INTRINSIC FUNCTIONS ON MATRICES OF REAL QUATERNIONS

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1. Introduction. It is well known that any semi-simple algebra over the real field R, or over the complex field C, is a direct sum (unique except for order) of simple algebras, and that a finite-dimensional simple algebra over a field $\mathfrak{F}$ is a total matrix algebra over a division algebra, or equivalently, a direct product of a division algebra over $\mathfrak{F}$ and a total matrix algebra over $\mathfrak{F}$ **(1)**. The only finite division algebras over R are R, C, and $\mathfrak{Q}$, the algebra of real quaternions, while the only finite division algebra over C is C. Thus the algebra of real quaternions and the algebra of matrices of quaternions hold important places in the structure theory of semi-simple algebras. $\mathfrak{Q}$ is the four-dimensional division algebra over R with basis $1, i_1, i_2, i_3 = i_1 i_2$, and with multiplication determined by the associative and distributive laws and $i_1 i_2 = -i_2 i_1$, $i_1^2 = i_2^2 = -1$.

Rinehart **(4)** has introduced and motivated the study of the class of intrinsic functions on a linear associative algebra $\mathfrak{A}$, with identity, over a field $\mathfrak{F}$. Let $\mathfrak{G}$ be the group of all automorphisms and anti-automorphisms of $\mathfrak{A}$ which leave $\mathfrak{F}$ element-wise invariant.

DEFINITION 1.1. *A set of elements $\mathfrak{D}$ will be called an intrinsic set of $\mathfrak{A}$ if $\Omega\mathfrak{D} = \mathfrak{D}$ for every Ω in $\mathfrak{G}$.*

DEFINITION 1.2. *The single-valued function F, with domain, $\mathfrak{D}$, and range in $\mathfrak{A}$, is called an intrinsic function on $\mathfrak{D}$ if $\mathfrak{D}$ is an intrinsic set of $\mathfrak{A}$ and if $Z \in \mathfrak{D}$ implies $F(\Omega Z) = \Omega F(Z)$ for all Ω in $\mathfrak{G}$.*

Intrinsic functions have already been characterized for the algebra of complex numbers over the real field **(4)**, for the algebra of real quaternions **(4)**, and for total matrix algebras over the real or complex fields **(5)**. In this paper we obtain a characterization for appropriately continuous intrinsic functions on the algebra of matrices of real quaternions. In all the cases previously studied the functional value has been a polynomial in the argument value, with coefficients from the ground field. For the algebra of matrices of quaternions this is not the case, as is shown in Section 4 by an example. In Section 5 we show that our results are consistent with the theory of primary functions **(4)**. A primary function on an algebra $\mathfrak{A}$ is a function, from $\mathfrak{A}$ to $\mathfrak{A}$, which arises from a function of a complex variable, called its stem function, by means of any one of the (essentially equivalent) extension techniques discussed in **(3)**.

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2. The structure of $\mathfrak{M}_{\mathfrak{Q}}$ and the canonical forms. Let $\mathfrak{M}_{\mathfrak{Q}} = \mathfrak{M}_{\mathfrak{Q}}^n$ denote the algebra of $n \times n$ matrices with quaternion elements. $\mathfrak{M}_{\mathfrak{Q}}$ is the direct product of the algebra of real quaternions and the total matrix algebra $\mathfrak{M}_R$ over the real field.

If μ is any unit-vector quaternion ($\mu^2 = -1$), then

$$C(\mu) = \{x_0 + p\mu \mid x_0, p \in R\}$$

is a subalgebra of $\mathfrak{Q}$ isomorphic to C. $C(\mu)$ will be called a complex field of $\mathfrak{Q}$. It is immediate that $\mathfrak{M}_{C(\mu)}$ is a subalgebra of $\mathfrak{M}_{\mathfrak{Q}}$ isomorphic to the total matrix algebra $\mathfrak{M}_C$.

