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THE DETERMINATION OF CALORIC MORPHISMS ON EUCLIDEAN DOMAINS

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Dedicated to Professor Masayuki Itô in honour of his sixtieth birthday

Abstract. Let D be a domain in $\mathbb{R}^{m+1}$ and E be a domain in $\mathbb{R}^{n+1}$. A pair of a smooth mapping $f : D \to E$ and a smooth positive function φ on D is called a caloric morphism if $\varphi \cdot u \circ f$ is a solution of the heat equation in D whenever u is a solution of the heat equation in E. We give the characterization of caloric morphisms, and then give the determination of caloric morphisms. In the case of $m < n$, there are no caloric morphisms. In the case of $m = n$, caloric morphisms are generated by the dilation, the rotation, the translation and the Appell transformation. In the case of $m > n$, under some assumption on f, every caloric morphism is obtained by composing a projection with a direct sum of caloric morphisms of $\mathbb{R}^{n+1}$.

§1. Introduction

For a non-negative integer k, $\mathbb{R}^{k+1}$ denotes the $k+1$-dimensional Euclidean space. The coordinates in $\mathbb{R}^{k+1}$ is denoted by (t, x) or (x_0, x) where $x = (x_1, \dots, x_k)$.

We shall use the following notation:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\right), \quad \Delta = \sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2}, \quad H = \frac{\partial}{\partial t} - \Delta.$$

A C^2-function h is said to be caloric if h satisfies the heat equation

$$Hh = 0.$$

Since the heat operator H is hypoelliptic (see, e.g. [9]), every caloric function is infinitely differentiable.

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Let m, n be positive integers and D a domain in $\mathbb{R}^{m+1}$. We denote by $(t,x) = (t, x_1, \dots, x_m)$, $(\tau, y) = (\tau, y_1, \dots, y_n)$ the points of $\mathbb{R}^{m+1}$, $\mathbb{R}^{n+1}$ respectively. We consider a mapping $f(t,x) = (f_0(t,x), f_1(t,x), \dots, f_n(t,x)) : D \to \mathbb{R}^{n+1}$ and a weight function φ which preserve solutions of the heat equation in the following sense. *A pair (f, φ) of C^2-mapping $f : D \to \mathbb{R}^{n+1}$ and a positive C^2-function φ on D is said to be a caloric morphism if $f(D)$ is a domain in $\mathbb{R}^{n+1}$ and if for every caloric function u on $f(D)$, $\varphi(t,x)(u \circ f)(t,x)$ is also a caloric function on D.*

In the case of $m = n$, the following three typical caloric morphisms are known.

The Appell transformation

Let $D = (0, \infty) \times \mathbb{R}^n$ (resp. $= (-\infty, 0) \times \mathbb{R}^n$). Put

$$f(t,x) = \left(-\frac{1}{t}, \frac{x}{t}\right), \quad \varphi(t,x) = \frac{1}{\sqrt{4\pi|t|}^{\,n}} e^{-|x|^2/4t}.$$

Then $f(D) = (-\infty, 0) \times \mathbb{R}^n$ (resp. $= (0, \infty) \times \mathbb{R}^n$) and (f, φ) is a caloric morphism.

The dilation and the rotation in x

Let $\lambda > 0$ and U be an (n,n)-orthogonal matrix. Put

$$f(t,x) = (\lambda^2 t, \lambda U x), \quad \varphi(t,x) = 1.$$

Then (f, φ) is a caloric morphism from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^{n+1}$.

The translation

Let $a \in \mathbb{R}$ and b, $c \in \mathbb{R}^n$. Put

$$f(t,x) = (t + a, x + tb + c), \quad \varphi(t,x) = e^{\frac{1}{4}|b|^2 t + \frac{1}{2} b \cdot x}.$$

Then (f, φ) is a caloric morphism from $\mathbb{R}^{n+1}$ onto $\mathbb{R}^{n+1}$.

We give two simple examples in the case of $m > n$.

Example 1. The symmetrization in $\mathbb{R}^m$ with respect to a subspace with codimension 2.

Let $m \geqq 4$, $n = m - 2$ and $D = \{(t,x)\ ;\ t > 0, |x'| > 0\}$ (resp. $D = \{(t,x); t < 0, |x'| > 0\}$), where $x' = (x_1, x_2, x_3, 0, \dots, 0)$ for $x = (x_1, \dots, x_m)$. Put

$$\begin{cases} f_0(t) = -t^{-1}, \\ f_1(t,x) = t^{-1}|x'|, \\ f_j(t,x) = t^{-1} x_{j+2}, \qquad 2 \leqq j \leqq n, \end{cases}$$

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$$\varphi(t,x) = |x'|^{-1}|t|^{-(m-2)/2}\exp\Big(-\frac{|x|^2}{4t}\Big).$$

Then $f(D) = \{(\tau, y); \tau < 0,\, y_1 > 0\}$ (resp. $f(D) = \{(\tau, y); \tau > 0,\, y_1 < 0\}$) and (f, φ) is a caloric morphism.

EXAMPLE 2. The projection in x.

Let h be an arbitrary positive caloric function on $\mathbb{R}^{m-n+1}$. Put

$$f(t, x_1, \dots, x_m) = (t, x_1, \dots, x_n), \quad \varphi(t,x) = h(t, x_{n+1}, \dots, x_m).$$

Then (f, φ) is a caloric morphism from $\mathbb{R}^{m+1}$ onto $\mathbb{R}^{n+1}$.

In the case of $m = n$, Leutwiler [7] proved that every caloric morphism has the following form:

$$f(t,x) = \Big(\frac{\alpha t + \beta}{\gamma t + \delta}, \frac{Rx + tv + w}{\gamma t + \delta}\Big),$$
$$\varphi(t,x) = \begin{cases} \frac{C}{|\gamma t+\delta|^{n/2}}\exp\Big[-\frac{|\gamma Rx+\gamma w-\delta v|^2}{\gamma|\gamma t+\delta|}\Big], & \gamma \neq 0, \\ C\exp\Big[\frac{|v|^2}{4}t + \frac{1}{2}v\cdot Rx\Big], & \gamma = 0, \end{cases} \tag{0}$$

where $\alpha, \beta, \gamma, \delta$ are real numbers with $\alpha\delta - \beta\gamma = 1$, $v, w \in \mathbb{R}^n$, R is an n-dimensional orthogonal matrix, $C > 0$ and $\cdot$ denotes the inner product of $\mathbb{R}^n$. It is a composition of the above three morphisms: the Appell transformation, the dilation, the translation.

The aim of this paper is to extend this to the case of $m \neq n$.

We first give a general characterization of caloric morphisms, which is essentially obtained by Leutwiler. As its corollary, there are no caloric morphism if $m < n$. Also by virtue of the characterization, we obtain a new systematic way to construct a caloric morphism by a "direct sum" of caloric morphisms in the case of $m > n$. It is remarkable that the direct sum gives caloric morphisms of new type such that f_0 is a sum of fractional linear functions. Note that in the case of $m = n$, f_0 is just a fractional linear function.

Our main result is the determination of caloric morphisms (f, φ) in the case of $m > n$ under the assumption that each f_i, $1 \leqq i \leqq n$ is a polynomial in x for every t and that f_0 is real analytic. Under the assumption, we can give an explicit form of caloric morphisms (Theorem 7 below). Although it seems to be complicated, it turns out to be a direct sum of the caloric morphisms of form (0) composed with a projection, as is shown in Corollary 10.

§2. Characterization of caloric morphisms

Definition 1. A pair (f, φ) of C^2-mapping $f : D \to \mathbb{R}^{n+1}$ and a positive C^2-function on D is said to be a caloric morphism, if $f(D)$ is a domain and if for every caloric function u on $f(D)$, $\varphi(t,x)(u \circ f)(t,x)$ is also a caloric function on D.

Remark 1. Using derivatives in the sense of distribution, we may assume f and φ to be continuous rather than of C^2. For the sake of simplicity, we assume here that f and φ are of C^2.

Theorem 1. *Let $f = (f_0, f_1, \ldots, f_n) : D \to \mathbb{R}^{n+1}$ be a C^2-mapping such that $f(D)$ is a domain and let φ be a positive C^2-function on D. Then the following statements are equivalent*:

(i) *(f, φ) is a caloric morphism.*

(ii) *For every polynomial $P(\tau, y)$ which is caloric and of degree $\leqq 4$,*

$$\varphi(t,x)(P \circ f)(t,x)$$

is caloric on D.

(iii) *f and φ satisfy the following equations*:

$$H\varphi = 0, \tag{1}$$

$$\varphi H f_i = 2\nabla\varphi \cdot \nabla f_i, \quad 1 \leqq i \leqq n, \tag{2}$$

$$\nabla f_0 = 0, \tag{3}$$

$$\nabla f_i(t,x) \cdot \nabla f_j(t,x) = \delta_{ij} \frac{df_0}{dt}(t), \quad 1 \leqq i, j \leqq n, \tag{4}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^m$.

(iv) *There exists a continuous function $\lambda(t) \geqq 0$ on D such that*

$$H\{\varphi(u \circ f)\}(t,x) = \lambda(t)^2 \varphi(t,x)(Hu \circ f)(t,x) \tag{5}$$

holds for every C^2 function u on $f(D)$ where H in the right hand side means the heat operator on $\mathbb{R}^{n+1}$.

Remark 2. By (3), f_0 depends only on t. And (4) shows that $df_0/dt \geqq 0$ and $|\nabla f_j(t,x)|^2$ is independent of x, where $|\cdot|$ denotes the norm of $\mathbb{R}^m$.

Proof.
(i)⇒(ii) is trivial.
(ii)⇒(iii): By the chain rule,

$$H\{\varphi(P\circ f)\} = H\varphi(P\circ f) + \sum_{i=0}^{n}(\varphi H f_i - 2\nabla\varphi\cdot\nabla f_i)\frac{\partial P}{\partial y_i}\circ f - \varphi\sum_{i,j=0}^{n}(\nabla f_i\cdot\nabla f_j)\frac{\partial^2 P}{\partial y_i\partial y_j}\circ f. \tag{6}$$

Let $P=1$. Then we have $H\varphi=0$. Let $P(y_0,y)=y_i$, $1\leqq i\leqq n$ in the equation (6). Then we obtain

$$\varphi H f_i = 2\nabla\varphi\cdot\nabla f_i, \quad 1\leqq i\leqq n.$$

Take a point $p\in D$ and put $q=f(p)$. Let $P(y_0,y)=(y_i-q_i)(y_j-q_j)$, $1\leqq i,j\leqq n$, $i\neq j$ in the equation (6). Since $(\partial^2 P/\partial y_i\partial y_j)(q)=1$ and the other derivatives of P vanish at q, we have

$$\nabla f_i(p)\cdot\nabla f_j(p)=0, \quad 1\leqq i,j\leqq n,\ i\neq j.$$

Since p is arbitrary,

$$\nabla f_i\cdot\nabla f_j=0, \quad 1\leqq i,j\leqq n,\ i\neq j, \tag{7}$$

in D. Let $P(y_0,y)=(y_0-q_0)^2+(y_0-q_0)(y_i-q_i)^2+\frac{1}{12}(y_i-q_i)^4$, $1\leqq i\leqq n$. Since $(\partial^2 P/\partial y_0^2)(q)=1$ and the other derivatives of order $\leqq 2$ vanish at q, we have

$$|\nabla f_0(p)|^2=0, \text{ and thus } \nabla f_0(p)=0. \tag{8}$$

Since p is arbitrary, (3) holds. Finally, let $P(y_0,y)=y_0-q_0+\frac{1}{2}(y_i-q_i)^2$, $1\leqq i\leqq n$. Since $(\partial P/\partial y_0)(q)=(\partial^2 P/\partial y_i^2)(q)=1$ and the other derivatives vanish at q, we have

$$\varphi(p)Hf_0(p)=\varphi(p)|\nabla f_i(p)|^2, \quad 1\leqq i\leqq n. \tag{9}$$

Combining (7), (8) and (9), we obtain (4).

