : r(g) \neq 0\},$$

and the *support group*, S.G.(r), is the subgroup generated by Supp(r).

PROPOSITION 4. (i) *If $rx = 1$ has a solution x, then it has a solution y for which*

$$\mathrm{S.G.}(y) \subseteq \mathrm{S.G.}(r).$$

(ii) *The left and right annihilator ideals of Δ coincide and are given by*

$$\Delta^* = \begin{cases} 0 \text{ if } G \text{ is infinite,} \\ A(g_1 + \ldots + g_n) \text{ if } G = \{g_1, \ldots, g_n\}. \end{cases}$$

In the latter case

$$\Delta \cap \Delta^* = \{a(g_1 + \ldots + g_n) : na = 0\}.$$

(iii) *Δ is a direct summand, that is, $R = \Delta \oplus J$ where J is a (necessarily two-sided) ideal, if and only if*

(a) *G is finite, say of order n, and*

(b) *n is a unit in A,*

and then $J = \Delta^*$, and A and Δ^* are isomorphic as rings.

Proof. (i) As Amitsur points out in **(1)**, one need merely put

$$y(g) = \begin{cases} x(g) & \text{if } g \in \mathrm{S.G.}(r), \\ 0 & \text{otherwise} \end{cases}$$

and the result follows by a simple calculation.

(ii) By Proposition 1, $\Delta^l = (\omega G)^l \neq 0$ implies G finite; similarly $\Delta^r \neq 0$ implies G finite.

Conversely let $G = \{1, g_2, \ldots, g_n\}$ and $r = a + a_2 g_2 + \ldots + a_n g_n \in \Delta^l$. Then

$$r \cdot (1 - g_i) = (a_i - a)g_i + \ldots = 0$$

so that $a_i = a$ and $r = a(g_1 + \ldots + g_n)$. And since $\{g_i g_j\}$ is a permutation of $\{g_i\}$ any such r annihilates each $1 - g_j$, and therefore is in Δ^l. By symmetry we thus have $\Delta^l = \Delta^* = \Delta^r$. Also $r \in \Delta$ if and only if $\delta(r) = na = 0$.

(iii) In general if $K \in \mathfrak{L}_r(S)$, there is a one-to-one correspondence between the left identity elements e of $K = eK = eS$ and the complementary summands $(1 - e)S$. If K is an ideal and a direct summand, there is just one e (a two-sided identity in K) and $(1 - e)S = K^r = K^l$ is also an ideal. Thus if $R = \Delta \oplus J$, we have $J = \Delta^*$, and since $J \neq 0$, G must be finite, say

$$G = \{g_1 = 1, g_2, \ldots, g_n\}.$$

Now

$$1 = x + j, \qquad x \in \Delta, \quad \in J,$$
$$= \sum_{i=2}^{n} a_i(1 - g_i) + a_1(g_1 + \ldots + g_n)$$
$$= \sum_{i=1}^{n} a_i + \sum_{i=2}^{n} (a_1 - a_i)g_i$$

and therefore $a_i = a_1$ for all i and $1 = na_1$, as required.

Conversely, since n is a unit, $\Delta \cap \Delta^* = 0$. Also

$$r = \sum r(g)g = na - \sum r(g)(1 - g), \qquad \text{where } na = \delta(r),$$
$$= \sum (a - r(g))(1 - g) + \sum ag \in \Delta + \Delta^*.$$

Hence $R = \Delta \oplus \Delta^*$, and since the complementary summand is uniquely determined, $J = \Delta^*$. If $nx = 1$, the correspondence $a \leftrightarrow xa(g_1 + \ldots + g_n)$ gives $A \approx \Delta^*$.

By the remarks on direct summands above and Proposition 1, we have immediately the following proposition.

PROPOSITION 5. *If ωH is a direct summand of R, then H is finite.*

3. Chain conditions. A ring is *artinian* (*noetherian*) if it has the minimum (maximum) condition on right ideals. An artinian ring is noetherian. Either property is inherited by factors.

THEOREM 1. *R is artinian if and only if A is artinian and G is finite.*

Proof. First let A be artinian and G finite. Then R is an A-module direct sum of a finite number of copies of A and therefore (**4**, p. 22) is an artinian A-module, a fortiori an artinian R-module, i.e., an artinian ring.

The proof of the converse must be deferred until after the proof of Theorem 5. (We do not use this result elsewhere in the paper.) However, it is interesting to see how far we can get on the basis of Proposition 1.

Since R is artinian, $A = R/\Delta$ is artinian, and since R is also noetherian a chain of subgroups

$$\ldots \subset H_n \subset H_{n+1} \subset \ldots$$

must be finite in both directions; otherwise

$$\ldots \subset \omega H_n \subset \omega H_{n+1} \subset \ldots$$

would violate a chain condition in R. Hence G has both the maximum and minimum conditions on subgroups, and it is an unsolved problem whether this implies that G is finite. Birkhoff (**3**, Problem 43) attributes the problem to Kaplansky, and Suzuki (**18**, p. 22) to Schur. We do not resolve the problem; we circumvent it.

THEOREM 2. (a) *R is noetherian if A is noetherian and G is finite.*

(b) Partial converse: *If R is noetherian, then A is noetherian and G has the maximum condition on subgroups.*

(c) *If G is abelian, then R is noetherian if and only if A is noetherian and G is finitely generated.*

Proof. (a) R is an A-module direct sum of a finite number of copies of A and therefore is a noetherian A-module, hence a noetherian ring.

(b) (By "partial converse" we do not mean to imply that it is necessarily this half which is susceptible of improvement.) $A = R/\Delta$ is noetherian. An ascending sequence of subgroups $H_1 \subset H_2 \subset \ldots$ gives rise to the chain $\omega H_1 \subset \omega H_2 \subset \ldots$ and so must terminate.

(c) G is a factor of a free abelian group F in a finite number of generators, say $x_1, \ldots, x_n$. Since $R(G, A)$ is a factor of $R(F, A)$, it is enough to prove the latter noetherian. Now regarding $x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}$ as $2n$ indeterminates, $R(F, A)$ is a factor of P, the polynomial ring in $2n$ indeterminates over A. (The indeterminates are meant to commute among themselves and with the elements of A.) Thus the result follows from the Hilbert basis theorem (**8**, p. 171):

LEMMA. *Let S be a noetherian ring and let $P = S[x_1, \ldots, x_n]$ be the polynomial ring over S in n commuting indeterminates. Then P is noetherian.*

We note that when A is noetherian and G is abelian, G finitely generated implies R noetherian, which in turn implies that G has the maximum condition on subgroups. Thus as a corollary we obtain the well-known result that in a finitely generated abelian group every subgroup is finitely generated. The corresponding statement for non-abelian groups is not true, and for this reason the lemma becomes false if we delete the word "commuting."

4. Complete reducibility. A ring S is *regular* (in the sense of von Neumann) if for every $x \in S$ there exists a $y \in S$ such that $xyx = x$. We say S is *completely reducible* (semi-simple in the sense of Bourbaki) if it satisfies any one of the many well-known equivalent conditions of which we mention the following (the notion semi-prime is defined in Section 6):

(1) S is the direct product of a finite number of full matrix rings over skew fields;
(2) S is regular and noetherian;
(3) S is semi-prime and artinian;
(4) every right ideal in S is a direct summand;
(5) every S-module is projective;
(6) every S-module is injective.

An element of a ring is *regular* if it is not a divisor of zero; we shall use this term only for elements in the centre so that the problem of distinguishing left and right divisors of zero will not arise.

We let $o(G)$ denote the set of orders of all finite subgroups in G. For example if G is torsion-free, $o(G) = \{1\}$. In Section 6 we shall require $\nu(G)$, which is the set of orders of all finite normal subgroups. For example, if G is a simple infinite group, then $\nu(G) = \{1\}$. Clearly, $\nu(G) \subseteq o(G)$ with equality if G is abelian. $o(G)$ will be used in the following contexts: *each $n \in o(G)$ is regular (a unit) in A*. These conditions are equivalent to, respectively: *each finite group element order is regular (a unit) in A*. If G is finite, say of order n, then G has an element of order p for each prime divisor p of n, and the conditions are equivalent to: *n is regular (a unit) in A.*

We require a preliminary result.

PROPOSITION 6. *The following statements are equivalent:*
(i) *$1 - g$ is a one-sided divisor of 0;*
(ii) *$1 - g$ is a divisor of 0;*
(iii) *g has finite order.*
When this is so, say g has order n,

$$\{1 - g\}^l = R(1 + g + \ldots + g^{n-1}),$$
$$\{1 - g\}^r = (1 + g + \ldots + g^{n-1})R.$$

Proof. If $g^n = 1$, then

$$(1 - g)(1 + g + \ldots + g^{n-1})r = 0,$$
$$r(1 + g + \ldots + g^{n-1})(1 - g) = 0$$

for all $r \in R$. Conversely suppose $(1 - g)(a_1g_1 + \ldots + a_kg_k) = 0$. Then in order that cancellation take place, the sequence $\{gg_i\}$ must be a permutation of the sequence $\{g_i\}$. Thus renumbering the g_i if necessary, $gg_1 = g_2$, $gg_2 = g_3, \ldots, gg_{n-1} = g_n$, $gg_n = g_1$ for some n. Hence $g^2g_n = g_2$, $g^3g_n = g_3, \ldots,$ $g^ng_n = g_n$, or $g^n = 1$ and g has finite order. In fact, since $g_1, \ldots, g_n$ are supposed distinct, g has order n. Thus, factoring the permutation into disjoint cycles, each cycle has length n, a typical one being $(h_i, gh_i, \ldots, g^{n-1}h_i)$, $h_i \in G$, and the elements g^jh_i are all distinct. We may write

$$a_1g_1 + \ldots + a_kg_k = (a_{10} + a_{11}g + \ldots + a_{1,n-1}g^{n-1})h_1 + \ldots + (a_{m0} + \ldots + a_{m,n-1}g^{n-1})h_m,$$

where $nm = k$ and the a_{ij} are the a_i rearranged. Multiplying on the left by $1 - g$, noting that each power of g must have coefficient 0, we have

$$\sum_{i=1}^{m} (a_{ij} - a_{i,j-1})h_i = 0, \qquad 1 \leqslant j \leqslant n,$$

where $a_{in} = a_{i0}$, and since the h_i are distinct, $a_{ij} = a_{i,j-1}$. Thus

$$a_1g_1 + \ldots + a_kg_k = (1 + g + \ldots + g^{n-1})(a_{10}h_1 + a_{20}h_2 + \ldots + a_{m0}h_m) \in (1 + g + \ldots + g^{n-1})R.$$

Clearly a similar argument works for a left annihilator of $1 - g$.