For any A in $\mathfrak{M}_{\mathfrak{Q}}$ we can write $A = A_1 + i_2 A_2$ where A_1 and A_2 are uniquely determined elements of $\mathfrak{M}_{C(i_1)}$. For each A in $\mathfrak{M}_{\mathfrak{Q}}$ let

$$\tilde{A} = \begin{bmatrix} A_1 & -\bar{A}_2 \\ A_2 & \bar{A}_1 \end{bmatrix}.$$

The mapping $A \leftrightarrow \tilde{A}$ is an isomorphism of $\mathfrak{M}_{\mathfrak{Q}}$ into $\mathfrak{M}_{C(i_1)}$ **(2)**. Wiegmann **(6)** has proved the following theorem.

THEOREM 2.1. *For any A in $\mathfrak{M}_{\mathfrak{Q}}$ there exists a non-singular matrix P in $\mathfrak{M}_{\mathfrak{Q}}$ such that*

$$\tilde{P}^{-1}\tilde{A}\tilde{P} = \tilde{J} = \begin{bmatrix} J & 0 \\ 0 & \bar{J} \end{bmatrix},$$

i.e., $P^{-1}AP = J$ where $J \in \mathfrak{M}_{C(i_1)}$ is in Jordan canonical form.

If μ and ν are any two-unit vector quaternions, then we can always find a unit-vector quaternion ρ such that $\rho^{-1}\mu\rho = \nu$. To see this let $\mu, \mu_2, \mu_3 = \mu\mu_2$ be an orthonormal basis for $\mathfrak{B}$, the vector subspace of $\mathfrak{Q}$. If $\nu = a\mu + b\mu_2 + c\mu_3$, then we can take

$$\rho = \begin{cases} \frac{1}{2}\sqrt{(2+2a)}\cdot\left[\mu + \dfrac{b}{1+a}\mu_2 + \dfrac{c}{1+a}\mu_3\right] & \text{if } a \neq -1, \\ \mu_2 & \text{if } a = -1. \end{cases}$$

If $P^{-1}AP = J_\mu$ is in Jordan canonical form in $\mathfrak{M}_{C(\mu)}$, and $\rho^{-1}\mu\rho = \nu$, then $(\rho I)^{-1}P^{-1}AP(\rho I) = \rho^{-1}J_\mu\rho = J_\nu$ is in Jordan canonical form in $\mathfrak{M}_{C(\nu)}$. J_ν is obtained from J_μ by the formal substitution of ν for μ.

From Theorem 2.1 and the foregoing discussion we have the following theorem.

THEOREM 2.2. *For any A in $\mathfrak{M}_{\mathfrak{Q}}$, and for any complex field $C(\mu)$ of $\mathfrak{Q}$, there exists a non-singular matrix P_μ in $\mathfrak{M}_{\mathfrak{Q}}$ such that $P_\mu^{-1}AP_\mu = J_\mu$ is in Jordan canonical form in $\mathfrak{M}_{C(\mu)}$. The canonical matrix in $\mathfrak{M}_{C(\nu)}$, where $C(\nu)$ is any other complex field of $\mathfrak{Q}$, is obtained from J_μ by formally replacing μ by ν.*

Since we are going to be dealing with function theory on $\mathfrak{M}_{\mathfrak{Q}}$, we shall need to introduce a few topological concepts. For every $Z = (z_{rs}) \in \mathfrak{M}_{\mathfrak{Q}}$

define $||Z|| = (1/n) \max |z_{ij}|$, where $|z_{ij}| = \sqrt{(z_{ij}\bar{z}_{ij})}$ is the usual norm for $\mathfrak{Q}$. $\mathfrak{M}_{\mathfrak{Q}}$ becomes a normed ring and hence a metric topological space with the topology induced by the norm. The concepts of neighbourhood, limit, continuity of functions, etc., are therefore well defined and the usual elementary processes of analysis are applicable.