(iii)⇒(iv): Let u be of C^2 in $f(D)$. By the chain rule

$$H\{\varphi(u\circ f)\} = H\varphi(u\circ f) + \sum_{i=0}^{n}(\varphi H f_i - 2\nabla\varphi\cdot\nabla f_i)\frac{\partial u}{\partial y_i}\circ f - \varphi\sum_{i,j=0}^{n}(\nabla f_i\cdot\nabla f_j)\frac{\partial^2 u}{\partial y_i\partial y_j}\circ f. \tag{10}$$

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Substituting (1)–(4) into (10), we have

$$H\{\varphi(u\circ f)\} = \varphi H f_0 \frac{\partial u}{\partial y_0}\circ f - \varphi \sum_{i=1}^{n} |\nabla f_i|^2 \frac{\partial^2 u}{\partial y_i^2}\circ f = \varphi \frac{df_0}{dt} Hu\circ f.$$

Putting $\lambda(t) = (df_0/dt(t))^{1/2}$, we obtain

$$H\{\varphi(u\circ f)\}(t,x) = \lambda(t)^2\varphi(t,x)(Hu\circ f)(t,x).$$

Note that $\lambda(t) = |\nabla f_i(t,x)|$ by (4).

(iv)⇒(i) is evident. □

COROLLARY 2. *For every caloric morphism* (f,φ), f *and* φ *are of* C^∞.

Proof. By (2), φf_i is caloric ($1 \leqq i \leqq n$), so φf_i is of C^∞. Since $\varphi > 0$ and φ is caloric, f_i is of C^∞, $1 \leqq i \leqq n$. f_0 is of C^∞ by (4). Thus f is a C^∞-mapping. □

COROLLARY 3. *Let* (f,φ) *be a caloric morphism from* D *to* $\mathbb{R}^{n+1}$. *Then for any* C^2*-function* u *on* $f(D)$, *we have the following implications*:

$$Hu \geqq 0 \Longrightarrow H\{\varphi(u\circ f)\} \geqq 0,$$
$$Hu \leqq 0 \Longrightarrow H\{\varphi(u\circ f)\} \leqq 0.$$

They immediately follow from (5).

COROLLARY 4. (i) *Let* $(f,\varphi) = ((f_0,\ldots,f_n),\varphi)$ *be a caloric morphism from* $D\subset\mathbb{R}^{m+1}$ *to* $\mathbb{R}^{n+1}$. *Then* $f_0'(t) > 0$ *on* D.

(ii) *If* $n > m$, *there are no caloric morphisms.*

Proof. (i) Suppose that $f_0'(t_0) = 0$ for some $(t_0,x_0)\in D$. Let $I\subset\mathbb{R}$ be the connected component of $\{t; f_0'(t) = 0\}$ such that $t_0 \in I$. Since f_0 is a non-decreasing function, $f_0(t) \neq f_0(t_0)$ for all $t\notin I$. So we have

$$f(\{(t,x)\in D; t\in I\}) = f(D)\cap\{(\tau,y)\in\mathbb{R}^{n+1}; \tau = f_0(t_0)\}.$$

Then by (4)

$$\nabla f_i(t,x) = 0, \quad (t,x)\in D,\ t\in I,\ 1\leqq i\leqq n.$$

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This and (2) imply

$$\frac{\partial f_i}{\partial t}(t,x) = 2\nabla \log \varphi \cdot \nabla f_i = 0, \quad (t,x) \in D,\, t \in I,\, 1 \leqq i \leqq n.$$

Therefore the set $f(\{(t,x) \in D; t \in I\})$ consists of one point. Thus the set $f(D) \cap \{(\tau,y); \tau = f_0(t_0)\}$ consists of one point. It is contrary to the condition that $f(D)$ is a domain. Therefore $f_0'(t) > 0$ for all t.

(ii) Let $m < n$. By virtue of (4), $\nabla f_1, \ldots, \nabla f_n$ are n orthogonal vectors in $\mathbb{R}^m$ with same length. Since $n > m$, we have $\nabla f_1 = \cdots = \nabla f_n = 0$ in D. Then (4) gives $f_0' = 0$ in D. This contradicts to (i). ∎

Let m, n, k be positive integers and let D, E be domains in $\mathbb{R}^{m+1}$, in $\mathbb{R}^{n+1}$, respectively. If $(f,\varphi) : E \to \mathbb{R}^{k+1}$ and $(g,\psi) : D \to \mathbb{R}^{n+1}$ are caloric morphisms such that $g(D) \subset E$, then we can make a caloric morphism $(F,\Phi) : D \to \mathbb{R}^{k+1}$ from (f,φ) and (g,ψ) by the composition $(F,\Phi) = (f \circ g, (\varphi \circ g)\psi)$.

The next proposition provides a manner for the construction of new caloric morphisms.

PROPOSITION 5. *Let $l, m_1, \ldots, m_l, n$ be positive integers and I be an open interval. For each $j = 1, \ldots, l$, suppose that D_j is a domain in $\mathbb{R}^{m_j}$ and that $(g_j, \varphi_j) = ((g_{j0}, g_{j1}, \ldots, g_{jn}), \varphi_j)$ is a caloric morphism : $I \times D_j \subset \mathbb{R}^{m_j+1} \to \mathbb{R}^{n+1}$. Put*

$$\begin{aligned}
&f_0(t) = g_{10}(t) + \cdots + g_{l0}(t),\\
&f_i(t, x_1, \ldots, x_{m_1+\cdots+m_l}) = g_{1i}(t, x_1, \ldots, x_{m_1})\\
&\qquad + g_{2i}(t, x_{m_1+1}, \ldots, x_{m_1+m_2}) + \cdots\\
&\qquad + g_{li}(t, x_{m_1+\cdots+m_{l-1}+1}, \ldots, x_{m_1+\cdots+m_l}), \quad 1 \leqq i \leqq n,\\
&\varphi(t, x_1, \ldots, x_{m_1+\cdots+m_l}) = \varphi_1(t, x_1, \ldots, x_{m_1})\varphi_2(t, x_{m_1+1}, \ldots, x_{m_1+m_2}) \cdots\\
&\qquad \varphi_l(t, x_{m_1+\cdots+m_{l-1}+1}, \ldots, x_{m_1+\cdots+m_l}).
\end{aligned}$$

Then $(f,\varphi) : I \times D_1 \times \cdots \times D_l \subset \mathbb{R}^{m_1+\cdots+m_l+1} \to \mathbb{R}^{n+1}$ is a caloric morphism.

We call the above caloric morphism (f,φ) the direct sum of $(g_1,\varphi_1), \ldots, (g_l,\varphi_l)$.

Proof. For each j, we denote by H_j, ∇_j and Δ_j the heat operator, the gradient and the Laplacian in $\mathbb{R}^{m_j+1}$. The heat operator, the gradient and the Laplacian in $\mathbb{R}^{m_1+\cdots+m_l+1}$ are denoted by H, ∇ and Δ. Since (g_j, φ_j) is a caloric morphism, (1), (2) and (4) show

$$H_j\varphi_j = 0, \quad \varphi_j H_j g_{ji} = 2\nabla_j\varphi_j \cdot \nabla_j g_{ji}, \quad \nabla_j g_{ji} \cdot \nabla_j g_{jk} = \delta_{ik}\frac{dg_{j0}}{dt}, \qquad 1 \leqq i, k \leqq n,\ 1 \leqq j \leqq l.$$

Using

$$\begin{aligned} \nabla f_i &= (\nabla_1 g_{1i}, \nabla_2 g_{2i}, \dots, \nabla_l g_{li}), \\ \nabla\varphi &= \varphi\Big(\frac{\nabla_1\varphi_1}{\varphi_1}, \frac{\nabla_2\varphi_2}{\varphi_2}, \dots, \frac{\nabla_l\varphi_l}{\varphi_l}\Big), \\ Hf_i &= H_1 g_{1i} + H_2 g_{2i} + \cdots + H_l g_{li}, \end{aligned}$$

we have

$$\begin{aligned} 2\nabla\varphi \cdot \nabla f_i &= \varphi\Big(\frac{2\nabla_1\varphi_1 \cdot \nabla_1 g_{1i}}{\varphi_1}, \frac{2\nabla_2\varphi_2 \cdot \nabla_2 g_{2i}}{\varphi_2}, \dots, \frac{2\nabla_l\varphi_l \cdot \nabla_l g_{li}}{\varphi_l}\Big) \\ &= \varphi(H_1 g_{1i} + H_2 g_{2i} + \cdots + H_l g_{li}) \\ &= \varphi H f_i, \quad 1 \leqq i \leqq n, \end{aligned}$$

and

$$\begin{aligned} \nabla f_i \cdot \nabla f_k &= \nabla_1 g_{1i} \cdot \nabla_1 g_{1k} + \nabla_2 g_{2i} \cdot \nabla_2 g_{2k} + \cdots + \nabla_l g_{li} \cdot \nabla_l g_{lk} \\ &= \delta_{ik}\Big(\frac{dg_{10}}{dt} + \frac{dg_{20}}{dt} + \cdots + \frac{dg_{l0}}{dt}\Big) \\ &= \delta_{ik}\frac{df_0}{dt}. \end{aligned}$$

On the other hand, since

$$\begin{aligned} \frac{\partial\varphi}{\partial t} &= \varphi\Big(\frac{1}{\varphi_1}\frac{\partial\varphi_1}{\partial t} + \frac{1}{\varphi_2}\frac{\partial\varphi_2}{\partial t} + \cdots + \frac{1}{\varphi_l}\frac{\partial\varphi_l}{\partial t}\Big), \\ \Delta\varphi &= \varphi\Big(\frac{\Delta_1\varphi_1}{\varphi_1} + \frac{\Delta_2\varphi_2}{\varphi_2} + \cdots + \frac{\Delta_l\varphi_l}{\varphi_l}\Big), \end{aligned}$$

we obtain $H\varphi = 0$. Thus (f, φ) is a caloric morphism. ∎

§3. Main result

In the case of $m = n$, the form of caloric morphism is explicitly determined by Leutwiler [7]. So hereafter, we assume $m > n$ in the rest of this paper.