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A group is *locally finite* if every finite subset generates a finite subgroup.

THEOREM 3. *R is regular if and only if*

(i) *A is regular,*

(ii) *G is locally finite, and*

(iii) *each $n \in o(G)$ is a unit in A.*

Remarks. (a) The result that (i)–(iii) imply R regular has been obtained by Auslander **(2)** and McLaughlin **(16)**, and we shall omit this part of the proof. In the other direction they proved (i) and (iii) but only that G is torsion; since the Burnside problem has been answered in the negative, this is definitely weaker than G being locally finite.

(b) In proving that (i)–(iii) imply R regular it is enough to assume that G is finite, for if R is regular "locally", then it is regular. We mean by this the following: *if $R(H, A)$ is regular for each finitely generated subgroup H of G, then $R(G, A)$ is regular.*

Other properties thus "reducible to the local case" are that of being semi-prime and semi-primitive. (See the corollary to Proposition 9 below.)

Proof. Let R be regular. Since a factor of a regular ring is regular, $A = R/\Delta$ is regular.

It is a standard result that in a regular ring every finitely generated right ideal is a direct summand. Thus if $\{g_1, \ldots, g_n\}$ generates the subgroup H, $\{1 - g_i\}$ generates the right ideal ωH (by Proposition 1), and ωH is a direct summand. By Proposition 5, H is finite and therefore G is locally finite.

Finally if g has order n we must show that n is a unit in A. There exists an $r \in R$ such that $(1 - g)r(1 - g) = 1 - g$, or $(1 - g)(1 - r(1 - g)) = 0$. By Proposition 6, $1 - r(1 - g) = (1 + g + \ldots + g^{n-1})r'$, for some $r' \in R$, and applying the norm epimorphism to this equation, in which all group elements are mapped onto 1, we obtain $1 = n\delta(r')$, where $\delta(r') \in A$, as required.

COROLLARY (the generalized Maschke theorem). *R is completely reducible if and only if*

(i) *G is finite,*

(ii) *A is completely reducible, and*

(iii) *the order of G is a unit in A.*

Proof. This follows from the above theorem and (the trivial parts of) Theorem 2. Thus if R is completely reducible it is noetherian and regular, hence A is noetherian and regular and therefore completely reducible. Also G has the maximum condition on subgroups and in particular is finitely generated, which together with local finiteness shows that G is finite. (iii) follows from Theorem 3.

The converse is also immediate.

Because of the independent interest of the corollary we give an alternative proof. First let R be completely reducible. This property is inherited by factors and therefore A is completely reducible. Every right ideal of R, in particular Δ, is a direct summand, and the result follows by Proposition 4.

The finiteness of G can also be seen as follows. $\Delta = \Delta_R$ is R-injective and the identity map $\epsilon: \Delta \to \Delta$ is extendible to the R-homomorphism $\epsilon': R \to \Delta$. Putting $\epsilon'(1) = d$, we see that $\epsilon'(r) = dr$ for all $r \in R$. In particular, since $1 - g \in \Delta$, $d(1 - g) = 1 - g$. Thus $(d - 1)(1 - g) = 0$ for all $g \in G$ and $d \neq 1$ since $d \in \Delta$. Putting $d' = g'(d - 1)$ for an appropriate $g' \in G$ we have

$$d'(1 - g) = 0, \qquad d'(1) \neq 0,$$

for all $g \in G$. Now $d'(1 - g)$ involves a term in g arising from $-d'(1)g$ and this must be cancelled by a term from $d' \cdot 1$. Hence $d'(g) \neq 0$ and since this is true for all $g \in G$, G is finite.

Conversely let (i)–(iii) of the corollary hold. Our proof will follow classical lines but will not introduce the notion of a representation.

Let J be a right ideal of R; we must prove that J is a direct summand of R. Now A is completely reducible and therefore every A-module, in particular $J = J_A$, is injective, and since an injective module is a direct summand of any containing module, we have $R = J \oplus K$, where K is an A-submodule of R, though not necessarily a right ideal. Thus for all $r \in R$, $r = \alpha(r) + \beta(r)$, where $\alpha(r) \in J$ and $\beta(r) \in K$ are uniquely determined by r. Clearly $\beta \in \mathrm{Hom}_A(R, K)$. Now define $\gamma: R \to R$ by

$$\gamma(r) = \frac{1}{n} \sum_{g \in G} \beta(r \cdot g) g^{-1},$$

where n is the order of G. Calculation shows that $\gamma \in \mathrm{Hom}_R(R, R)$ so that $J' = \gamma(R)$ is a right ideal in R. We shall prove that $R = J \oplus J'$, thus showing that R is completely reducible.

We may write $r = (r - \gamma(r)) + \gamma(r)$; now

$$r - \gamma(r) = \frac{1}{n} \sum_{g \in G} (rg - \beta(rg)) g^{-1},$$

but $rg - \beta(rg) = \alpha(rg) \in J$. Hence $r - \gamma(r) \in J$, and since $\gamma(r) \in J'$ we have $R = J + J'$. It remains to prove that $J \cap J' = 0$. If $j \in J$, then $jg \in J$, $\beta(jg) = 0$, and $\gamma(j) = 0$; thus for any $r \in R$, $\gamma(r - \gamma(r)) = 0$, i.e., $\gamma^2(r) = \gamma(r)$. Suppose $j \in J \cap J'$; then since $j \in J'$, $j = \gamma(r)$ for some r and therefore $j = \gamma(r) = \gamma^2(r) = \gamma(j) = 0$, as required, since $j \in J$ and therefore $\gamma(j) = 0$.

5. Self-injectivity. Let G be a finite group of order n and let A be a field whose characteristic does not divide n. Then the Maschke theorem says that R is completely reducible, and therefore every R-module is injective. In particular R is R-injective, that is, R is self-injective. Dr. Lambek pointed out to me that the latter result is true even if the characteristic does divide n. We shall obtain below (Theorem 4) a comprehensive generalization of this fact after some preliminary results.

Since A is a subring of R, an R-module M is canonically an A-module by restriction of the operator ring.

PROPOSITION 7. *Let M be an R-module and let $\phi \in \mathrm{Hom}_R(M, R)$. Define $t(\phi) = \tilde{\phi}$ by*

$$\tilde{\phi}(m) = \phi(m)(1), \qquad \textit{for all} \quad m \in M,$$

i.e., $\tilde{\phi}(m)$ is the trace of $\phi(m)$. Then

$$\tilde{\phi} \in \mathrm{Hom}_A(M, A)$$

and

$$t\colon \mathrm{Hom}_R(M, R) \to \mathrm{Hom}_A(M, A)$$

is an abelian group monomorphism; when G is finite it is an isomorphism.

Proof. A simple computation shows that $\tilde{\phi} \in \mathrm{Hom}_A(M, A)$. Now let g, h, k represent elements of G and let

$$r = \sum r(g)g^{-1} \in R.$$

(It will be convenient to let $r(g)$ denote the coefficient of g^{-1} rather than g in the course of this proof.) Then

$$rh = \sum r(g)g^{-1}h = \sum r(hk)k^{-1},$$

where $k^{-1} = g^{-1}h$, $g = hk$, and we have the formula

$$(rh)(k) = r(hk). \tag{15}$$

Next,

$$\begin{aligned} \tilde{\phi}(mg) &= (\phi(mg))(1) \\ &= ((\phi m)g)(1), \qquad \text{since } \phi \text{ is an } R\text{-homomorphism}, \\ &= (\phi m)(g), \qquad \text{by (15)}. \end{aligned}$$

Thus

$$(\phi m)(g) = \tilde{\phi}(mg). \tag{16}$$

Hence ϕ is uniquely determined by $\tilde{\phi}$ and t is faithful; it is readily verified that $t(\phi - \phi') = t\phi - t\phi'$.

It remains to show that when G is finite t is onto. For any $\psi \in \mathrm{Hom}_A(M, A)$ define, following (16), $\phi\colon M \to R$ by

$$(\phi m)(g) = \psi(mg).$$

Since G is finite, $(\phi m)(g) \neq 0$ for at most a finite number of g and therefore $\phi m \in R$. Straightforward calculation shows that $\phi \in \mathrm{Hom}_R(M, R)$ and that $\tilde{\phi} = \psi$.

PROPOSITION 8. *If M is any A-module, $H = \mathrm{Hom}_A(R, M)$ can be made into an R-module in a natural way. If M is A-injective, then H is R-injective.*

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Proof. The proposition actually follows from Cartan and Eilenberg **(5**, Proposition 2.3a, p. 166**)**, but since a proof is not given there we give the outline of one.

As is well known, H is an abelian group with the obvious definitions. Using the notation $\alpha \in H$, $r \in R$, $\alpha[r] \in H$, $\alpha(r) \in M$, we define $\alpha[r]$ by

$$\alpha[r](r') = \alpha(rr').$$

H thus becomes an R-module.

Suppose M is A-injective. Let $K \subset L$ be R-modules and let $\phi \in \mathrm{Hom}_R(K, H)$. We wish to extend ϕ to $\phi' \in \mathrm{Hom}_R(L, H)$. If $x \in K$, then $\phi(x) \in H$ and we define ψ by

$$\psi(x) = \phi(x)(1).$$

Then $\psi \in \mathrm{Hom}_A(K, M)$. Since M is A-injective, ψ has an extension to $\psi' \in \mathrm{Hom}_A(L, M)$. Finally, define ϕ' by

$$\phi'(x)(r) = \psi'(xr), \qquad \text{for all } x \in L.$$

THEOREM 4. 1. *If A is self-injective and G is finite, then R is self-injective.*

2. Partial converse: *If R is self-injective, then A is self-injective and G is torsion.*

Remark. The referee informed me that part (1) (in the case when A is commutative) was obtained by Eilenberg and Nakayama **(7)**.

Proof. 1. Since $\mathrm{Hom}_R(R, R) \approx R$, putting $M = R$ in Proposition 7, we have the abelian group isomorphism

$$t: R \approx \mathrm{Hom}_A(R, A) = H,$$

where $t(r)(r') = (rr')(1)$. Now by Proposition 8 with $M = A$, R and H are R-modules and t is actually an R-module isomorphism, for

$$\begin{aligned} t(rr')(r'') &= (rr'r'')(1) \\ &= t(r)(r'r'') \\ &= (t(r)[r'])(r'') \end{aligned}$$

so that $t(rr') = t(r)[r']$. By Proposition 8, if A is self-injective, then H is R-injective and therefore R is R-injective, that is, R is self-injective.