In the sequel, we shall determine caloric morphisms (f, φ), $f = (f_0, f_1, \ldots, f_n)$ in the case that f_i, $1 \leqq i \leqq n$ is a polynomial of x for each t and that f_0 is real analytic.

PROPOSITION 6. *Let (f, φ) be a caloric morphism and assume that f_i, $1 \leqq i \leqq n$ is a polynomial of x for each fixed t. Then*

$$f_i(t, x) = \sum_{j=1}^{m} a_{ij}(t) x_j + b_i(t), \quad 1 \leqq i \leqq n,$$

where a_{ij}, b_i, $1 \leqq i \leqq n$, $1 \leqq j \leqq m$ are C^∞-functions.

Proof. Let t be fixed. Suppose that $f_i(t, x)$ is a polynomial of degree $l \geqq 1$. Write $f_i(t, x) = h(t, x) + g(t, x)$, where h is a homogeneous polynomial of degree l and g is a polynomial of degree $\leqq l - 1$. Since $\nabla h \neq 0$, the degree of the polynomial $|\nabla f_i|^2 = |\nabla h|^2 + 2\nabla h \cdot \nabla g + |\nabla g|^2$ is equal to $2l - 2$. On the other hand, $|\nabla f_i|^2$ is of degree 0 by (4) of Theorem 1. Thus $\deg f_i \leqq 1$. □

Remark 3. We cannot replace real analytic functions in the place of polynomials in the above proposition. In the above Example 1, f_1 is not a polynomial.

Main result of this paper is the following

THEOREM 7. *Let $(f, \varphi) = ((f_0, f_1, \ldots, f_n), \varphi)$ be a caloric morphism defined on a domain $D \subset \mathbb{R}^{m+1}$. Assume that for each $1 \leqq i \leqq n$ and each t, $f_i(t, x)$ is a polynomial of x and that $f_0(t)$ is real analytic.*

Then there exist a positive integer $k \leqq m/n$ and an orthogonal coordinate of $\mathbb{R}^m$ denoted by $(x_1, \ldots, x_m)$ again with four families α_i, $1 \leqq i \leqq k$, β_i, $1 \leqq i \leqq k$, δ_i, $0 \leqq i \leqq n$ and γ_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$ of real numbers satisfying $\alpha_i > 0$ and $\beta_i \neq \beta_j$, $i \neq j$, and a positive caloric function $h = h(t, x_{kn+1}, \ldots, x_m)$ (in the case of $m = nk$, h is a positive constant) such that f and φ are of form (I) *or* (II).

(I)

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_i(t,x) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i, \quad 1 \leqq i \leqq n,$$

$$\varphi(t,x) = h \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{n/2}} \exp \sum_{i=1}^{n} \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)},$$

(II)

$$f_0(t) = \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_i(t,x) = \alpha_1(x_i + \gamma_{i1}t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i, \quad 1 \leqq i \leqq n,$$

$$\varphi(t,x) = h \exp \sum_{i=1}^{n} \left[\frac{\gamma_{i1}^2}{4}t + \frac{\gamma_{i1}}{2}x_i\right] \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{n/2}} \exp \sum_{i=1}^{n} \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)}.$$

First we shall prove the assertion of the theorem in the case of $n = 1$ under the assumption that $\log \varphi$ is a polynomial of x of degree $\leqq 2$.

LEMMA 8. *Let $(f,\varphi) = ((f_0, f_1), \varphi)$ be a caloric morphism from $D \subset \mathbb{R}^{m+1}$ to $\mathbb{R}^{1+1}$. Assume that f_1 and φ are of the following form*:

$$f_1(t,x) = \sum_{j=1}^{m} a_j(t)x_j + b(t),$$

$$\varphi(t,x) = \exp\left(\frac{1}{4}x \cdot U(t)x + v(t)\cdot x + w(t)\right),$$

where $a_1, \ldots, a_m$, b and w are C^∞-functions, v is a C^∞-vector and where U is a symmetric (m,m)-matrix of C^∞-functions.

Then there exist a positive integer $k \leqq m$ and an orthogonal coordinate of $\mathbb{R}^m$ denoted by $(x_1, \ldots, x_m)$ again with four families α_i, $1 \leqq i \leqq k$, β_i, $1 \leqq i \leqq k$, δ_i, $i = 0, 1$ and γ_i, $1 \leqq i \leqq k$ of real numbers satisfying $\alpha_i > 0$ and $\beta_i \neq \beta_j$, $i \neq j$, and a positive caloric function $h = h(t, x_{k+1}, \ldots, x_m)$

(in the case of $m = k$, h is a positive constant) such that f and φ are of form (1) *or* (2).

(1)

$$\begin{aligned} f_0(t) &= \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, \\ f_1(t,x) &= \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1, \\ \varphi(t,x) &= h(t,x) \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)}, \end{aligned}$$

if $U(t_0)$ is invertible or $a(t_0)$ is orthogonal to the zero-eigenspace of $U(t_0)$ for some t_0.

(2)

$$\begin{aligned} f_0(t) &= \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0, \\ f_1(t,x) &= \alpha_1(x_1 + \gamma_1 t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1, \\ \varphi(t,x) &= h(t,x) \exp\left[\frac{\gamma_1^2}{4}t + \frac{\gamma_1}{2}x_1\right] \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)}, \end{aligned}$$

otherwise.

Proof of Lemma 8. We may assume $t_0 = 0$ by some translation of t. Since (f, φ) is a caloric morphism, f_1 and $\log \varphi$ satisfy the equations

$$\begin{aligned} &\frac{\partial \log \varphi}{\partial t} - \Delta \log \varphi - |\nabla \log \varphi|^2 = 0, \\ &H f_1 = 2 \nabla \log \varphi \cdot \nabla f_1, \end{aligned}$$

by (1) and (2). Then we have the following differential equations

$$\begin{aligned} &U' = U^2, \quad v' = Uv, \quad w' = \frac{|v|^2}{4} + \frac{\operatorname{tr} U}{2}, \\ &a' = Ua, \quad b' = a \cdot v, \end{aligned}$$

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where $a = (a_1, \dots, a_m)$ and $\operatorname{tr} U$ denotes the trace of the matrix U.

Since $U(0)$ is real symmetric, we have the spectral decomposition $U(0) = \sum_{j=1}^{l} \lambda_j P_j$, where λ_j is a real eigenvalue of $U(0)$ with multiplicity n_j, and P_j is the orthogonal projection of $\mathbb{R}^m$ to the corresponding eigenspace. Since $U(t)$ is the solution of $U' = U^2$,

$$U(t) = \sum_{j=1}^{l} \frac{\lambda_j}{1 - \lambda_j t} P_j,$$

and so the solutions of $a' = Ua$, $v' = Uv$ are

$$a(t) = \sum_{j=1}^{l} \frac{1}{1 - \lambda_j t} P_j a_0, \quad v(t) = \sum_{j=1}^{l} \frac{1}{1 - \lambda_j t} P_j v_0,$$

where $a_0 = a(0)$ and $v_0 = v(0)$.

Let k be the cardinal of $\{P_j; P_j a_0 \neq 0\}$ (note that $a_0 \neq 0$ because of (4) and Corollary 4). We may assume $P_j a_0 \neq 0$, $1 \leqq j \leqq k$, $P_j a_0 = 0$, $k < j \leqq l$ and $\lambda_j \neq 0$, $1 < j \leqq k,\ k+1 < j \leqq l$ by some rearrangement of $\lambda_1, \dots, \lambda_l$, if necessary.

Assume that $U(0)$ is invertible. Then $\lambda_j \neq 0$ for all j and the solutions of $b' = a \cdot v$ and $w' = |v|^2/4 + \operatorname{tr} U/2$ are

$$b(t) = \sum_{j=1}^{k} \frac{P_j a_0 \cdot P_j v_0}{\lambda_j (1 - \lambda_j t)} + \delta_1,$$

$$w(t) = \sum_{j=1}^{l} \Big(\frac{|P_j v_0|^2}{4\lambda_j (1 - \lambda_j t)} - \frac{n_j}{2} \log(1 - \lambda_j t) \Big) + \delta_2$$

with some constants δ_1 and δ_2. By $f_0' = |\nabla f_1|^2$ we have

$$f_0(t) = \int |a(t)|^2 \, dt = \sum_{j=1}^{k} \frac{|P_j a_0|^2}{\lambda_j (1 - \lambda_j t)} + \delta_0$$

with some constant δ_0. Put

$$\alpha_j = \frac{|P_j a_0|}{|\lambda_j|} > 0, \quad e_j = \frac{\lambda_j P_j a_0}{|\lambda_j P_j a_0|} \in \mathbb{R}^m, \quad \beta_j = \frac{1}{\lambda_j}, \quad 1 \leqq j \leqq k.$$

Note that $\beta_1, \dots, \beta_k$ are mutually distinct. Adding $m - k$ eigenvectors of $U(0)$ to $\{e_1, \dots, e_k\}$, in the case of $m > k$, we obtain an orthonormal basis

$\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$. For $j > k$, we denote by λ_j the eigenvalue of $U(0)$ corresponding to e_j and put $\beta_j = \dfrac{1}{\lambda_j}$. By the orthogonal coordinate of $\mathbb{R}^m$ defined by $\{e_1, \ldots, e_m\}$, we write $x = (x_1, \ldots, x_m)$ again for every $x \in \mathbb{R}^m$. Putting $\gamma_j = e_j \cdot \sum_{i=1}^{l} P_i v_0/\lambda_i$, $1 \leqq j \leqq m$, we obtain

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_1(t,x) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1,$$

$$\varphi(t,x) = C \prod_{j=1}^{m} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)},$$

where C is a positive constant.
Put

$$h(t,x) = C \prod_{k<j\leqq m} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)}.$$

Then $h = h(t, x_{k+1}, \ldots, x_m)$ is a positive caloric function and

$$\varphi(t,x) = h \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j + \gamma_j)^2}{4(\beta_j - t)}.$$

Assume that $U(0)$ is not invertible. Then there are two cases: a_0 is not orthogonal to the zero-eigenspace of $U(0)$, or a_0 is orthogonal to the zero-eigenspace. They are equivalent to $\lambda_1 = 0$, or $\lambda_{k+1} = 0$, respectively.