2. Now suppose that R is self-injective. If J is a right ideal in A and $\phi: J \to A$ is an A-homomorphism, to show that A is self-injective we must show that there exists an $a \in A$ such that $\phi j = aj$ for all $j \in J$. We define $\phi': \omega_1 J \to R$ by

$$\phi' \sum r(g)g = \sum \phi(r(g))g,$$

where, by (8), $r(g) \in J$ for all $g \in G$. Clearly ϕ' is an R-homomorphism and since R is self-injective, there exists an $r = a_1 + a_2 g_2 + \ldots \in R$ such that $\phi' j' = rj'$ for all $j' \in \omega_1 J$. Thus for all $j \in J$

$$\phi j = rj = a_1 j + a_2 j g_2 + \ldots .$$

But $\phi j \in A$ so that $\phi j = a_1 j, a_2 j = \ldots = 0$, and we have the required $a = a_1$.

Finally suppose that G has an element g of infinite order. We must show that R is not self-injective. By Proposition 6, $1 - g$ is not a one-sided divisor of zero and since $1 - g$ is in the proper ideal Δ, the equation $x(1 - g) = 1$ has no solution. Let $J = (1 - g)R$ and define $\phi: J \to R$ by $\phi((1 - g)r) = r$. Since $(1 - g)r \neq 0$, ϕ is well-defined and is clearly an R-homomorphism. If R were self-injective, there would exist an $x \in R$ such that

$$\phi((1 - g)r) = r = x(1 - g)r$$

for all $r \in R$; in particular for $r = 1$ we have $1 = x(1 - g)$, which we know is impossible.

6. On the radicals. A ring is *prime* if 0 is a prime ideal; this means that for non-zero ideals J and K

$$JK \neq 0, \qquad J^l = 0. \tag{17}$$

A ring S is *primitive* if there exists a faithful irreducible S-module (see **9**); a primitive ring is prime.

S is *simple* if its only ideals are 0 and S; a simple ring is primitive.

If S is artinian the notions prime, primitive, and simple all coincide.

The *prime radical* $\mathfrak{P}(S)$ is the intersection of all prime ideals in S; equivalently $\mathfrak{P}(S)$ consists of all strongly nilpotent $x \in S$, an element $x \in S$ being called *strongly nilpotent* if every sequence $\{x_n\}$, where $x_0 = x$, $x_{n+1} = x_n y_n x_n$, $y_n \in S$ arbitrary, is ultimately 0. If S is commutative (more generally for elements in the centre of S) strong nilpotency is equivalent to nilpotency. $\mathfrak{P}(S)$ is also the intersection of the *radical ideals* of S, that is, ideals J for which $\mathfrak{P}(S/J) = 0$. $\mathfrak{P}(S)$ is also called the *lower nil* or *McCoy radical.*

The *Jacobson radical* $\mathfrak{R}(S)$ is the intersection of the maximal right ideals; equivalently $\mathfrak{R}(S)$ consists of all $x \in S$ such that for all $y \in S$, $1 - xy$ has a right inverse. It turns out that $\mathfrak{R}(S)$ is also the intersection of the maximal left ideals, and if $x \in \mathfrak{R}(S)$, then $1 - xy$ is actually a unit for all $y \in S$.

The *simplicial radical* $\mathfrak{S}(S)$ is the intersection of all maximal ideals. If S is commutative, $\mathfrak{S}(S) = \mathfrak{R}(S)$.

In any case these ideals satisfy

$$\mathfrak{P}(S) \subseteq \mathfrak{R}(S) \subseteq \mathfrak{S}(S). \tag{18}$$

All three radicals coincide if S is artinian.

S is *semi-prime* if $\mathfrak{P}(S) = 0$; this occurs if and only if S is a subdirect product of prime rings. In the commutative case the prime rings are the integral domains and the factors of the product may be taken to be S/P_i, where the P_i are the prime ideals of S. Any subdirect product of semi-prime rings is semi-prime. If S is a subring of T, then

$$\mathfrak{P}(S) \supseteq S \cap \mathfrak{P}(T) \tag{19}$$

with equality if S is contained in the centre of T, or if S is an ideal. In particular, a subring of a commutative semi-prime ring is semi-prime.

S is *semi-primitive* (semi-simple in the sense of Jacobson) if $\mathfrak{R}(S) = 0$. S is semi-primitive if and only if it is a subdirect product of primitive rings. In the commutative case the primitive rings are the fields, and the factors of the product may be taken to be S/M_i, where M_i ranges over the maximal ideals of S. Any subdirect product of semi-primitive rings is semi-primitive.

Thirdly, S is *semi-simple* if $\mathfrak{S}(S) = 0$; S is semi-simple if and only if it is a subdirect product of simple rings (which in the commutative case are the fields), and a subdirect product of semi-simple rings is again semi-simple.

We have

$$\mathfrak{P}(S/\mathfrak{P}(S)) = 0, \qquad \mathfrak{R}(S/\mathfrak{R}(S)) = 0, \qquad \mathfrak{S}(S/\mathfrak{S}(S)) = 0. \tag{20}$$

An ideal J of S is *nilpotent* if for some integer n, $J^n = 0$. $\mathfrak{P}(S)$ contains all nilpotent ideals; in fact S is semi-prime if and only if it contains no nilpotent $J \neq 0$. J is *nil* if every $x \in J$ is nilpotent, i.e., for every $x \in J$ there exists an $n = n(x)$ such that $x^n = 0$. A nilpotent ideal is nil; $\mathfrak{R}(S)$ contains all nil ideals.

It is readily verified that if A is a subdirect product of the rings A_i, then $R(G, A)$ is a subdirect product of the rings $R(G, A_i)$.

PROPOSITION 9. *Let B be a subring of A and let H be a subgroup of G so that $R_1 = R(G, B)$ and $R' = R(H, A)$ are subrings of $R = R(G, A)$. Then*

$$\mathfrak{P}(R_1) \supseteq R_1 \cap \mathfrak{P}(R), \tag{21}$$

with equality if B is contained in the centre of A. Also

$$\mathfrak{P}(R') \supseteq R' \cap \mathfrak{P}(R), \tag{22}$$

with equality if $H \subseteq C$. For example, putting $H = 1$,

$$\mathfrak{P}(A) = A \cap \mathfrak{P}(R); \tag{23}$$

thus if R is semi-prime, so is A. Next,

$$\mathfrak{R}(R') \supseteq R' \cap \mathfrak{R}(R). \tag{24}$$

Putting $H = 1$, we have

$$\mathfrak{R}(A) \supseteq A \cap \mathfrak{R}(R), \tag{25}$$

there being equality in (25) *if either*

(i) *A is artinian, or*

(ii) *G is locally finite.*

Proof. (21) and (22) are examples of (19) (whose proof is trivial). We must show the opposite inclusions when B and H are central. (Since R_1 and R' need not be commutative, the equality cases of (21) and (22) do not come under the first equality case of (19).)

Let B be central. If J is an ideal in R, clearly $K = R_1 \cap J$ is an ideal in R_1 and we prove that if J is prime so is K. If $uR_1v \subseteq K$, where $u, v \in R_1$, we must show that either $u \in K$ or $v \in K$. But $uRv \subseteq J$ since

$$\begin{aligned} urv &= u(a_1g_1 + a_2g_2 + \ldots)v \\ &= ua_1g_1v + \ldots = ug_1va_1 + \ldots \end{aligned}$$

and $ug_iv \in K \subseteq J$, whence $ug_iva_i \in J$. By the primality of J it follows that, say, $u \in J$ and therefore $u \in K$, and K is prime. Hence

$$\begin{aligned} \mathfrak{P}(R_1) &= \cap\{P: P \text{ a prime ideal in } R_1\} \\ &\subseteq \cap\{R_1 \cap J: J \text{ a prime ideal in } R\} \\ &= R_1 \cap \mathfrak{P}(R), \end{aligned}$$

which proves equality in (21).

Now let H be central; by Proposition 3

$$\begin{aligned} \mathfrak{P}(R') &= \cap\{P: P \text{ prime in } R'\} \\ &\subseteq \cap\{\Omega_H P: P \text{ prime in } R\} \\ &= R' \cap \mathfrak{P}(R) \end{aligned}$$

and therefore we have equality in (22).

If $r' \in R' \cap \mathfrak{R}(R)$, $1 - r'x$ has an inverse in R for all $x \in R'$ (indeed for all $x \in R$); but $1 - r'x \in R'$ and therefore by Proposition 4(i), $1 - r'x$ has an inverse in R', hence $r' \in \mathfrak{R}(R')$ and (24) follows. (The analogue of (19) is not generally true for the Jacobson radical.)

If A is artinian,

$$\mathfrak{R}(A) = \mathfrak{P}(A) = A \cap \mathfrak{P}(R) \subseteq A \cap \mathfrak{R}(R),$$

which is the inclusion opposite to (25) and we have equality in this case.

Now suppose that G is locally finite and $a \in \mathfrak{R}(A)$. We must prove that $a \in \mathfrak{R}(R)$, that is, for any $r \in R$ we must find an $x \in R$ such that $(1+ar)x=1$. Let S.G.$(r) = H = \{1, g_2, \ldots, g_n\}$ (which is finite by the assumption on G). If

$$\begin{aligned} r &= a_1 + a_2g_2 + \ldots + a_ng_n, \\ x &= x_1 + x_2g_2 + \ldots + x_ng_n, \end{aligned}$$

we must solve

$$((1 + aa_1) + aa_2g_2 + \ldots + aa_ng_n)(x_1 + \ldots + x_ng_n) = 1,$$

or

$$\begin{aligned} (1 + aa_1)x_1 + aa_{12}x_2 + \ldots + aa_{1n}x_n &= 1, \\ aa_{21}x_1 + (1 + aa_1)x_2 + \ldots + aa_{2n}x_n &= 0, \\ \vdots \qquad\qquad & \quad \vdots \\ aa_{n1}x_1 + aa_{n2}x_2 + \ldots + (1 + aa_1)x_n &= 0, \end{aligned}$$

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where in the ith row, $a_{i1}, \ldots, a_{i,i-1}, a_{i,i+1}, \ldots, a_{in}$ are a permutation of $a_2, a_3, \ldots, a_n$. If A is commutative, the determinant of this system is clearly of the form $1 + aa'$, where $a' \in A$, and since $a \in \mathfrak{R}(A)$, $1 + aa'$ is a unit so that by a standard theorem of linear algebra the system has a solution, as required.