If $\lambda_1 = 0$, then $b(t)$, $w(t)$ are given by

$$b(t) = P_1 a_0 \cdot P_1 v_0 t + \sum_{1<j\leqq k} \frac{P_j a_0 \cdot P_j v_0}{\lambda_j(1 - \lambda_j t)} + \delta_1,$$

$$w(t) = \frac{|P_1 v_0|^2}{4} t + \sum_{1<j\leqq l} \left(\frac{|P_j v_0|^2}{4\lambda_j(1 - \lambda_j t)} - \frac{n_j}{2} \log(1 - \lambda_j t) \right) + \delta_2$$

with some constants δ_1 and δ_2. Thus

$$f_0(t) = |P_1 a_0|^2 t + \sum_{1<j\leqq k} \frac{|P_j a_0|^2}{\lambda_j(1 - \lambda_j t)} + \delta_0$$

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with some constant δ_0. Put

$$\alpha_j = \begin{cases} |P_j a_0|, & j = 1, \\ \dfrac{|P_j a_0|}{|\lambda_j|}, & j > 1, \end{cases} \qquad e_j = \begin{cases} \dfrac{P_j a_0}{|P_j a_0|}, & j = 1, \\ \dfrac{\lambda_j P_j a_0}{|\lambda_j P_j a_0|}, & j > 1, \end{cases}$$

$$\beta_j = \frac{1}{\lambda_j}, \quad 1 < j \leqq k.$$

Note that β_j are mutually distinct. Adding $m-k$ eigenvectors of $U(0)$ to $\{e_1, \ldots, e_k\}$, in the case of $m > k$, we obtain an orthonormal basis $\{e_1, \ldots, e_m\}$ of $\mathbb{R}^m$. If $j > k$ and $U(0)e_j = \lambda_i e_j$ for some $\lambda_i \neq 0$, we put $\beta_j = 1/\lambda_i$. By the orthogonal coordinate of $\mathbb{R}^m$ defined by $\{e_1, \ldots, e_m\}$, we write $x = (x_1, \ldots, x_m)$ again for every $x \in \mathbb{R}^m$.
Putting $\gamma_j = e_j \cdot (P_1 v_0 + \sum_{1<i\leqq l} P_i v_0/\lambda_i)$, $1 \leqq j \leqq m$, we obtain

$$f_0(t) = \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_1(t,x) = \alpha_1(x_1 + \gamma_1 t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1,$$

$$\varphi(t,x) = C \prod_{j\in J_0} \exp\left[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\right] \prod_{j\in J_1} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)},$$

where $J_0 = \{j; U(0)e_j = 0\}$, $J_1 = \{j; U(0)e_j \neq 0\}$ and where C is a positive constant.
Put

$$h(t,x) = C \prod_{\substack{j\in J_0 \\ k<j\leqq m}} \exp\left[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\right] \prod_{\substack{j\in J_1 \\ k<j\leqq m}} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

Then $h = h(t, x_{k+1}, \ldots, x_m)$ is a positive caloric function and

$$\varphi(t,x) = h \exp\left[\frac{\gamma_1^2}{4}t + \frac{\gamma_1}{2}x_1\right] \prod_{1<j\leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp\frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

Finally, if $\lambda_{k+1} = 0$, then $b(t)$, $w(t)$ are given by

$$b(t) = \sum_{1\leqq j\leqq k} \frac{P_j a_0 \cdot P_j v_0}{\lambda_j(1-\lambda_j t)} + \delta_1,$$

$$w(t) = \frac{|P_{k+1}v_0|^2}{4}t + \sum_{j \neq k+1} \left(\frac{|P_j v_0|^2}{4\lambda_j(1-\lambda_j t)} - \frac{n_j}{2} \log(1-\lambda_j t) \right) + \delta_2$$

with some constants δ_1 and δ_2. Thus

$$f_0(t) = \sum_{1 \leqq j \leqq k} \frac{|P_j a_0|^2}{\lambda_j(1-\lambda_j t)} + \delta_0$$

with some constant δ_0. Put

$$\alpha_j = \frac{|P_j a_0|}{|\lambda_j|}, \quad e_j = \frac{\lambda_j P_j a_0}{|\lambda_j P_j a_0|}, \quad \beta_j = \frac{1}{\lambda_j}, \quad 1 \leqq j \leqq k.$$

Note that β_j are mutually distinct. Adding $m-k$ eigenvectors of $U(0)$ to $\{e_1, \dots, e_k\}$, in the case of $m > k$, we obtain an orthonormal basis $\{e_1, \dots, e_m\}$ of $\mathbb{R}^m$. If $j > k$ and $U(0)e_j = \lambda_i e_j$ for some $\lambda_i \neq 0$, we put $\beta_j = 1/\lambda_i$. By the orthogonal coordinate of $\mathbb{R}^m$ defined by $\{e_1, \dots, e_m\}$, we write $x = (x_1, \dots, x_m)$ again for every $x \in \mathbb{R}^m$.
Putting $\gamma_j = e_j \cdot (P_{k+1}v_0 + \sum_{1 \leqq i \leqq l,\, i \neq k+1} P_i v_0/\lambda_i)$, $1 \leqq j \leqq m$, we obtain

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_1(t,x) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_j + \gamma_j) + \delta_1,$$

$$\varphi(t,x) = C \prod_{j \in J_0} \exp\left[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\right] \prod_{j \in J_1} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j+\gamma_j)^2}{4(\beta_j - t)},$$

where $J_0 = \{j; U(0)e_j = 0\}$, $J_1 = \{j; U(0)e_j \neq 0\}$ and where C is a positive constant.
Since $1, \dots, k \in J_1$,

$$h(t,x) = C \prod_{j \in J_0} \exp\left[\frac{\gamma_j^2}{4}t + \frac{\gamma_j}{2}x_j\right] \prod_{\substack{j \in J_1 \\ k < j \leqq m}} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

is a positive caloric function and

$$\varphi(t,x) = h \prod_{1 \leqq j \leqq k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_j+\gamma_j)^2}{4(\beta_j - t)}.$$

∎

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For the proof of Theorem 7, we may assume that f is a caloric morphism of the form

$$f_i(t,x) = \sum_{j=1}^{m} a_{ij}(t)x_j + b_i(t), \quad 1 \leqq i \leqq n, \tag{11}$$

by virtue of Proposition 6. Denote by $a_i(t)$ the row-vector $(a_{i1}(t), \dots, a_{im}(t))$.

We introduce the functions $p_k(t)$, $q_k(t)$, $k \geqq 1$ which will be used in the proof of Theorem 7. We define $p_1(t)$ and $q_1(t)$ by

$$p_1(t) = \frac{f_0''(t)}{2f_0'(t)}, \quad q_1(t) = \frac{1}{\sqrt{3}}(p_1'(t) - p_1(t)^2)^{1/2}.$$

(Recall that $f_0'(t) > 0$ for all t by virtue of Corollary 4). For $k \geqq 2$, we define $p_k(t)$ and $q_k(t)$ inductively by

$$p_k(t) = \frac{q_{k-1}'(t)}{kq_{k-1}(t)} + \frac{k-2}{k}p_{k-1}(t), \tag{12}$$

$$q_k(t) = \frac{k}{\sqrt{2k+1}}\Big(p_k'(t) - p_k^2(t) + \frac{2k-3}{(k-1)^2}q_{k-1}^2(t)\Big)^{1/2}, \tag{13}$$

if $q_{k-1}(t) \neq 0$. We put $r_i(t) \in \mathbb{R}^m$, $1 \leqq i \leqq n$ by

$$r_i(t) = \frac{1}{|a_i(t)|}a_i(t),$$

(Note that $|a_i(t)| = \sqrt{f_0'(t)} > 0$ for all i and t because of (4)). And we put $r_{n+1}(t), \dots, r_{kn}(t)$ inductively by

$$r_{i+n}(t) = \begin{cases} \dfrac{1}{q_1(t)}r_i'(t), & 1 \leqq i \leqq n, \\ \dfrac{1}{q_j(t)}(r_i'(t) + q_{j-1}(t)r_{i-n}(t)), & \\ \qquad (j-1)n+1 \leqq i \leqq jn, \ 2 \leqq j \leqq k-1, & \end{cases} \tag{14}$$

if $q_j(t) \neq 0$, $1 \leqq j \leqq k-1$.

The following is the key lemma to prove Theorem 7.

LEMMA 9. *Let l be a positive integer. Assume that $q_1, \dots, q_l$ are defined on an open interval $I \subset \mathbb{R}$. Then the following statements hold.*

(i) *If $q_l \neq 0$ on I, then $r_1(t), \dots, r_{(l+1)n}(t)$ defined in (14) are orthonormal C^∞-vectors of $\mathbb{R}^m$. Adding arbitrary C^∞-vectors $r_{(l+1)n+1}(t), \dots, r_m(t)$*

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such that $\{r_1(t), \ldots, r_m(t)\}$ *forms an orthonormal basis of* $\mathbb{R}^m$ *for each* $t \in I$*, in the case of* $m \geqq (l+1)n+1$*, we take the change of variables*

$$\begin{cases} \tau = t, \\ \xi_j = r_j(t) \cdot x, \quad 1 \leqq j \leqq m, \end{cases}$$

on $D \cap (I \times \mathbb{R}^m)$*. Then there exists a* C^∞*-function* $\psi_{l+1}(\tau, \xi_{ln+1}, \ldots, \xi_m)$ *on* $D \cap (I \times \mathbb{R}^m)$ *such that*

$$\log \varphi(\tau, \xi) = \sum_{k=1}^{l} \Big(\sum_{i=(k-1)n+1}^{kn} \frac{1}{4} p_k(\tau) \xi_i^2 + \frac{1}{2k} q_k(\tau) \xi_i \xi_{i+n} + \beta_i(\tau) \xi_i + \rho_i(\tau) \Big) + \psi_{l+1}(\tau, \xi_{ln+1}, \ldots, \xi_m),$$

$$\frac{\partial \psi_{l+1}}{\partial \xi_i} = \frac{1}{2} p_{l+1}(\tau) \xi_i + \frac{1}{2(l+1)} \sum_{j=ln+1}^{m} (r_i'(\tau) \cdot r_j(\tau)) \xi_j + \beta_i(\tau), \quad ln+1 \leqq i \leqq (l+1)n,$$

and

$$\frac{\partial \psi_{l+1}}{\partial \tau} - \Delta_\xi \psi_{l+1} - \sum_{k=ln+1}^{m} \frac{\partial \psi_{l+1}}{\partial \xi_k} \Big(\frac{\partial \psi_{l+1}}{\partial \xi_k} - \sum_{j=ln+1}^{m} (r_k'(\tau) \cdot r_j(\tau)) \xi_j \Big) + \sum_{i=ln+1}^{(l+1)n} \Big(\frac{2l-1}{4l^2} q_l(\tau)^2 \xi_i^2 + \frac{l-1}{l} q_l(\tau) \beta_{i-n}(\tau) \xi_i \Big) = 0,$$

where

$$\beta_i = \begin{cases} \dfrac{b_i'}{2\sqrt{f_0'}}, & 1 \leqq i \leqq n, \\ \dfrac{1}{2q_1}(\beta_{i-n}' - p_1 \beta_{i-n}), & n+1 \leqq i \leqq 2n, \\ \dfrac{k}{(k+1)q_k}\Big(\beta_{i-n}' - p_k \beta_{i-n} + \dfrac{k-2}{k-1} q_{k-1} \beta_{i-2n}\Big), & kn+1 \leqq i \leqq (k+1)n, 2 \leqq k \leqq l, \end{cases}$$

and

$$\rho_i(\tau) = \int \Big(\frac{1}{2} p_k(\tau) + \beta_i^2(\tau) \Big) d\tau, \quad (k-1)n+1 \leqq i \leqq kn, 1 \leqq k \leqq l. \tag{15}$$