In general, let $B = A_n$, where the subscript n denotes the ring of n by n matrices; then **(9**, p. 11**)** $\mathfrak{R}(B) = \mathfrak{R}(A)_n$. In matrix notation the above system is

$$(1 + b)v = u,$$

where v is the column vector $(x_1, \ldots, x_n)$, u is the column vector $(1, 0, \ldots, 0)$, and $b \in \mathfrak{R}(B)$. Hence $1 + b$ is a unit and the equation has the solution

$$v = (1 + b)^{-1}u,$$

which completes the proof.

We will see below extensive classes of R for which $\mathfrak{R}(R) = 0$ but $\mathfrak{R}(A) \neq 0$. If $\mathfrak{R}_A(R)$ denotes the radical of R regarded as an A-module **(4**, p. 63**)**, it is not hard to see that

$$\mathfrak{R}(A) = A \cap \mathfrak{R}_A(R).$$

COROLLARY **(**cf. Amitsur **1)**. *In order that $R(G, A)$ be semi-prime (semi-primitive), it is sufficient that $R' = R(H, A)$ be semi-prime (semi-primitive) for each finitely generated subgroup H of G.*

For if $r \in \mathfrak{P}(R)$, then $r \in \mathfrak{P}(R) \cap R'$, where $H = \text{S.G.}(r)$, and the result for $\mathfrak{P}(R)$ follows by (22); similarly the result for $\mathfrak{R}(R)$ follows by (24).

We recall that $\nu(G)$ denotes the set of orders of the finite normal subgroups in G.

THEOREM 5. *Let A be commutative. Then R is semi-prime if and only if A is semi-prime and each $n \in \nu(G)$ is regular in A.*

Remark. This theorem is due to Passman **(17)** (though his proof is somewhat different from that which follows), and I am indebted to the referee for giving me this reference. I did not have the complete theorem (see the remarks following Lemma 3 below).

Proof. First we show that R semi-prime implies the conditions stated. By Proposition 9, $\mathfrak{P}(A) = 0$. Let $H = \{g_1, \ldots, g_n\}$ be a finite normal subgroup and suppose $na = 0$, $a \neq 0$. Then $J = \omega H$ is an ideal,

$$a(g_1 + \ldots + g_n) = -a(1 - g_1) - \ldots - a(1 - g_n) \in J^l \cap J,$$

and

$$(J^l \cap J)^2 \subseteq J^l J = 0.$$

Hence $J^l \cap J$ is a non-zero nilpotent ideal and R is not semi-prime. (For this part we did not need A commutative.)

The converse is proved with the aid of three lemmas.

LEMMA 1. *Let $T = M^{-1}S$ be the classical ring of quotients of the ring S with respect to the multiplicatively closed set M $(1 \in M, 0 \notin M)$ contained in the centre of S. Then*

$$\mathfrak{P}(T) = M^{-1}\mathfrak{P}(S) = \{x/u : x \in \mathfrak{P}(S), u \in M\},$$

and therefore (using (19))

$$\mathfrak{P}(S) = S \cap \mathfrak{P}(T).$$

Proof. First if $x/u \in M^{-1}\mathfrak{P}(S)$ we must show that $x/u \in \mathfrak{P}(T)$, i.e., any sequence $\{x_n\}$ where $x_0 = x/u$, $x_{n+1} = x_n y_n x_n$, $y_n \in T$ arbitrary, is ultimately zero. Now $y_n = s_n/u_n$, where $s_n \in S$, $u_n \in M$ so that

$$x_1 = xs_0x/u^2u_0 = x_1'/u^2u_0,$$
$$x_2 = x_1's_1x_1'/u^4u_0^2u_1 = x_2'/u^4u_0^2u_1,$$

etc., and since x_n' is ultimately zero so is x_n. Thus $M^{-1}\mathfrak{P}(S) \subseteq \mathfrak{P}(T)$.

Conversely an element of the ideal $\mathfrak{P}(T)$, being in T, has the form s/u, hence $s \in \mathfrak{P}(T)$. Since s is strongly nilpotent in T it is strongly nilpotent in S, and therefore $s \in \mathfrak{P}(S)$. Thus $\mathfrak{P}(T) \subseteq M^{-1}\mathfrak{P}(S)$.

Let M be the multiplicatively closed set generated by the regular elements $\nu(G)$, and put $B = M^{-1}A$. Hence $R(G, B) = M^{-1}R(G, A)$ and applying the lemma to these two cases we see that $\mathfrak{P}(B) = 0$, each $n \in \nu(G)$ is a unit in B, and it is sufficient to prove that $R(G, B)$ is semi-prime. Since B is commutative, it is a subdirect product of the integral domains $D_i = B/P_i$, where P_i ranges over the proper prime ideals of B, and since $R(G, B)$ is a subdirect product of the rings $R(G, D_i)$, it is sufficient to prove that $R(G, D_i)$ is semi-prime; note that each $n \in \nu(G)$ is a unit in D_i. Applying the lemma once more, it is sufficient to prove that $R(G, F)$ is semi-prime, where F denotes the field of quotients of D_i; note that the characteristic p of F divides no $n \in \nu(G)$. The two cases $p = 0$ and $p > 0$ are treated differently.

First let $p = 0$ and put $R^* = R(G, F^*)$, where F^* is the algebraic closure of F. By Proposition 9 (the equality case of (21)) it is sufficient to prove that $\mathfrak{P}(R^*) = 0$. Since the prime radical is a nil ideal this follows from the following lemma.

LEMMA 2. *R^* contains no non-zero nil ideals.*

Proof. F^* is obtained from a real closed field P by the adjunction of $i = \sqrt{(-1)}$, and an element $a \in F^*$ can be written in the form $a = b + ci$ where $b, c \in P$; we denote the conjugate of a by $\bar{a} = b - ci$. If

$$r = \sum r(g)g \neq 0$$

is an element of the ideal J in R^*, define

$$\bar{r} = \sum \overline{r(g)}\, g^{-1};$$

then $s = r\bar{r} \in J$. Since P is formally real and $s(1)$ is a sum of squares in P, we have $s(1) > 0$. Also it is easy to see that $s(g^{-1}) = \overline{s(g)}$. Now if t is any element such that $t(1) > 0$ and $t(g^{-1}) = \overline{t(g)}$, it is readily verified that t^2 has the same properties. It follows that $s^{2^n} \neq 0$ for all n, hence J is not nil.

Passing to finite characteristics, the proof of the theorem is completed by the following lemma.

LEMMA 3. *Let $p > 0$ divide no $n \in \nu(G)$. Then $R(G, F)$ contains no nilpotent ideals.*

Remark. This result is the crux of the theorem; the proof is given in **(17)**. I had obtained this result (actually that there are no non-zero nil ideals) in the very much easier circumstance when p divides no $n \in o(G)$.

We need an extension of the theorem to non-commutative A in order to complete the proof of Theorem 1.

PROPOSITION 10. *If $\mathfrak{S}(A) = 0$ and each $n \in \nu(G)$ is regular in A, then $\mathfrak{P}(R) = 0$.*

Proof. As in the previous theorem we go to the classical ring of quotients with respect to the multiplicative system generated by $\nu(G)$; thus we may assume to begin with that each $n \in \nu(G)$ is a unit in A. Now A is a subdirect product of various simple rings $A_i = A/M_i$, M_i being a maximal ideal, and it is sufficient to prove $\mathfrak{P}(R(G, A_i)) = 0$. Since A_i is simple, its centre is a field F, and each $n \in \nu(G)$ is a unit in A_i; A_i is a central simple F-algebra.

LEMMA. *If S is a central simple F-algebra and T any F-algebra, then*

$$\mathfrak{P}(S \otimes_F T) = S \otimes_F \mathfrak{P}(B).$$

Proof. Let $\mathfrak{L}$ designate the lattice of ideals. In Jacobson **(9**, p. 109**)** it is shown that the correspondence

$$J \leftrightarrow S \otimes_F J, \qquad J \in \mathfrak{L}(T),$$

yields a lattice isomorphism

$$\mathfrak{L}(T) \approx \mathfrak{L}(S \otimes_F T).$$

Moreover a simple calculation shows that products are preserved. Hence prime ideals correspond to prime ideals and

$$\begin{aligned}\mathfrak{P}(S \otimes_F T) &= \bigcap \{P : P \text{ prime in } S \otimes_F T\}\\ &= \bigcap \{S \otimes_F P : P \text{ prime in } T\}\\ &= S \otimes_F \bigcap \{P : P \text{ prime in } T\}\\ &= S \otimes_F \mathfrak{P}(T).\end{aligned}$$

We now apply the lemma to the case $S = A_i$, $T = R(G, F)$. Since $R(G, F)$

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is a free F-module, with basis $\{g\}$, each element of $A_i \otimes_F R(G, F)$ has a unique representation $\sum a \otimes g$ and it is readily verified that the correspondence

$$\sum ag \leftrightarrow \sum a \otimes g, \qquad a \in A_i,$$

yields the ring isomorphism

$$R(G, A_i) \approx A_i \otimes_F R(G, F). \tag{26}$$

((26) is valid more generally for any ring A_i and any subring F contained in its centre.) Since $\mathfrak{P}(R(G, F)) = 0$ by the previous theorem, the result follows.

Completion of the proof of Theorem 1. When R is artinian we have already observed that A is artinian and G has, in particular, the ascending chain condition on normal subgroups. It remains to prove that G is finite. We may assume to begin with that A is completely reducible; (for $A/\mathfrak{P}(A)$ is completely reducible and $R(G, A/\mathfrak{P}(A)) = R(G, A)/\omega_1\mathfrak{P}(A)$ is artinian). Then in particular, $\mathfrak{S}(A) = 0$. We put $G_1 = G$ and $R_1 = R$.

If $\mathfrak{P}(R_1) = 0$, the result follows by the generalized Maschke theorem. Otherwise, by the preceding proposition, G_1 contains a finite normal subgroup H_1 (whose order is a zero divisor in A). Now $R_2 = R(G_1/H_1, A) = R_1/\omega H_1$ is artinian, and iterating the argument, either $\mathfrak{P}(R_2) = 0$ or $G_2 = G_1/H_1$ contains a non-trivial finite normal subgroup H_2. Continuing in this way we obtain a chain of groups $G_{i+1} = G_i/H_i$ and artinian rings $R_i = R(G_i, A)$. Clearly $G_i = G/N_i$ where $N_1 \subset N_2 \subset \ldots$, and therefore the process must terminate, which means that at some point $\mathfrak{P}(R_n) = 0$, whence by the converse of the Maschke theorem G_n is finite; but $G_n = G_{n-1}/H_{n-1}$, and H_{n-1} is finite. Thus G_{n-1} is finite, and working backward we eventually get $G = G_1$ finite, as required. (For an alternative proof see the remarks after the corollary to Theorem 8.)

We now pass on to consideration of the Jacobson radical. The next result, preliminary to Theorem 6, exhibits a curious interplay between the prime and Jacobson radicals. Since there exist A for which $\mathfrak{P}(A) = 0$, $\mathfrak{R}(A) \neq 0$, for example local domains, the proposition affords examples of R with $\mathfrak{R}(R) = 0$ but $\mathfrak{R}(A) \neq 0$; Theorem 6 supplies further examples. Note, by an *ordered group* we mean a $G \neq 1$ which is totally ordered.

PROPOSITION 11. *Let A be commutative and G ordered. Then R is semi-primitive if and only if A is semi-prime.*

Proof. First, $\mathfrak{R}(R) = 0$ implies $\mathfrak{P}(R) = 0$, which implies $\mathfrak{P}(A) = 0$ by Proposition 9.

Conversely, A is a subdirect product of integral domains A_i and it is sufficient to show that $\mathfrak{R}(R(G, A_i)) = 0$. Suppose $rr' = 1$, where

$$r = a_1g_1 + \ldots + a_mg_m, \qquad a_j \in A_i, a_j \neq 0, g_1 < \ldots < g_m,$$
$$r' = b_1h_1 + \ldots + b_nh_n, \qquad b_j \in A_i, b_j \neq 0, h_1 < \ldots < h_n.$$

Then

$$rr' = a_1b_1g_1h_1 + \ldots + a_mb_ng_mh_n,$$

where $a_1b_1 \neq 0, a_mb_n \neq 0, g_1h_1 < \ldots < g_mh_n$. Thus $rr' = 1$ implies $m = n = 1$. Suppose now that $r \in \Re(R)$. Then $1 + r\cdot g$ has an inverse for all $g \in G$; hence $1 + r\cdot g$ involves only one element of G. Since there are $g \neq 1$ this clearly implies $r = 0$.

In discussing $\Re(R)$, the non-commutativity of A and G seems to introduce serious difficulties; we can at least give a complete answer when they are commutative. The division in the following theorem of abelian G into two classes would indicate that the final analysis of $\Re(R)$ cannot be too simple.

By *not torsion* we mean that G has at least one element of infinite order.

THEOREM 6. *Let A and G be commutative.*

(i) *G torsion: R is semi-primitive if and only if A is semi-primitive and each $n \in o(G)$ is regular in A.*

(ii) *G not torsion: R is semi-primitive if and only if A is semi-prime and each $n \in o(G)$ is regular in A.*

Remarks. If A is a field of characteristic 0, or a subdirect product of such, the theorem gives a result of Amitsur **(1)** and Villamayor **(19, 20)**. (Actually they deal with a more general class of G.) Since our result treats arbitrary commutative A, and clarifies the role of $o(G)$, and since our proof is on more direct lines, we feel it worth while to give here. Note that the theorem includes examples such as: $\Re(R(G, Z)) = 0$ *for any abelian G.*

Proof. We first establish two lemmas.

LEMMA 1. *Let $s_1, \ldots, s_n$ be finitely many regular elements of the commutative semi-primitive ring S. Then S is a subdirect product of the fields S/M_i, where M_i runs over those maximal ideals of S which do not contain any of the elements $s_1, \ldots, s_n$.*

Proof. Let $\{M_i: i \in I\}$ be the set of maximal ideals so described and denote the remaining by M_j. Define

$$\phi : S \to \prod_{i\in I} S/M_i$$

in the canonical way so that

$$K = \text{Ker } \phi = \bigcap_{i\in I} M_i.$$

$s = s_1s_2 \ldots s_n$ is regular. If $x \in K$, then $xs \in K$; but xs is also in each M_j since M_j contains one of $s_1, \ldots, s_n$. Thus xs is in every maximal ideal, i.e., $xs \in \Re(S)$; but $\Re(S) = 0$ so that $xs = 0$ and by the regularity of s, $x = 0$. Hence ϕ is faithful, and the result clearly follows.

LEMMA 2. *Let A and G be commutative, let $\Re(A) = 0$, and let each $n \in o(G)$ be regular in A. Then $\Re(R) = 0$.*

Proof. By the corollary to Proposition 9 we may assume that G is finitely generated, and then, by the fundamental theorem on abelian groups, G is the direct product of a finite number of cyclic groups; it follows that $o(G)$ is finite. By Lemma 1, A is a subdirect product of the fields A/M_i, where none of the M_i contains an element of $o(G)$. Hence the characteristic p_i of A/M_i divides no $n \in o(G)$ or, what is the same thing, $p_i \notin o(G)$. Since it is sufficient to prove that $\mathfrak{R}(R(G, A/M_i)) = 0$, the proof of the lemma is reduced to proving the following statement:

Let G be a finitely generated abelian group and let A be a field whose characteristic $p \notin o(G)$. Then $\mathfrak{R}(R) = 0$.

Now by the fundamental theorem on abelian groups G is generated by a number of generators of finite order and by m, say, generators of infinite order, and no relation obtains among these generators. The proof will proceed by induction on m.

If $m = 0$, G is finite and the result follows by the Maschke theorem. Inductively, let G have $m \geqslant 1$ generators of infinite order, one of them being g, and let $r \in \mathfrak{R}(R)$. Then $r \cdot g^k \in \mathfrak{R}(R)$ for any power g^k of g. Thus we may assume that r is a polynomial in g:

$$r = x_0 + x_1 g + \ldots + x_s g^s,$$

where the $x_i \in R(G, A)$ do not involve g. Choose any $t > s$ not a multiple of p, and let N be the subgroup generated by g^t. By Proposition 1 we have the epimorphism

$$R = R(G, A) \to R(G, A)/\omega N = R(G/N, A) = \bar{R}$$

and therefore $\overline{\mathfrak{R}(R)} \subseteq \mathfrak{R}(\bar{R})$, letting bars signify images under the epimorphism. But G/N has $m - 1$ generators of infinite order and by our choice of t, $p \notin o(G/N)$. Hence by induction $\mathfrak{R}(\bar{R}) = 0$; finally

$$\begin{aligned} 0 = \bar{r} &= \bar{x}_0 + \bar{x}_1 \bar{g} + \ldots + \bar{x}_s \bar{g}^s \\ &= x_0 + x_1 g + \ldots + x_s g^s = r, \end{aligned}$$

since the x_i do not involve g (the epimorphism simply amounts to replacing g^t by 1) and since $t > s$. Hence $\mathfrak{R}(R) = 0$ and the proof of Lemma 2 is complete.

We now return to the two cases of the theorem.

(i) Since G is abelian, *torsion* is equivalent to *locally finite*. Thus $\mathfrak{R}(R) = 0$ implies $\mathfrak{R}(A) = 0$ by (25), equality case (ii). Since also $\mathfrak{P}(R) = 0$, each $n \in \nu(G) = o(G)$ is regular by (the easy half of) Theorem 5. The converse follows immediately from Lemma 2.

(ii) If $\mathfrak{R}(R) = 0$, then $\mathfrak{P}(R) = 0$ so that $\mathfrak{P}(A) = 0$ and, as in case (i), each $n \in o(G)$ is regular.

Conversely let $g \in G$ be an element of infinite order. It will be sufficient to show that $\mathfrak{R}(R(H, A)) = 0$ for every finitely generated subgroup H of G which contains g. For if $r \in \mathfrak{R}(R(G, A))$, then $r \in R(H, A) = R'$, where H is the subgroup generated by the finite set $\{g\} \cup \mathrm{Supp}(r)$, and the result will follow from (24).

Now $H = G_1 \times G_2 \times \ldots \times G_k$, where the G_i are cyclic groups at least one of which, say G_1, is infinite. Put

$$A_1 = R(G_1, A), \quad A_2 = R(G_2, A_1), \quad \ldots \quad A_k = R(G_k, A_{k-1});$$

clearly $A_k \approx R(H, A)$. Now $\mathfrak{P}(A) = 0$ and G_1 is ordered so that, by Proposition 11, $\mathfrak{R}(A_1) = 0$. It is readily verified that each $n \in o(G)$ is regular in each A_i. Thus by successive applications of Lemma 2, $\mathfrak{R}(A_2) = 0$, $\mathfrak{R}(A_3) = 0, \ldots, \mathfrak{R}(A_k) = 0$, and the proof of the theorem is complete.

Let $\mathfrak{C}$ denote the class of all groups G with the following property:

If F is a field whose characteristic $\notin o(G)$, then $\mathfrak{R}(R(G, F)) = 0$.

Since a regular ring is semi-primitive, Theorem 3 shows that $\mathfrak{C}$ contains all locally finite groups; by Proposition 11, $\mathfrak{C}$ contains all ordered groups; and by Theorem 6, $\mathfrak{C}$ contains all abelian groups.

Proposition 12. (i) *If $\mathfrak{S}(A) = 0$, each $n \in o(G)$ is a unit in A, and $G \in \mathfrak{C}$ then* $\mathfrak{R}(R) = 0$. (Note this includes the case of commutative A with $\mathfrak{R}(A) = 0$.)

(ii) *If each finitely generated subgroup of G is in $\mathfrak{C}$, then $G \in \mathfrak{C}$.*

(iii) *If $G, H \in \mathfrak{C}$, then their direct product $G \times H \in \mathfrak{C}$.*

(iv) *If G is arbitrary and G_0 is the infinite cyclic group, then $G \times G_0 \in \mathfrak{C}$.*

Proof. (i) A is a subdirect product of various simple rings $B = A/M$, M being a maximal ideal, and it suffices to prove $\mathfrak{R}(R(G, B)) = 0$. The centre F of B is a field, say of characteristic p, and since each $n \in o(G)$ is a unit in B, $p \notin o(G)$. (Since $o(G)$ is not necessarily finite we do not have available the device of Theorem 6, using Lemma 1; thus we must assume that each $n \in o(G)$ is a unit, rather than a regular element.)