(ii) *If* $q_l(t) = 0$ *for all* $t \in I$*, then* $r_1(t), \dots, r_{ln}(t)$ *defined in* (14) *are orthonormal* C^∞*-vectors of* $\mathbb{R}^m$ *and satisfies the equations*

$$(16)\quad r'_{(l-1)n+i}(t) = \begin{cases} 0, & \text{if } l = 1, \\ -q_{l-1}(t) r_{(l-2)n+i}(t), & \text{if } l \geqq 2, \end{cases} \qquad 1 \leqq i \leqq n,$$

for all $t \in I$*. Add arbitrary* C^∞*-vectors* $r_{ln+1}(t), \dots, r_m(t)$ *such that* $\{r_1(t), \dots, r_m(t)\}$ *forms an orthonormal basis of* $\mathbb{R}^m$ *for each* $t \in I$*, if necessary. We take the change of variables* $(t, x) \mapsto (\tau, \xi)$ *defined in* (1)*. Then there exists a* C^∞*-function* $\psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m)$ *on* $D \cap (I \times \mathbb{R}^m)$ *such that*

$$(17)\ \log \varphi(\tau, \xi) = \sum_{k=1}^{l-1} \Big(\sum_{i=(k-1)n+1}^{kn} \frac{1}{4} p_k(\tau) \xi_i^2 + \frac{1}{2k} q_k(\tau) \xi_i \xi_{i+n} + \beta_i(\tau) \xi_i + \rho_i(\tau) \Big) + \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau) \xi_i^2 + \beta_i(\tau) \xi_i + \rho_i(\tau) \Big) + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m),$$

and

$$(18)\ \frac{\partial \psi_{l+1}}{\partial \tau} - \Delta_\xi \psi_{l+1} - |\nabla_\xi \psi_{l+1}|^2 + \sum_{j,k=ln+1}^{m} (r'_k(\tau) \cdot r_j(\tau)) \xi_j \frac{\partial \psi_{l+1}}{\partial \xi_k} = 0,$$

where β_i *and* ρ_i, $1 \leqq i \leqq ln$ *are defined in* (i).

Proof. We shall show the lemma by induction.

First we shall deal with the case of $l = 1$. By (4) and Corollary 4,

$$a_i(t) \cdot a_j(t) = \nabla f_i(t, x) \cdot \nabla f_j(t, x) = \delta_{ij} f'_0(t) > 0, \quad 1 \leqq i \leqq n,$$

which shows that $\{r_1(t), \dots, r_n(t)\}$ is an orthonormal system of $\mathbb{R}^m$ for each t. Let $r_{n+1}(t), \dots, r_m(t)$ be $m - n$ orthonormal C^∞-vectors such that $\{r_1(t), \dots, r_m(t)\}$ is an orthonormal basis of $\mathbb{R}^m$. By the chain rule,

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial t} \frac{\partial}{\partial \xi_j} = \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} r'_j(\tau) \cdot x \frac{\partial}{\partial \xi_j} = \frac{\partial}{\partial \tau} + \sum_{j,k=1}^{m} (r'_j(\tau) \cdot r_k(\tau)) \xi_k \frac{\partial}{\partial \xi_j},$$

$$\frac{\partial}{\partial x_i} = \frac{\partial \tau}{\partial x_i} \frac{\partial}{\partial \tau} + \sum_{j=1}^{m} \frac{\partial \xi_j}{\partial x_i} \frac{\partial}{\partial \xi_j} = \sum_{j=1}^{m} r_{ji}(\tau) \frac{\partial}{\partial \xi_j},$$

where $r_i(\tau) = (r_{i1}(\tau), \dots, r_{im}(\tau))$, $1 \leqq i \leqq m$. Since $r_1(\tau), \dots, r_m(\tau)$ is orthonormal, we have

$$\Delta_x = \Delta_\xi,$$
$$\nabla_x u \cdot \nabla_x v = \nabla_\xi u \cdot \nabla_\xi v.$$

Since (f, φ) is a caloric morphism, Theorem 1 (2) and Proposition 6 imply

$$2\nabla \log \varphi \cdot \nabla f_i = \frac{\partial f_i}{\partial t}, \quad 1 \leqq i \leqq n. \tag{19}$$

By (11) we have

$$f_i(\tau, \xi) = \sqrt{f_0'(\tau)}\xi_i + b_i(\tau) \tag{20}$$

and hence

$$Hf_i = \frac{\partial f_i}{\partial t} = \frac{f_0''(\tau)}{2\sqrt{f_0'(\tau)}}\xi_i + \sqrt{f_0'(\tau)}\sum_{j=1}^{m}(r_i'(\tau) \cdot r_j(\tau))\xi_j + b_i'(\tau).$$

Then (19) becomes

$$\frac{\partial \log \varphi}{\partial \xi_i} = \frac{1}{2}p_1(\tau)\xi_i + \frac{1}{2}\sum_{j=1}^{m}(r_i'(\tau) \cdot r_j(\tau))\xi_j + \beta_i(\tau). \tag{21}$$

Hence we have

$$r_i'(\tau) \cdot r_j(\tau) = r_i(\tau) \cdot r_j'(\tau), \quad 1 \leqq i, j \leqq n, \tag{22}$$

because $(\partial/\partial \xi_j)(\partial \log \varphi/\partial \xi_i) = r_i'(\tau) \cdot r_j(\tau)$. On the other hand, $r_i(\tau) \cdot r_j(\tau) = \delta_{ij}$ implies

$$r_i'(\tau) \cdot r_j(\tau) = -r_i(\tau) \cdot r_j'(\tau), \quad 1 \leqq i, j \leqq m. \tag{23}$$

Therefore

$$r_i'(\tau) \cdot r_j(\tau) = 0, \quad 1 \leqq i, j \leqq n. \tag{24}$$

Then by (21) and (24),

$$\psi_2 = \log \varphi - \sum_{i=1}^{n}\left(\frac{1}{4}p_1(\tau)\xi_i^2 + \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau) \cdot r_j(\tau))\xi_i\xi_j + \beta_i(\tau)\xi_i + \rho_i(\tau)\right)$$

is a C^∞-function of $\tau, \xi_{n+1}, \dots, \xi_m$. Thus we have

$$\text{(25)}\quad \log\varphi(\tau,\xi) = \sum_{i=1}^{n}\Big(\frac{1}{4}p_1(\tau)\xi_i^2 + \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_i\xi_j + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_2(\tau,\xi_{n+1},\dots,\xi_m).$$

On the other hand, $\psi_1 := \log\varphi$ satisfies

$$\frac{\partial\psi_1}{\partial t} - \Delta\psi_1 - |\nabla\psi_1|^2 = 0$$

because φ is a positive caloric function. In the coordinate $(\tau,\xi_1,\dots,\xi_m)$, the above equation is

$$\text{(26)}\qquad \frac{\partial\psi_1}{\partial\tau} + \sum_{j,k=1}^{m}(r_j'(\tau)\cdot r_k(\tau))\xi_k\frac{\partial\psi_1}{\partial\xi_j} - \Delta_\xi\psi_1 - |\nabla_\xi\psi_1|^2 = 0.$$

Then from (25), we have

$$\frac{\partial\psi_1}{\partial\tau} = \sum_{i=1}^{n}\Big(\frac{1}{4}p_1'(\tau)\xi_i^2 + \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))'\xi_i\xi_j + \beta_i'(\tau)\xi_i + \rho_i'(\tau)\Big) + \frac{\partial\psi_2}{\partial\tau},$$

$$\frac{\partial\psi_1}{\partial\xi_k} = \begin{cases} \frac{1}{2}p_1(\tau)\xi_k + \frac{1}{2}\sum_{j=n+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \beta_k(\tau), & 1 \leqq k \leqq n, \\ \frac{1}{2}\sum_{i=1}^{n}(r_i'(\tau)\cdot r_k(\tau))\xi_i + \frac{\partial\psi_2}{\partial\xi_k}, & n+1 \leqq k \leqq m, \end{cases}$$

$$\Delta_\xi\psi_1 = \frac{n}{2}p_1(\tau) + \Delta_\xi\psi_2.$$

Substituting these into (26) and comparing the coefficients with respect to $\xi_1,\dots,\xi_n$, we obtain the following:

$$\text{(27)}\quad \frac{1}{4}(p_1'(\tau) - p_1^2(\tau))\delta_{ij} - \frac{3}{4}\sum_{k=n+1}^{m}(r_i'(\tau)\cdot r_k(\tau))(r_j'(\tau)\cdot r_k(\tau)) = 0, \qquad 1 \leqq i,j \leqq n,$$

$$\text{(28)}\quad \frac{1}{2}\sum_{j=n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))'\xi_j + (\beta_i'(\tau) - p_1(\tau)\beta_i(\tau))$$

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$$
\begin{aligned}
&- 2 \sum_{k=n+1}^{m} (r_i'(\tau) \cdot r_k(\tau)) \frac{\partial \psi_2}{\partial \xi_k} \\
&+ \frac{1}{2} \sum_{j,k=n+1}^{m} (r_i'(\tau) \cdot r_k(\tau))(r_k'(\tau) \cdot r_j(\tau))\xi_j = 0, \quad 1 \leqq i \leqq n,
\end{aligned}
$$

and

$$
\begin{aligned}
(29) \quad \frac{\partial \psi_2}{\partial \tau} - \Delta_\xi \psi_2 - \sum_{k=n+1}^{m} \frac{\partial \psi_2}{\partial \xi_k}\Big(\frac{\partial \psi_2}{\partial \xi_k} - \sum_{j=n+1}^{m} (r_k'(\tau) \cdot r_j(\tau))\xi_j\Big) \\
+ \frac{1}{4} \sum_{i=1}^{n} \sum_{j,k=n+1}^{m} (r_i'(\tau) \cdot r_j(\tau))(r_i'(\tau) \cdot r_k(\tau))\xi_j \xi_k = 0.
\end{aligned}
$$

Since $r_i'(\tau) \cdot r_j(\tau) = 0$, $1 \leqq i, j \leqq n$, $r_i'(\tau) = \sum_{k=n+1}^{m} (r_i'(\tau) \cdot r_k(\tau)) r_k(\tau)$ for $1 \leqq i \leqq n$. Hence (27) gives

$$
(30) \qquad r_i'(\tau) \cdot r_j'(\tau) = q_1(\tau)^2 \delta_{ij}, \quad 1 \leqq i, j \leqq n.
$$

(Note that $q_1(\tau)^2 = |r_i'(\tau)|^2 \geqq 0$.)