If F' is an extension field of F, its characteristic is again p, and since $G \in \mathfrak{C}$, using (26) we have

$$\mathfrak{R}(R(G, F) \otimes_F F') = \mathfrak{R}(R(G, F')) = 0.$$

Hence $R(G, F)$ is a separable F-algebra; also $\mathfrak{R}(B) = 0$. By (**4**, p. 93),

$$\mathfrak{R}(R(G, B)) = \mathfrak{R}(R(G, F) \otimes_F B) = 0.$$

(ii) This follows from the corollary to Proposition 9.

(iii) Let A be a field whose characteristic p is not in $o(G \times H)$, and therefore p is in neither $o(G)$ nor $o(H)$. We must show that $\mathfrak{R}(R(G \times H, A)) = 0$.

$R(G, A) \otimes_A R(H, A)$ can be made into a ring in the usual way and since both factors are free A-modules, each element of the tensor product has a unique representation in the form (with a self-explanatory notation)

$$\sum a(g \otimes h).$$

The correspondence

$$\sum a(g \otimes h) \leftrightarrow \sum a(g, h)$$

leads immediately to the ring isomorphism

$$R(G, A) \otimes_A R(H, A) \approx R(G \times H, A).$$

$R(G, A)$ is separable, and, since $H \in \mathfrak{C}$, $\mathfrak{R}(R(H, A)) = 0$. Again the conclusion follows by (**4**, p. 93).

(iv) We require three lemmas (we state the second in more generality than needed).

LEMMA 1. *If A has no non-zero nil ideals, then $R(G_0, A)$ is semi-primitive.*

We omit the proof, which is almost identical to that for the polynomial ring $A[x]$; see (**9**, p. 12).

LEMMA 2. *Let A be a field and B an algebraic extension (of finite or infinite degree). Then $R(G, B)$ semi-primitive implies $R(G, A)$ semi-primitive. The converse implication holds provided that in addition B is a separable extension.*

Proof of Lemma 2. Using

$$R(G, B) \approx R(G, A) \otimes_A B,$$

from (**4**, pp. 83-85) we have

$$\mathfrak{R}(R(G, A)) = R(G, A) \cap \mathfrak{R}(R(G, B)),$$

and when B is separable,

$$\mathfrak{R}(R(G, B)) = B \otimes_A \mathfrak{R}(R(G, A));$$

the lemma follows immediately.

Returning to the proof of (iv), if F is a field whose characteristic

$$p \notin o(G \times G_0) = o(G)$$

we must prove that $\mathfrak{R}(R(G \times G_0, F)) = 0$.

First, if $p = 0$, we may assume by Lemma 2 that F is algebraically closed; by Lemma 2 of Theorem 5, $R(G, F)$ contains no nil ideals, and by Lemma 1, $R(G_0, R(G, F)) \approx R(G \times G_0, F)$ is semi-primitive, as required.

Finally, if $p > 0$, the result follows in the same way from the following lemma.

LEMMA 3. *If F is a field of characteristic $p > 0$, and p divides no $n \in o(G)$, then $R(G, F)$ contains no non-zero nil ideals.*

This fact has already been alluded to in the remarks to Lemma 3 of Theorem

5. Since a proof of this result also is given by Passman **(17)**, we shall not give one here.

Various previous results, along with arguments similar to those already encountered, combine to describe the situation for finite groups; since finite groups are often of particular interest, we state these facts in a theorem (omitting the proof).

THEOREM 7. *Let A be commutative and let G be finite of order n. Then R is semi-prime (semi-primitive) if and only if A is semi-prime (semi-primitive) and n is regular in A.*

7. Primality. In the last section we asked when R is semi-prime (semi-primitive) and it is natural now to ask when R is prime (primitive). The commutative case is easily disposed of (we omit the details):

(i) R is an integral domain if and only if A is an integral domain and G is abelian torsion-free.

(ii) R is never a field (indeed a skew field) except in trivial cases when $G = 1$, $R = A$.

We attempt only the "prime" half of the general question.

Let $\rho(G)$ be the set of $g \in G$ which have only a finite number of conjugates and let $\sigma(G)$ denote the set of $g \in \rho(G)$ of finite order. Then, using Dietzmann's lemma **(22**, p. 154**)**, one sees that $\rho(G)$ and $\sigma(G)$ are characteristic subgroups. Groups with $\rho(G) = G$ are known as *FC-groups* **(**see **22**, Appendix n for references**)**; groups with $\sigma(G) = G$ are called *locally normal*, an equivalent definition being **(22**, p. 154**)**: *every finite subset is contained in a finite normal subgroup.* We define G to be *prime* if it satisfies either one of the following two conditions, whose equivalence is easily proved using Dietzmann's lemma:

(i) $\sigma(G) = 1$,

(ii) $\nu(G) = \{1\}$, i.e., G contains no finite normal subgroup except 1.

THEOREM 8. *R is prime if and only if A is prime and G is prime.*

Proof. (i) If G is not prime it contains a non-trivial finite normal subgroup H, so by Proposition 1, ωH is a non-trivial ideal and $(\omega H)^l \neq 0$; hence R is not prime.

If A is not prime there exist non-zero ideals J and K in A with $JK = 0$. By Proposition 2, $\omega_1(JK) = \omega_1 J \cdot \omega_1 K = 0$ and since ω_1 is faithful and $\omega_1 J$ and $\omega_1 K$ are ideals, R is not prime.

(ii) Conversely if R is not prime but G is prime, we must prove that A is not prime. Put $\rho(G) = G_1$, $R_1 = R(G_1, A)$ and define the A-epimorphism

$$\psi: R \rightarrow R_1$$

by

$$\psi \sum r(g)g = \sum_{g \in G_1} r(g)g;$$

that is, ψ deletes those terms $r(g)g$ of r for which $g \notin G_1$.

LEMMA 1. *If J and K are non-zero ideals in R, then ψJ and ψK are non-zero ideals in R_1; and if $JK = 0$, then $\psi J \cdot \psi K = 0$.*

Remark. Passman defined ψ in **(17)**, and he points out (unpublished) that this lemma can be proved in the same way as his Lemma 8 **(17)**. Note that we do not need to assume A commutative.

We thus have R_1 not prime; clearly $\rho(G_1) = G_1$ and $\sigma(G_1) = \sigma(G) = 1$, i.e., G_1 is a torsion-free FC-group.

LEMMA 2. *For any G, $\rho(G)/\sigma(G)$ is abelian torsion-free.*

This striking result is due to B. H. Neumann **(21)**.

Thus G_1 is abelian torsion-free and therefore **(3**, p. 224**)** can be ordered. Let J be a non-zero ideal in R_1 with $J^l \neq 0$, say

$$r = a_1 g_1 + a_2 g_2 + \ldots + a_n g_n \in J^l,$$

where

$$a_1 \neq 0, \qquad g_1 < g_2 < \ldots < g_n.$$

Define K to be the set consisting of 0 together with all "leading coefficients" of elements of J:

$$K = \{a\colon \text{there exists } j = ag + a'g' + \ldots \in J, g < g' < \ldots\}.$$

Obviously K is a non-zero ideal in A. Now

$$\begin{aligned} 0 = rj &= (a_1 g_1 + \ldots)(ag + \ldots) \\ &= a_1 a g_1 g + \ldots \end{aligned}$$

and $a_1 a g_1 g$ is the "smallest" summand on the right-hand side. Hence $a_1 a = 0$; thus $a_1 \in K^l$, and therefore A is not prime, as required.

COROLLARY. *If G is prime, then R is semi-prime if and only if A is semi-prime.*

Remarks. The class of prime groups contains all torsion-free groups and therefore all ordered groups (incidentally, not all torsion-free groups can be ordered). Thus when A is commutative it is interesting to compare the corollary with Proposition 11.

This corollary affords us an alternative method of completing the proof of Theorem 1: we may assume that A is completely reducible; if $\mathfrak{P}(R) \neq 0$, then G is not prime and therefore contains a non-trivial finite normal subgroup H_1. The argument is completed as before.

One would conjecture that the commutativity condition on A in Theorem 5 can be dropped, and this corollary shows that this is so at least in the case when $\nu(G) = \{1\}$.

Proof. If R is semi-prime, then A is semi-prime by Proposition 9. Conversely by the subdirect product argument we may assume that A is prime; but then by the theorem R is prime and therefore semi-prime.

8. Properties of the fundamental ideal. Kaplansky **(12**, p. 2**)** raised the question: *When is Δ a nil ideal?* We obtain a partial result on this; however, when the stronger condition that Δ be nilpotent is imposed, we are able to give a complete answer (generalizing a result of Losey **(14)**). We also look into a number of other questions concerning Δ.

We require a combinatorial lemma. If $k > 0$,

$$\binom{k}{0} + \binom{k}{1} + \ldots + \binom{k}{k} = (1+1)^k = 2^k,$$

$$\binom{k}{0} - \binom{k}{1} + \ldots \pm \binom{k}{k} = (1-1)^k = 0,$$

so that, adding and subtracting, we have

$$f_0 = \binom{k}{0} + \binom{k}{2} + \ldots = 2^{k-1},$$

$$f_1 = \binom{k}{1} + \binom{k}{3} + \ldots = 2^{k-1}.$$

Hence $f = \text{g.c.d.}\{f_i\} = 2^{k-1}$ so that f is a power of 2 which is large when k is large. We shall require the following generalization.

PROPOSITION 13. *Let p be a prime, k and s positive integers, and for $i = 0, 1, \ldots, p^s - 1$ put*

$$f_i = \binom{k}{i} - \binom{k}{p^s + i} + \binom{k}{2p^s + i} - \ldots$$

except that when $p = 2$ the signs are all to be taken as positive. If $f = \text{g.c.d.}\{f_i\}$, then $f = p^t$ and $t \to \infty$ as $k \to \infty$.

Proof. Let ζ be a primitive p^s-root of unity. Since

$$(x - \zeta)(x - \zeta^2) \ldots (x - \zeta^{p^s-1}) = \frac{x^{p^s} - 1}{x - 1} = x^{p^s-1} + \ldots + x + 1,$$

we have

$$(1 - \zeta)(1 - \zeta^2) \ldots (1 - \zeta^{p^s-1}) = p^s. \tag{27}$$

Now

$$(1 - \zeta^n)^k = f_0 - \zeta^n f_1 + \ldots \pm \zeta^{n(p^s-1)} f_{p^s-1} \tag{28}$$

and multiplying these equations together ($n = 1, 2, \ldots, p^s - 1$) we have by (27)

$$p^{sk} = (f_0 - \zeta f_1 + \ldots) \ldots (f_0 - \zeta^{p^s-1} f_1 + \ldots).$$

Since f divides each f_i, f divides p^{sk} in the cyclotomic domain and therefore in Z. Hence $f = p^t$ for some t.

Multiplying the equations (28) ($n = 0, 1, \ldots, p^s - 1$) by appropriate powers of ζ, adding the results and dividing by p^s, using

$$(29) \qquad 1 + \zeta + \ldots + \zeta^{p^s - 1} = 0,$$

we obtain the following formulas:

$$(30) \qquad f_i = \frac{(-1)^i}{p^s} \sum_{n=1}^{p^s-1} \zeta^{u(i,n)} (1 - \zeta^n)^k,$$

where the $u(i, n)$ are integers. Now

$$(1 - \zeta)(1 + \zeta + \ldots + \zeta^{n-1}) = 1 - \zeta^n$$

so that, taking out a common factor, (30) becomes

$$(31) \qquad f_i = \frac{(1 - \zeta)^k}{p^s} F_i,$$

where F_i is a polynomial in ζ with integral coefficients.

If α is an element of the cyclotomic field, we denote its (field) norm by $N\alpha$. Since the field is of degree $p^{s-1}(p - 1)$, for any rational x

$$(32) \qquad Nx = x^{p^{s-1}(p-1)},$$

and

$$N(1 - \zeta) = \prod_{n=1}^{p^s} \{1 - \zeta^n : \text{g.c.d.}\ (n, p) = 1\}$$

$$= \frac{(1 - \zeta)(1 - \zeta^2) \ldots (1 - \zeta^{p^s - 1})}{(1 - \zeta^p)(1 - \zeta^{2p}) \ldots (1 - \zeta^{p^s - p})}.$$

But ζ^p is a primitive p^{s-1}-root of unity so that applying (27) to both the numerator and the denominator of the above fraction gives

$$(33) \qquad N(1 - \zeta) = \frac{p^s}{p^{s-1}} = p.$$

Applying the norm to equation (31) gives, by (32) and (33),

$$f_i^{p^{s-1}(p-1)} = p^{k - sp^{s-1}(p-1)} NF_i.$$

Since F_i is an algebraic integer, NF_i is a rational integer and therefore p^t divides f_i, where (brackets denote the integral part)

$$t \geqslant \left[\frac{k - sp^{s-1}(p - 1)}{p^{s-1}(p - 1)}\right],$$

and $t \to \infty$ as $k \to \infty$, as required.

The *Wedderburn radical* $\mathfrak{W}(S)$ of the ring S is the union of the nilpotent ideals in S; it coincides with the union of the nilpotent right ideals and with the union of the nilpotent left ideals. We have (cf. (18))

$$(34) \qquad \mathfrak{W}(S) \subseteq \mathfrak{P}(S)$$

with equality if S is commutative, or (by Levitzki's theorem) if S is noetherian. However, there are examples of $\mathfrak{W}(S/\mathfrak{W}(S)) \neq 0$ and therefore of $\mathfrak{W}(S) \neq \mathfrak{P}(S)$. In any case $\mathfrak{W}(S) = 0$ if and only if $\mathfrak{P}(S) = 0$; thus Theorem 5 can be regarded as a theorem concerning $\mathfrak{W}(R)$.

If $x \in \mathfrak{P}(S)$, then x is strongly nilpotent, that is, all the sequences x_n, defined earlier, are ultimately zero; if $x \in \mathfrak{W}(S)$, then the indices at which these sequences become zero are bounded.

$x \in \mathfrak{W}(S)$ if and only if the principal ideal J generated by x is nilpotent; equivalently, there exists an n such that

$$xs_1xs_2 \dots xs_n = 0$$

for all choices of the $s_i \in S$. Thus if T is a subring of S,

$$\mathfrak{W}(T) \supseteq T \cap \mathfrak{W}(S) \tag{35}$$

with equality if T is contained in the centre or if T is an ideal (cf. (19)). We can also prove equality when $S = R$, $T = A$: Let $aa_1 \dots aa_n = 0$ for all $a_i \in A$; now $ar_1 \dots ar_n$, where $r_i \in R$, is a sum of terms of the form $aa_1 \dots aa_n g$ and therefore vanishes. Hence $a \in \mathfrak{W}(R)$ and we have the following proposition.

PROPOSITION 14. $\mathfrak{W}(A) = A \cap \mathfrak{W}(R)$.

By a *p-group* we mean a group in which the orders of all elements are powers of the fixed prime p. A standard result of group theory states that except in the trivial case $G = 1$ (which we tacitly exclude from the following discussion) the centre of a finite p-group is distinct from 1.

Locally normal groups were defined in Section 7. Since a p-group is torsion, a locally normal p-group is the same thing as an FC p-group, examples of this class being all finite p-groups and all abelian p-groups.

A well-known result states that if A is a field of characteristic p and Δ is nil, then G is a p-group. It does not seem to have been noticed before that the conclusion does not depend on Δ being nil, but can be proved assuming only that it is contained in the Jacobson radical.

PROPOSITION 15. (i) *If $\Delta \subseteq \mathfrak{R}(R)$, then G is a p-group and $p \in \mathfrak{R}(A)$.*
(ii) *If Δ is nil, then G is a p-group and $p \in \mathfrak{P}(A)$.*
(iii) *If $\Delta \subseteq \mathfrak{W}(R)$, then G is a locally normal p-group and $p \in \mathfrak{W}(A)$.*

Remark. Since p is in the centre of A, the following statements are equivalent:
(a) $p \in \mathfrak{P}(A)$,
(b) $p \in \mathfrak{W}(A)$,
(c) p is nilpotent in A.

Proof. (i) We first prove that G cannot have an element of infinite order. Thus suppose that g has infinite order so that

$$(1 - g) + (1 - g^2) = 2 - g - g^2 \in \Delta \subseteq \mathfrak{R}(R),$$

and therefore

$$1 - (2 - g - g^2) = -1 + g + g^2$$

has a right inverse, say

$$(-1 + g + g^2)x = 1.$$

By Proposition 4(i) we may suppose that S.G.(x) is contained in the cyclic subgroup generated by g. Thus by multiplying through an appropriate power of g we have

$$(-1 + g + g^2)(a_0 + a_1 g + \ldots + a_n g^n) = g^k$$

for some $n > 0$ and k, where $a_0 \neq 0$ and $a_n \neq 0$. The left-hand side contains the two distinct terms $-a_0$ and $a_n g^{n+2}$, contradicting the equality.

Thus all $g \in G$ have finite order. Choosing any such order n and any prime p dividing n there exists a $g \in G$ with order p. Now for any $a \in A$

$$1 + (1 - g)\{p + (p - 1)g + (p - 2)g^2 + \ldots + g^{p-1}\}a$$

has a right inverse x. As before, x may be taken to be a polynomial in g, and the equation

$$\{1 + (1 - g)(p + \ldots + g^{p-1})a\}x = 1$$

may be thought of as an equation in $R = R(G_p, A)$, where G_p is the cyclic group of order p. If J is the ideal in R generated by the element $1 + g + g^2 + \ldots + g^{p-1}$, passing to the factor ring R/J amounts to replacing g^{p-1} by $-1 - g - \ldots - g^{p-2}$. Then the above equation becomes

$$(1 + pa)\bar{x} = 1,$$

where $\bar{x} = a_0 + a_1 g + \ldots + a_{p-2} g^{p-2}$, say, and therefore

$$(1 + pa)a_0 = 1.$$

Since a was arbitrary, this implies $p \in \mathfrak{R}(A)$. No other prime $q \in \mathfrak{R}(A)$ for then $lp + mq = 1 \in \mathfrak{R}(A)$, for appropriate integers l and m, contradicting the fact that $\mathfrak{R}(A)$ is a proper ideal. Thus p is the only prime dividing any group element order and G is a p-group.

(ii) Since the Jacobson radical contains all nil ideals, it follows from (i) that G is a p-group, and it remains to prove that p is nilpotent in A. Choosing any $g \neq 1$, say g has order p^s, in the notation of Proposition 13

$$(1 - g)^k = f_0 - f_1 g + \ldots = 0$$

for some k, whence $f_0 = f_1 = \ldots = 0$ in A, and their g.c.d. p^t, which is a linear combination of the f_i, is 0 in A, as required.

(iii) Since $\mathfrak{W}(R)$ is a nil ideal, by (ii) it remains to prove that G is locally normal, and this follows immediately from the following lemma.

LEMMA. *If* $1 - g \in \mathfrak{W}(R)$, *then* $g \in \sigma(G)$.

Proof. Let H be the normal closure of $\{g\}$, that is the smallest normal subgroup containing g, and let J be the principal ideal generated by $1 - g$; since ωH is an ideal and contains $1 - g$, $J \subseteq \omega H$. Conversely,

$$g_1(1 - g)g_1^{-1} = 1 - g_1 g g_1^{-1} \in J$$

so that $1 - \bar{g} \in J$ for all conjugates $\bar{g}$ of g. But H is generated by the $\bar{g}$ and therefore, by Proposition 1, ωH is generated by $\{1 - \bar{g}\}$; hence $\omega H \subseteq J$, and therefore $\omega H = J$. Since $1 - g \in \mathfrak{W}(R)$, J is nilpotent; thus $(\omega H)^l \neq 0$, so that by Proposition 1, H is finite. Hence $g \in \sigma(G)$.

THEOREM 9. *Δ is nilpotent if and only if*
(i) *G is a finite p-group, and*
(ii) *p is nilpotent in A.*

Remark. Since a finite p-group is nilpotent (i.e. its lower central series becomes stationary with the value 1 after a finite number of steps), the nilpotency of Δ implies that of G. A more general statement is made in Proposition 17 below.

Proof. First suppose that $\Delta = \omega G$ is nilpotent. Then $(\omega G)^l \neq 0$ so that by Proposition 1, G is finite; the rest follows by Proposition 15. (Looking into the matter a little more closely, if $\Delta^n = 0$, it is easy to see that the group order is $\leqslant 2^{n-1}$, and that any independent set of generators of G contains at most $n - 1$ elements.)

Conversely let $p^t = 0$ in A and let us first suppose that G is an abelian finite p-group. The factors of $r = (1 - g_1) \ldots (1 - g_n)$ commute and since there is only a finite number of distinct factors possible, given k we may choose n large enough so that r contains a factor of the form $(1 - g)^k$. Thus it will be sufficient to prove that $(1 - g)^k = 0$ for all g and some fixed k, for then $\Delta^n = 0$. But by Proposition 13 (where g has order p^s)

$$(1 - g)^k = f_0 - f_1 g + \ldots$$

and by choosing k large enough we may make each f_i divisible by p^t, i.e., $f_i = 0$ in A, as required. (Since G is finite, a sufficiently large k will do for all g.)