If $q_1 \neq 0$ on an open interval I, then (24) and (30) show that $r_1(\tau), \ldots, r_n(\tau), r_1'(\tau), \ldots, r_n'(\tau)$ are linearly independent for all $\tau \in I$. Therefore $m \geqq 2n$. Putting

$$
r_{i+n}(\tau) = \frac{r_i'(\tau)}{q_1(\tau)}, \quad 1 \leqq i \leqq n,
$$

we have an orthonormal system $\{r_1(\tau), \ldots, r_{2n}(\tau)\}$ of $\mathbb{R}^m$. Adding $m - 2n$ C^∞-vectors $r_{2n+1}(\tau), \ldots, r_m(\tau)$ if $m \geqq 2n + 1$, we obtain an orthonormal basis $\{r_1(\tau), \ldots, r_m(\tau)\}$ of $\mathbb{R}^m$. Then

$$
r_i'(\tau) \cdot r_j(\tau) = q_1(\tau) r_{i+n}(\tau) \cdot r_j(\tau) = q_1(\tau) \delta_{i+n,\, j}, \quad 1 \leqq i \leqq n,\ n+1 \leqq j \leqq m.
$$

By (25), (28) and (29)

$$
\begin{aligned}
\log \varphi(\tau, \xi) = &\sum_{i=1}^{n} \Big(\frac{1}{4} p_1(\tau) \xi_i^2 + \frac{1}{2} q_1(\tau) \xi_i \xi_{i+n} + \beta_i(\tau) \xi_i + \rho_i(\tau)\Big) \\
&+ \psi_2(\tau, \xi_{n+1}, \ldots, \xi_m),
\end{aligned}
$$

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$$\frac{1}{2}q_1'(\tau)\xi_{i+n} + \beta_i'(\tau) - p_1(\tau)\beta_i(\tau) - 2q_1(\tau)\frac{\partial\psi_2}{\partial\xi_{i+n}} + \frac{1}{2}q_1(\tau)\sum_{j=n+1}^{m}(r_{i+n}'(\tau)\cdot r_j(\tau))\xi_j = 0, \qquad 1 \leqq i \leqq n,$$

and

$$\frac{\partial\psi_2}{\partial\tau} - \Delta_\xi\psi_2 - \sum_{k=n+1}^{m}\frac{\partial\psi_2}{\partial\xi_k}\Big(\frac{\partial\psi_2}{\partial\xi_k} - \sum_{j=n+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\Big) + \frac{1}{4}q_1(\tau)^2\sum_{i=1}^{n}\xi_{i+n}^2 = 0.$$

If $q_1(\tau) = 0$ for all $\tau \in I$, then by (30), $r_i' = 0$, $1 \leqq i \leqq n$ on I so that

$$\log\varphi(\tau,\xi) = \sum_{i=1}^{n}\Big(\frac{1}{4}p_1(\tau)\xi_i^2 + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_2(\tau,\xi_{n+1},\dots,\xi_m),$$

and

$$\frac{\partial\psi_2}{\partial\tau} - \Delta_\xi\psi_2 - |\nabla_\xi\psi_2|^2 + \sum_{j,k=n+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\frac{\partial\psi_2}{\partial\xi_k} = 0.$$

Thus the assertion in the case of $l = 1$ is shown.

Assume $l \geqq 2$ and that the assertion for $1,\dots,l-1$ holds. Suppose that $q_1 \neq 0,\dots,q_{l-1} \neq 0$ on some open interval I. Then q_l is defined on I and $r_1(\tau),\dots,r_{ln}(\tau)$ defined in (14) are orthonormal C^∞-vectors on $\mathbb{R}^m$. By the assumption on $1,\dots,l-1$, there exists a C^∞-function $\psi_l(\tau,\xi_{(l-1)n+1},\dots,\xi_m)$ such that

$$\text{(31)}\quad \log\varphi(\tau,\xi) = \sum_{k=1}^{l-1}\Big(\sum_{i=(k-1)n+1}^{kn}\frac{1}{4}p_k(\tau)\xi_i^2 + \frac{1}{2k}q_k(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_l(\tau,\xi_{(l-1)n+1},\dots,\xi_m),$$

$$\text{(32)}\qquad \frac{\partial\psi_l}{\partial\xi_i} = \frac{1}{2}p_l(\tau)\xi_i + \frac{1}{2l}\sum_{j=(l-1)n+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_j + \beta_i(\tau), \qquad (l-1)n+1 \leqq i \leqq ln,$$

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and

$$(33)\quad \frac{\partial \psi_l}{\partial \tau} - \Delta_\xi \psi_l - \sum_{k=(l-1)n+1}^{m} \frac{\partial \psi_l}{\partial \xi_k}\Big(\frac{\partial \psi_l}{\partial \xi_k} - \sum_{j=(l-1)n+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j\Big)$$
$$+ \sum_{i=(l-1)n+1}^{ln} \Big(\frac{2l-3}{4(l-1)^2} q_{l-1}(\tau)^2 \xi_i^2 + \frac{l-2}{l-1} q_{l-1}(\tau)\beta_{i-n}(\tau)\xi_i\Big) = 0.$$

By (23) and (32)

$$(34)\qquad r_i'(\tau)\cdot r_j(\tau) = 0, \quad (l-1)n+1 \leqq i,\, j \leqq ln$$

for $\tau \in I$. Put

$$(35)\quad \psi_{l+1}$$
$$= \psi_l - \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau)\xi_i^2 - \frac{1}{2l} \sum_{j=ln+1}^{m} (r_i'(\tau)\cdot r_j(\tau))\xi_i\xi_j + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big).$$

Then ψ_{l+1} is a C^∞-function of $\tau, \xi_{ln+1}, \ldots, \xi_m$ (in the case of $m = ln$, we have $(1/2l)\sum_{j=ln+1}^{m}(r_i'(\tau)\cdot r_j(\tau))\xi_j = 0$ and ψ_{l+1} depends only on τ). From (35) follow

$$\frac{\partial \psi_l}{\partial \tau} = \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l'(\tau)\xi_i^2 + \frac{1}{2l} \sum_{j=ln+1}^{m} (r_i'(\tau)\cdot r_j(\tau))'\xi_i\xi_j + \beta_i'(\tau)\xi_i + \rho_i'(\tau)\Big)$$
$$+ \frac{\partial \psi_{l+1}}{\partial \tau},$$

$$\frac{\partial \psi_l}{\partial \xi_k} = \begin{cases} \dfrac{1}{2} p_l(\tau)\xi_k + \dfrac{1}{2l} \displaystyle\sum_{j=ln+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j + \beta_k(\tau), \\ \qquad\qquad\qquad\qquad (l-1)n+1 \leqq k \leqq ln, \\ \dfrac{1}{2l} \displaystyle\sum_{i=(l-1)n+1}^{ln} (r_i'(\tau)\cdot r_k(\tau))\xi_i + \dfrac{\partial \psi_{l+1}}{\partial \xi_k}, \quad ln+1 \leqq k \leqq m, \end{cases}$$

$$\frac{\partial\psi_l}{\partial\xi_k} - \sum_{j=(l-1)n+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j$$
$$= \begin{cases} \dfrac{1}{2}p_l(\tau)\xi_k - \dfrac{2l-1}{2l}\displaystyle\sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \beta_k(\tau), \\ \qquad\qquad (l-1)n+1 \leqq k \leqq ln, \\ \dfrac{2l+1}{2l}\displaystyle\sum_{i=(l-1)n+1}^{ln}(r_i'(\tau)\cdot r_k(\tau))\xi_i - \sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j + \dfrac{\partial\psi_{l+1}}{\partial\xi_k}, \\ \qquad\qquad ln+1 \leqq k \leqq m, \end{cases}$$

and

$$\Delta_\xi\psi_l = \frac{n}{2}p_l(\tau) + \Delta_\xi\psi_{l+1}.$$

Substituting these into (33) and comparing the coefficients with respect to $\xi_{(l-1)n+1}, \ldots, \xi_{ln}$, we obtain the following:

$$(36)\quad \frac{1}{4}\Big(p_l'(\tau) - p_l(\tau)^2 + \frac{2l-3}{(l-1)^2}q_{l-1}(\tau)^2\Big)\delta_{ij} - \frac{2l+1}{4l^2}\sum_{k=ln+1}^{m}(r_i'(\tau)\cdot r_k(\tau))(r_j'(\tau)\cdot r_k(\tau)) = 0, \qquad (l-1)n+1 \leqq i,j \leqq ln,$$

$$(37)\quad \frac{l+1}{l}\sum_{k=ln+1}^{m}(r_i'(\tau)\cdot r_k(\tau))\frac{\partial\psi_{l+1}}{\partial\xi_k}$$
$$= \frac{1}{2l}\sum_{j=ln+1}^{m}\{(r_i'(\tau)\cdot r_j(\tau))' + (l-1)p_l(\tau)(r_i(\tau)'\cdot r_j(\tau))\}\xi_j$$
$$+ \frac{1}{2l}\sum_{j,k=ln+1}^{m}(r_i'(\tau)\cdot r_k(\tau))(r_k'(\tau)\cdot r_j(\tau))\xi_j$$
$$+ \beta_i'(\tau) - p_l(\tau)\beta_i(\tau) + \frac{l-2}{l-1}q_{l-1}(\tau)\beta_{i-n}(\tau), \qquad (l-1)n+1 \leqq i \leqq ln,$$

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and

$$\frac{\partial \psi_{l+1}}{\partial \tau} - \Delta_\xi \psi_{l+1} - \sum_{k=ln+1}^{m} \frac{\partial \psi_{l+1}}{\partial \xi_k}\Big(\frac{\partial \psi_{l+1}}{\partial \xi_k} - \sum_{j=ln+1}^{m} (r_k'(\tau)\cdot r_j(\tau))\xi_j\Big) + \frac{2l-1}{4l^2} \sum_{i=(l-1)n+1}^{ln} \sum_{j,k=ln+1}^{m} (r_i'(\tau)\cdot r_j(\tau))(r_i'(\tau)\cdot r_k(\tau))\xi_j\xi_k + \frac{l-1}{l} \sum_{i=(l-1)n+1}^{ln} \sum_{j=ln+1}^{m} \beta_i(\tau)(r_i'(\tau)\cdot r_j(\tau))\xi_j = 0. \tag{38}$$

Let $P_l = P_l(\tau)$ be the orthogonal projection of $\mathbb{R}^m$ to the orthogonal complement of the subspace generated by $\{r_1(\tau), \dots, r_{ln}(\tau)\}$. By (36) and (13), we have

$$P_l r_i' \cdot P_l r_j' = q_l^2 \delta_{ij}, \quad (l-1)n+1 \leqq i, j \leqq ln. \tag{39}$$

We shall show that

$$P_l r_i' = r_i' + q_{l-1} r_{i-n}, \quad (l-1)n+1 \leqq i \leqq ln. \tag{40}$$

By recalling the definition of P_l, (34) implies

$$P_l r_i' = r_i' - \sum_{j=1}^{(l-1)n} (r_i' \cdot r_j) r_j.$$

If $1 \leqq j \leqq (l-1)n$, then by (14),

$$r_j' = \begin{cases} q_1 r_{j+n}, & 1 \leqq j \leqq n, \\ q_k r_{j+n} - q_{k-1} r_{j-n}, & (k-1)n+1 \leqq j \leqq kn,\ 2 \leqq k \leqq l-1, \end{cases}$$

and so

$$r_i' \cdot r_j = -r_i \cdot r_j' = -q_{l-1}\delta_{i,\,j+n}, \quad (l-1)n+1 \leqq i \leqq ln,\ 1 \leqq j \leqq (l-1)n. \tag{41}$$

Thus (40) holds.