Turning to the general case we proceed by induction on the group order. Let $R = R(G, A)$, $R' = R(G/C, A)$, where C is the centre of G, have fundamental ideals Δ and Δ' respectively. It is readily verified that in the canonical epimorphism $\phi: R \rightarrow R'$ we have $\phi(\Delta^k) = \Delta'^k$ for $k = 1, 2, \ldots$. Since $C \neq 1$, G/C is a p-group of smaller order than G and by induction $\Delta'^n = 0$ for some n. Hence $\Delta^n \subseteq \operatorname{Ker} \phi = \omega C$ and it remains to prove that ωC is nilpotent. Now the elements of ωC are sums of terms of the form $(1 - c)r$, where $c \in C$ and $r \in R$, and since $1 - c$ is in the centre of R, an element of $(\omega C)^m$ is a sum of terms each of which contains a factor of the form $(1 - c_1) \ldots (1 - c_m)$. But from the abelian case dealt with above, if Δ'' denotes the fundamental ideal of $R(C, A)$, $\Delta''^m = 0$ for some m, and it follows that $(\omega C)^m = 0$, which completes the proof.

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A ring is *locally nilpotent* if every finitely generated subring is nilpotent. (For the purpose of this definition we must depart from our convention that a ring must have a unit element.)

COROLLARY. *Δ is locally nilpotent if and only if*
(i) *G is a locally finite p-group, and*
(ii) *p is nilpotent in A.*

The deduction of this result from the theorem, which we omit, is analogous to that given by Losey **(14)**.

If G is a finite p-group and A is a field of characteristic p, the Maschke theorem says that the radical of R is non-zero, and it is natural to try to determine it. A classical result says that in this case the radical coincides with Δ. In the next theorem we generalize this result and give the converse.

However when we depart from finite G and artinian A, "radical" can be interpreted in several ways. The natural choice in this context appears to be the Wedderburn radical.

THEOREM 10. *The following statements are equivalent:*
(i) $\mathfrak{W}(R) = \Delta$;
(ii) $\mathfrak{P}(R) = \mathfrak{W}(R) = \Delta$;
(iii) *G is a locally normal p-group, A is semi-prime, and $p = 0$ in A.*

Remark. The conditions under (iii) certainly do not imply R is commutative or noetherian, so that there is no a priori reason why $\mathfrak{W}(R) = \mathfrak{P}(R)$.

Proof. (i) $\Rightarrow$ (ii). First $\Delta \subseteq \mathfrak{P}(R)$ by (34). Conversely $\mathfrak{W}(A) = A \cap \Delta = 0$ by Proposition 14; hence $0 = \mathfrak{P}(A) = \mathfrak{P}(R/\Delta)$, so that Δ is a radical ideal and therefore $\Delta \supseteq \mathfrak{P}(R)$.

(ii) $\Rightarrow$ (iii). We have just seen that (ii) implies A semi-prime; the rest follows from Proposition 15.

(iii) $\Rightarrow$ (i). If $g \in G$ the normal closure H of $\{g\}$ is a finite p-group and by Theorem 9, $\Delta_1 = \Delta(H, A)$ is nilpotent, say $\Delta_1{}^n = 0$. Now an element of the ideal $(\omega H)^n$ of R is a sum of terms of the form

$$x = (1 - g_1)r_1(1 - g_2)r_2 \dots (1 - g_n)r_n, \qquad g_i \in H, r_i \in R,$$

and by the normality of H it follows easily that x is a sum of terms of the form

$$y = (1 - g_1')(1 - g_2') \dots (1 - g_n')r, \qquad g_i' \in H, r \in R;$$

hence $y = 0$, since $(1 - g_1') \dots (1 - g_n') \in \Delta_1{}^n$. Thus $1 - g$ is contained in the nilpotent ideal ωH; hence $1 - g \in \mathfrak{W}(R)$ for all g, and therefore

$$\Delta \subseteq \mathfrak{W}(R).$$

Finally, since $\mathfrak{P}(A) = \mathfrak{P}(R/\Delta) = 0$, Δ is a radical ideal; hence

$$\Delta \supseteq \mathfrak{P}(R) \supseteq \mathfrak{W}(R).$$

This theorem gives necessary and sufficient conditions for $\mathfrak{W}(R) = \Delta$, but only sufficient conditions for $\mathfrak{P}(R) = \Delta$. In the next proposition we give a partial converse for the latter case, and look into the two other questions suggested by Proposition 15. For convenience we restate two of our previous results.

PROPOSITION 16. (i) *If Δ is nil, then G is a p-group and p is nilpotent in A.*

(ii) *If G is a locally finite p-group and p is nilpotent in A, then Δ is nil.*

(iii) *If $\mathfrak{R}(R) = \Delta$, then G is a p-group, $\mathfrak{R}(A) = 0$, and $p = 0$ in A.*

(iv) *If G is a locally finite p-group, $\mathfrak{R}(A) = 0$, and $p = 0$ in A, then $\mathfrak{R}(R) = \Delta$.*

(v) *If $\mathfrak{P}(R) = \Delta$, then G is a p-group, $\mathfrak{P}(A) = 0$, and $p = 0$ in A.*

(vi) *If G is a locally normal p-group, $\mathfrak{P}(A) = 0$, and $p = 0$ in A, then $\mathfrak{P}(R) = \Delta$.*

Remarks. Kaplansky **(12)** states as an open problem whether Δ nil implies G locally finite. (He restricts A to be a field of characteristic p.) This would follow, by the corollary to Theorem 9, from the unproven conjecture in ring theory that a nil ring is locally nilpotent.

Novikov's counterexamples to Burnside's conjecture show that not all p-groups are locally finite.

Combining (iv) with the previous theorem we have a substantial class of R for which

$$\mathfrak{R}(R) = \mathfrak{P}(R) = \mathfrak{W}(R) = \Delta.$$

Proof. (i) occurs in Proposition 15.

(ii) follows from the corollary to Theorem 9; it also follows directly from the theorem: having Δ nil is a property reducible to the local case; that is, if $\Delta(H, A)$ is nil for each finitely generated subgroup H of G, then $\Delta(G, A)$ is nil, as is clear. The present case is thus reduced to proving that Δ is nil if G is a finite p-group and p is nilpotent in A; by the theorem, Δ is in fact nilpotent.

(iv) By (ii), Δ is nil and therefore $\Delta \subseteq \mathfrak{R}(R)$. On the other hand,

$$\delta(\mathfrak{R}(R)) \subseteq \mathfrak{R}(A)(=0)$$

since δ is an epimorphism, so that $\mathfrak{R}(R) \subseteq \operatorname{Ker} \delta = \Delta$.

(v) $\mathfrak{P}(A) = \mathfrak{P}(R/\mathfrak{P}(R)) = 0$, and since the prime radical is always nil, the remaining facts follow from (i).

(vi) occurs in Theorem 10.

The definition of powers of Δ can be continued transfinitely by putting $\Delta^{\nu+1} = \Delta^{\nu}\Delta$, and

$$\Delta^{\nu} = \bigcap_{\mu<\nu} \Delta^{\mu}$$

if ν is a limit ordinal. Let ω_0 denote the first infinite ordinal. Jennings **(11)** has shown that $\Delta^{\omega_0} = 0$ if G is a finitely generated torsion-free nilpotent group and A is a field of characteristic 0. Losey **(13)** proved that $\Delta^{\omega_0} = 0$ if G is a finitely generated nilpotent group and $A = Z$. The reason for the various hypotheses is somewhat clarified by the following proposition. We let $G = G_1 \supseteq G_2 \supseteq \ldots \supseteq G_\nu \supseteq \ldots$ denote the lower central series of G.

PROPOSITION 17. (i) *For any ordinal* ν

$$\Omega\Delta^\nu \supseteq G_\nu$$

or, equivalently,

$$\Delta^\nu \supseteq \omega G_\nu.$$

In particular, $\Delta^{\omega_0} = 0$ *implies* $G_{\omega_0} = 1$ *(hence G is "residually nilpotent").*

(ii) *If $o(G)$ contains an $n > 1$ which is a unit in A, then $\Omega\Delta^\nu \neq 0$ for all ν; hence $\Delta_\nu \neq 0$, and, in particular,* $\Delta^{\omega_0} \neq 0$

Remark. Losey **(13)** has proved the remarkable result that if $A = Z$, then $\Omega\Delta^n = G_n$ (for all G).

Proof. (i) By (4) and (5) the two inclusions are equivalent. Losey **(13)** has proved the result when ν is finite, and his method (which goes back to Jennings **(10)**) extends readily to the general case.

(ii) The following simple proof is due to D. Sussman. If g has order n and n is a unit in A, we shall show that each Δ^ν contains $1 - g \neq 0$.

(a) $1 - g \in \Delta = \Delta^1$;

(b) if $1 - g \in \Delta^\nu$, then

$$\frac{n - (1 + g + \ldots + g^{n-1})}{n} \cdot (1 - g) = 1 - g \in \Delta^{\nu+1},$$

the first factor on the left being in Δ since it has norm 0;

(c) if $1 - g \in \Delta^\mu$ for each $\mu < \nu$, ν a limit ordinal, then clearly $1 - g \in \Delta^\nu$.

Part (ii) can also be derived from the formula

$$(36) \qquad (1 - g)(1 - g^2) \ldots (1 - g^{p-1}) = p - (1 + g + \ldots + g^{p-1}),$$

where g has order p, a prime. Sussman noted the following elegant proof of (36). Clearly we may think of (36) as an equation in $R(G, Z)$, where G is the cyclic group of order p. Now $r = (1 - g) \ldots (1 - g^{p-1}) \in \Delta$ so that r has norm 0. Also each group automorphism $g \to g^i$, $0 < i < p$, leaves r invariant so that all the coefficients $r(g^i)$, other than $r(1)$, must be equal, and we have $r = mp - m(1 + g + \ldots + g^{p-1})$ for some integer m. If ζ is a pth root of unity, the mapping $g \to \zeta$ extends obviously to a ring homomorphism ϕ of R into the complex field. By (27), $\phi r = (1 - \zeta) \ldots (1 - \zeta^{p-1}) = p$, and by (29), $\phi(mp - m(1 + \ldots + g^{p-1})) = mp$. Hence $p = mp$, $m = 1$, and (36) is proved.

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McGill University,
Montreal, Quebec