If $q_l(t) \neq 0$ for all $t \in I$, then (39) and (41) imply that $r_1(\tau), \dots, r_{(l+1)n}(\tau)$ defined in (14) are orthonormal C^∞-vectors of $\mathbb{R}^m$ on I where

$$r_{i+n}(\tau) = \frac{1}{q_l(\tau)}(r_i'(\tau) + q_{l-1}(\tau) r_{i-n}(\tau)), \quad (l-1)n+1 \leqq i \leqq ln.$$

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In the case of $m > (l+1)n$, we choose arbitrary C^∞-vectors $r_{(l+1)n+1}(\tau), \dots,$ $r_m(\tau)$ such that $\{r_1(\tau), \dots, r_m(\tau)\}$ forms an orthonormal basis of $\mathbb{R}^m$ for each $t \in I$. Then we have

$$r_i'(\tau) \cdot r_j(\tau) = q_l(\tau)\delta_{i+n,\,j} \quad (l-1)n+1 \leqq i \leqq ln,\ ln+1 \leqq j \leqq m.$$

From (35) follows

$$\begin{aligned}\psi_l(\tau, \xi_{(l-1)n+1,\dots,\xi_m}) \\ = \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4}p_l(\tau)\xi_i^2 - \frac{1}{2l}q_l(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\ + \psi_{l+l}(\tau, \xi_{ln+1}, \dots, \xi_m),\end{aligned}$$

which implies

$$\begin{aligned}\log\varphi(\tau,\xi) \\ = \sum_{k=1}^{l} \sum_{i=(k-1)n+1}^{kn} \Big(\frac{1}{4}p_k(\tau)\xi_i^2 + \frac{1}{2k}q_k(\tau)\xi_i\xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\ + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m).\end{aligned}$$

From (37) and (38) follow

$$\begin{aligned}\frac{\partial\psi_{l+1}}{\partial\xi_i} &= \frac{1}{2(l+1)}\Big(\frac{q_l'(\tau)}{q_l(\tau)} - (l-1)p_l(\tau)\Big)\xi_i \\ &\quad + \frac{1}{2(l+1)} \sum_{j=(l+1)n+1}^{m} (r_i'(\tau)\cdot r_j(\tau))\xi_j + \beta_i(\tau), \\ &\qquad\qquad ln+1 \leqq i \leqq (l+1)n,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\psi_{l+1}}{\partial\tau} - \Delta_\xi\psi_{l+1} - \sum_{k=ln+1}^{m} \frac{\partial\psi_{l+1}}{\partial\xi_k}\Big(\frac{\partial\psi_{l+1}}{\partial\xi_k} - \sum_{j=ln+1}^{m}(r_k'(\tau)\cdot r_j(\tau))\xi_j\Big) \\ + \sum_{i=ln+1}^{(l+1)n}\Big(\frac{2l-1}{4l^2}q_l(\tau)^2\xi_i^2 + \frac{l-1}{l}q_l(\tau)\beta_{i-n}(\tau)\xi_i\Big) = 0.\end{aligned}$$

Assume $q_l(t) = 0$ for all $t \in I$. Then (39) gives

$$P_l r_i' = 0, \quad (l-1)n+1 \leqq i \leqq ln.$$

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This and (40) show

$$r_i'(\tau) = -q_{l-1}(\tau) r_{i-n}(\tau), \quad (l-1)n+1 \leqq i \leqq ln.$$

Substituting this into (35), we have

$$\psi_l(\tau, \xi_{(l-1)n+1}, \dots, \xi_m) = \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau)\xi_i^2 + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m),$$

which implies

$$\begin{aligned}
&\log \varphi(\tau, \xi) \\
&= \sum_{k=1}^{l-1} \sum_{i=(k-1)n+1}^{kn} \Big(\frac{1}{4} p_k(\tau)\xi_i^2 + \frac{1}{2k} q_k(\tau)\xi_i \xi_{i+n} + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) \\
&\quad + \sum_{i=(l-1)n+1}^{ln} \Big(\frac{1}{4} p_l(\tau)\xi_i^2 + \beta_i(\tau)\xi_i + \rho_i(\tau)\Big) + \psi_{l+1}(\tau, \xi_{ln+1}, \dots, \xi_m).
\end{aligned}$$

From (38) follows

$$\frac{\partial \psi_{l+1}}{\partial \tau} - \Delta_\xi \psi_{l+1} - |\nabla_\xi \psi_{l+1}|^2 + \sum_{j,k=ln+1}^{m} (r_k'(\tau) \cdot r_j(\tau)) \xi_j \frac{\partial \psi_{l-1}}{\partial \xi_k} = 0.$$

Thus the assertion for l is shown. □

Proof of Theorem 7. For each $t \in D$, there exists a positive integer $l \leqq m/n$ such that $q_l(t) = 0$. In fact, if $q_1(t) \neq 0, \dots, q_k(t) \neq 0$, then by Lemma 9, $(k+1)n \leqq m$.

Assume that $q_1 \neq 0, \dots, q_{l-1} \neq 0$ and $q_l = 0$ on an open interval I. Then by (14) and (16), we obtain n systems of linear differential equations:

$$(42) \quad \frac{d}{dt} \begin{pmatrix} r_i \\ r_{n+i} \\ \vdots \\ r_{(l-1)n+i} \end{pmatrix} = \begin{pmatrix} 0 & q_1 & & \mathbf{0} \\ -q_1 & 0 & \ddots & \\ & \ddots & \ddots & q_{l-1} \\ \mathbf{0} & & -q_{l-1} & 0 \end{pmatrix} \begin{pmatrix} r_i \\ r_{n+i} \\ \vdots \\ r_{(l-1)n+i} \end{pmatrix} =: Q \begin{pmatrix} r_i \\ r_{n+i} \\ \vdots \\ r_{(l-1)n+i} \end{pmatrix},$$

for $1 \leqq i \leqq n$. Fix arbitrary $t_0 \in I$ and let $S(t) = (s_{jk}(t))_{j,k=1}^{l}$ be the solution of the initial value problem

$$\left\{ \begin{array}{l} \dfrac{d}{dt} S(t) = Q(t) S(t), \\ S(t_0) = I_l, \end{array} \right. \tag{43}$$

where I_l is the (l,l) unit matrix. Then $S(t)$ is an orthogonal matrix for every $t \in I$, because $Q(t)$ is skew symmetric. Then by (42), we have

$$\begin{pmatrix} r_i(t) \\ r_{n+i}(t) \\ \vdots \\ r_{(l-1)n+i}(t) \end{pmatrix} = S(t) \begin{pmatrix} r_i(t_0) \\ r_{n+i}(t_0) \\ \vdots \\ r_{(l-1)n+i}(t_0) \end{pmatrix}, \quad 1 \leqq i \leqq n.$$

This means that $r_1(t), r_2(t), \ldots, r_{ln}(t)$ are contained in the ln-dimensional space V spanned by the constant vectors $r_1(t_0), r_2(t_0), \ldots, r_{ln}(t_0)$ for every t. Therefore we can choose constant vectors $r_{ln+1}, \ldots, r_m$ which are the orthonormal basis of the orthogonal complement of V. Put $x_j = r_j(t_0) \cdot x$, $1 \leqq j \leqq m$ for $x \in \mathbb{R}^m$. Then

$$\xi_{(j-1)n+i} = \sum_{k=1}^{l} s_{jk}(t) x_{(k-1)n+i}, \quad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq l, \tag{44}$$

and if $m \geqq ln + 1$,

$$\xi_j = x_j, \quad ln + 1 \leqq j \leqq m.$$

Then ψ_{l+1} is a C^∞-function of $t, x_{ln+1}, \ldots, x_m$ and so the equation (18) reduces to

$$\frac{\partial \psi_{l+1}}{\partial t} - \Delta \psi_{l+1} - |\nabla \psi_{l+1}|^2 = 0.$$

Therefore $\varphi_{l+1}(t, x_{ln+1}, \ldots, x_m) = \exp \psi_{l+1}$ is a positive caloric function (in the case of $m = ln$, ψ_{l+1} is equal to a constant). From (20) follows

$$f_i = \sum_{k=1}^{l} \lambda(t) s_{1k}(t) x_{(k-1)n+i} + b_i(t),$$

where $\lambda(t) = \sqrt{f_0'(t)}$. On the other hand, by (17) and (44) we have

$$\log \varphi = \sum_{i=1}^{n} \Big[\sum_{j,k=1}^{l} \frac{1}{4} u_{jk}(t) x_{(j-1)n+i} x_{(k-1)n+i} + \sum_{j=1}^{l} \frac{1}{2} v_{ij}(t) x_{(j-1)n+i} + w_i(t) \Big] + \psi_{l+1},$$

where

$$u_{ij} = \sum_{k=1}^{l} p_k s_{ki} s_{kj} + \sum_{k=1}^{l-1} \frac{q_k}{k} (s_{ki} s_{k+1,j} + s_{k+1,i} s_{kj}), \quad 1 \leqq i, j \leqq l,$$

and

$$v_{ij} = \sum_{k=1}^{l} 2\beta_{(k-1)n+i} s_{kj}, \quad w_i = \sum_{k=1}^{l} \rho_{(k-1)n+i}, \quad 1 \leqq i \leqq n, 1 \leqq j \leqq l.$$

Put

$$g_{i1}(t, x_1, \dots, x_l) = \sum_{j=1}^{l} \lambda(t) s_{1j}(t) x_j + b_i(t), \quad 1 \leqq i \leqq n, \tag{45}$$

$$g_i(t, x_1, \dots, x_l) = (f_0(t), g_{i1}(t, x_1, \dots, x_l)), \quad 1 \leqq i \leqq n,$$

$$\varphi_i(t, x_1, \dots, x_l) = \exp \Big[\sum_{j,k=1}^{l} \frac{1}{4} u_{jk}(t) x_j x_k + \sum_{j=1}^{l} \frac{1}{2} v_{ij}(t) x_j + w_i(t) \Big], \quad 1 \leqq i \leqq n. \tag{46}$$

Then

$$f_i(t, x) = g_{i1}(t, x_i, x_{n+i}, \dots, x_{(l-1)n+i}),$$

$$\varphi(t, x) = \varphi_{l+1} \prod_{i=1}^{n} \varphi_i(t, x_i, x_{n+i}, \dots, x_{(l-1)n+i}).$$

We shall prove that each pair (g_i, φ_i), $1 \leqq i \leqq n$ is a caloric morphism from $I \times \mathbb{R}^l$ to $\mathbb{R}^{1+1}$. By $Hg_{i1} = \partial g_{i1}/\partial t$ and (43), we have

$$\begin{aligned} Hg_{i1} &= \sum_{j=1}^{n} (\lambda'(t) s_{1j}(t) x_j + \lambda(t) s'_{1j}(t) x_j) + b'_i(t) \\ &= \sum_{j=1}^{n} (\lambda'(t) s_{1j}(t) x_j + \lambda(t) q_1(t) s_{2j}(t) x_j) + b'_i(t). \end{aligned}$$

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On the other hand,

$$2\nabla \log \varphi_i \cdot \nabla g_{i1} = \sum_{j,k=1}^{l} \frac{1}{2}\lambda(u_{jk}s_{1k} + u_{kj}s_{1k})x_j + \sum_{j=1}^{l} \lambda v_{ij}s_{1j}$$
$$= \sum_{j=1}^{l} \lambda(p_1 s_{1j}x_j + q_1 s_{2j}x_j + 2\beta_i),$$

because $u_{ij} = u_{ji}$ and S is orthogonal. Hence

$$Hg_{i1} = 2\nabla \log \varphi_i \cdot \nabla g_{i1}, \quad 1 \leqq i \leqq n.$$

Since $f_0' = \lambda^2$,

$$\frac{df_0}{dt} = |\nabla g_{i1}|^2.$$

By the assumption, $\varphi(t,x)$ and φ_{l+1} are caloric functions, φ_{l+1} is independent of $x_1, \ldots, x_{ln}$ and

$$\prod_{i=1}^{n} \varphi_i(t, x_i, x_{n+1}, \ldots, x_{(l-1)n+i})$$

is a caloric function. Hence we have

$$\sum_{i=1}^{n} (K\varphi_i)(t, x_i, x_{n+i}, \ldots, x_{(l-1)n+i}) = 0,$$

where $K\varphi_i = (1/\varphi_i)H\varphi_i$. We have also $K\varphi_i = (\partial \log \varphi_i/\partial t) - \Delta \log \varphi_i - |\nabla \log \varphi_i|^2$. Comparing the coefficients with respect to x_j, we see that $K\varphi_i$ depends only on t. Therefore

$$\frac{\partial \log \varphi_i}{\partial t} - \Delta \log \varphi_i - |\nabla \log \varphi_i|^2 = \sum_{j=1}^{l} \left(\rho'_{(j-1)n+i} - \frac{1}{2}u_{jj} - \frac{1}{4}v_{ij}^2\right).$$

Since

$$\begin{pmatrix} u_{11} & \ldots & u_{1l} \\ \vdots & \ddots & \vdots \\ u_{l1} & \ldots & u_{ll} \end{pmatrix} = {}^tS \begin{pmatrix} p_1 & q_1 & & \mathbf{0} \\ q_1 & p_2 & \ddots & \\ & \ddots & \ddots & \frac{q_{l-1}}{l-1} \\ \mathbf{0} & & \frac{q_{l-1}}{l-1} & p_l \end{pmatrix} S,$$

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and

$$(v_{i1}, \dots, v_{il}) = 2(\beta_i, \beta_{n+i}, \dots, \beta_{(l-1)n+i})S,$$

we have

$$\sum_{j=1}^{l} \left(\rho'_{(j-1)n+i} - \frac{1}{2}u_{jj} - \frac{1}{4}v_{ij}^2\right) = \sum_{j=1}^{l} \left(\rho'_{(j-1)n+i} - \frac{p_j}{2} - \beta_{(j-1)n+i}^2\right) = 0$$

by the definition of ρ_j in (15). Therefore each φ_i is a positive caloric function. Thus (g_i, φ_i) is a caloric morphism. By (45) and (46), each (g_i, φ_i) satisfies the assumption of Lemma 8. Therefore there exist a positive integer $k \leqq l$, an orthogonal coordinate of $\mathbb{R}^m$ denoted by $(x_1, \dots, x_m)$ again and positive caloric functions $h_i = h_i(t, x_{kn+i}, \dots, x_{(l-1)n+i})$, $1 \leqq i \leqq n$ (in the case of $k = l$, $h_1, \dots, h_n$ are positive constants) such that f and φ are of form (1) or (2) with four families α_i, $1 \leqq i \leqq k$, β_i, $1 \leqq i \leqq k$, δ_i, $0 \leqq i \leqq n$ and γ_{ij}, $1 \leqq i \leqq n$, $1 \leqq j \leqq k$ of real numbers satisfying $\alpha_i > 0$ and $\beta_i \neq \beta_j$, $i \neq j$:

(1)

$$f_0(t) = \sum_{j=1}^{k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$f_i(t, x) = g_{i1}(t, x_i, \dots, x_{(l-1)n+i}) = \sum_{j=1}^{k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i,$$

$$\begin{aligned}\varphi(t, x) &= \varphi_{l+1} \prod_{i=1}^{n} \varphi_i(t, x_i, \dots, x_{(l-1)n+i}) \\ &= \varphi_{l+1} \prod_{i=1}^{n} h_i \prod_{j=1}^{k} \frac{1}{|\beta_j - t|^{1/2}} \exp \frac{(x_{(j-1)n+i} + \gamma_{ij})^2}{4(\beta_j - t)},\end{aligned}$$

(2)

$$f_0(t) = \alpha_1^2 t + \sum_{1<j\leqq k} \frac{\alpha_j^2}{\beta_j - t} + \delta_0,$$

$$\begin{aligned}f_i(t, x) &= g_{i1}(t, x_i, \dots, x_{(l-1)n+i}) \\ &= \alpha_1(x_i + \gamma_{i1}t) + \sum_{1<j\leqq k} \frac{\alpha_j}{\beta_j - t}(x_{(j-1)n+i} + \gamma_{ij}) + \delta_i,\end{aligned}$$

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$$\begin{aligned}\varphi(t,x) &= \varphi_{l+1}\prod_{i=1}^{n}\varphi_i(t,x_i,\ldots,x_{(l-1)n+i})\\ &= \varphi_{l+1}\prod_{i=1}^{n}h_i\exp\left[\frac{\gamma_{i1}^2}{4}t+\frac{\gamma_{i1}}{2}x_i\right]\\ &\qquad\times\prod_{1<j\leqq k}\frac{1}{|\beta_j-t|^{1/2}}\exp\frac{(x_{(j-1)n+i}+\gamma_{ij})^2}{4(\beta_j-t)}.\end{aligned}$$

Put $h=\varphi_{l+1}h_1\cdots h_n$. Then $h=h(t,x_{kn+1},\ldots,x_m)$ is a positive caloric function. We obtain the required form of (f,φ) on $D\cap(I\times\mathbb{R}^m)$. Since f_0 is of C^∞, the form of (f,φ) holds on the closure $\bar{I}$ of I, if $\bar{I}$ is contained in the interval where f_0 is defined. Thus (f,φ) has the required form on each open interval where $q_1>0,\ldots,q_{l-1}>0$. Fix an open interval I such that $q_1>0,\ldots,q_{l-2}>0$. The analyticity of f_0 and (13) implies that q_{l-1} is an analytic function on I. Therefore, the zero-points of q_{l-1} is discrete, which is denoted by $\{\sigma_\nu\}_{\nu=M}^{N}$ (M,N may be $-\infty,\infty$, respectively). For each ν, f_0 is of form

$$f_0(t)=\begin{cases}\displaystyle\sum_{j=1}^{k}\frac{\alpha_j^2}{\beta_j-t}+\delta_0, & t\in(\sigma_{\nu-1},\sigma_\nu],\\ \displaystyle\sum_{j=1}^{\tilde{k}}\frac{\tilde{\alpha}_j^2}{\tilde{\beta}_j-t}+\tilde{\delta}_0, & t\in[\sigma_\nu,\sigma_{\nu+1}),\end{cases}$$

in the case of (1). Then $\tilde{k}=k$, $\tilde{\alpha}_j=\alpha_j$, $\tilde{\beta}_j=\beta_j$ and $\tilde{\delta}_0=\delta_0$, because f_0 is of C^∞. Therefore (f,φ) has the required form on each interval where $q_1>0,\ldots,q_{l-2}>0$. In the case of (2), the same argument holds. Consequently, (f,φ) is of a required form on D. This completes the proof of Theorem 7. ∎

COROLLARY 10. *Let (f,φ) be the same as in Theorem 7. Then (f,φ) is equal to the composition of a the direct sum of k caloric morphisms of $\mathbb{R}^{n+1}$ and a projection $\mathbb{R}^{m+1}\to\mathbb{R}^{kn+1}$.*

Proof. In the case of (I), we put

$$g_{j0}(t)=\begin{cases}\dfrac{\alpha_1^2}{\beta_1-t}+\delta_0, & j=1,\\ \dfrac{\alpha_j^2}{\beta_j-t}, & j>1,\end{cases}$$

$$g_{ji}(t,x_1,\ldots,x_n)=\begin{cases}\dfrac{\alpha_1}{\beta_1-t}(x_i+\gamma_{ij})+\delta_i, & j=1,\\ \dfrac{\alpha_j}{\beta_j-t}(x_i+\gamma_{ij}), & j>1,\end{cases}$$

$$\varphi_j(t,x_1,\ldots,x_n)=\frac{1}{|\beta_j-t|^{n/2}}\exp\sum_{i=1}^{n}\frac{(x_i+\gamma_{ij})^2}{4(\beta_j-t)},$$

for $1\leqq i\leqq n$ and $1\leqq j\leqq k$. In the case of (II), we put

$$g_{j0}(t)=\begin{cases}\alpha_1^2t+\delta_0, & j=1,\\ \dfrac{\alpha_j^2}{\beta_j-t}, & j>1,\end{cases}$$

$$g_{ji}(t,x_1,\ldots,x_n)=\begin{cases}\alpha_1(x_i+\gamma_{i1}t)+\delta_1, & j=1,\\ \dfrac{\alpha_j}{\beta_j-t}(x_i+\gamma_{ij}), & j>1,\end{cases}$$

$$\varphi_j(t,x_1,\ldots,x_n)=\begin{cases}\exp\displaystyle\sum_{i=1}^{n}\Big[\frac{\gamma_{i1}^2}{4}t+\frac{\gamma_{i1}}{2}x_i\Big], & j=1,\\ \dfrac{1}{|\beta_j-t|^{n/2}}\exp\displaystyle\sum_{i=1}^{n}\frac{(x_i+\gamma_{ij})^2}{4(\beta_j-t)}, & j>1,\end{cases}$$

for $1\leqq i\leqq n$ and $1\leqq j\leqq k$. Then each pair $(g_j,\varphi_j)=((g_{j0},\ldots,g_{jn}),\varphi_j)$, $1\leqq j\leqq k$ is a caloric morphism. (g_1,φ_1) is defined on $\mathbb{R}^n\setminus\{t\neq\beta_1\}$ in the case of (I) and on $\mathbb{R}^n$ in the case of (I). For $j>1$, (g_j,φ_j) is defined on $\mathbb{R}^n\setminus\{t\neq\beta_j\}$. Let (p,ψ) be the projection $\mathbb{R}^{m+1}\to\mathbb{R}^{kn+1}$ such that $p_0(t)=t$, $p_i(t,x_1,\ldots,x_m)=x_i$, $1\leqq i\leqq kn$ and $\psi(t,x_1,\ldots,x_m)=h(t,x_{kn+1},\ldots,x_m)$. Then (f,φ) is equal to the composition of the direct sum of (g_1,φ_1),…,(g_k,φ_k) and (p,ψ). □

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