The topology induced on $\mathfrak{M}_{\mathfrak{Q}}$ by our norm in turn induces a topology in each of the subalgebras $\mathfrak{M}_{C(\mu)}$, the open sets of $\mathfrak{M}_{C(\mu)}$ being the intersections of the open sets of $\mathfrak{M}_{\mathfrak{Q}}$ with $\mathfrak{M}_{C(\mu)}$. The topology on $\mathfrak{M}_{C(\mu)}$ described above is precisely the topology used in **(5)** to study intrinsic functions on $\mathfrak{M}_{C(\mu)}$.

Before discussing intrinsic functions on $\mathfrak{M}_{\mathfrak{Q}}$ we need to discuss the associated group of automorphisms and anti-automorphisms. Let $\mathfrak{G}$ be the group of all automorphisms and anti-automorphisms of $\mathfrak{M}_{\mathfrak{Q}}$, and let $\mathfrak{H}$ be the subgroup of all inner automorphisms of $\mathfrak{M}_{\mathfrak{Q}}$. $\mathfrak{M}_{\mathfrak{Q}}$ is a normal simple algebra over R and it is known **(1)** that every automorphism of a normal simple algebra is an inner automorphism, so $\mathfrak{H}$ contains all the automorphisms in $\mathfrak{G}$. It is easy to see that $\mathfrak{H}$ is normal in $\mathfrak{G}$ and that $\mathfrak{H}$ is of index two in $\mathfrak{G}$. Since Hermitian conjugation ($A \leftrightarrow A^* = \bar{A}^T$) is known to be an anti-automorphism of $\mathfrak{M}_{\mathfrak{Q}}$ **(2)**, any anti-automorphism of $\mathfrak{M}_{\mathfrak{Q}}$ is a product of an inner automorphism and Hermitian conjugation.

3. Intrinsic functions on $\mathfrak{M}_{\mathfrak{Q}}$. Let F be an intrinsic function defined on a domain (open set) $\mathfrak{D}$ of $\mathfrak{M}_{\mathfrak{Q}}$. F must admit all automorphisms and anti-automorphisms of $\mathfrak{M}_{\mathfrak{Q}}$. That is, we must have $F(P^{-1}AP) = P^{-1}F(A)P$ for every non-singular P in $\mathfrak{M}_{\mathfrak{Q}}$, and we must also have $F(A^*) = F(A)^*$.

By Theorem 2.2 every matrix in $\mathfrak{D}$ is similar to a matrix in $\mathfrak{D}_\mu = \mathfrak{D} \cap \mathfrak{M}_{C(\mu)}$, for any unit-vector quaternion μ. Thus F is determined in all of $\mathfrak{D}$ once F is known in any $\mathfrak{D}_\mu$. We now prove the following important theorem.

THEOREM 3.1. *If μ is any unit-vector quaternion, and F is an intrinsic function on a domain $\mathfrak{D}$ of $\mathfrak{M}_{\mathfrak{Q}}$, then F maps the domain $\mathfrak{D}_\mu = \mathfrak{D} \cap \mathfrak{M}_{C(\mu)}$ into $\mathfrak{M}_{C(\mu)}$.*

Proof. Assume that $J_\mu \in \mathfrak{M}_{C(\mu)}$ but that the element in the i, j position of $F(J_\mu)$ is $q \notin C(\mu)$. Let $P = (\mu I)$, so that

$$PJ_\mu P^{-1} = (\mu I)J_\mu(-\mu I) = J_\mu.$$

We must have

$$F(J_\mu) = F(PJ_\mu P^{-1}) = PF(J_\mu)P^{-1}$$

since F is intrinsic. Comparing the elements in the i, j position of $F(J_\mu)$ and $PF(J_\mu)P^{-1}$ we see that we must have $q = -\mu q \mu$, which can be true only if $q \in C(\mu)$. This establishes the theorem.

For $n = 1$, Theorem 3.1 yields an alternative, non-geometric proof of the result, proved in **(4)**, that an intrinsic function on $\mathfrak{Q}$ maps every complex field of $\mathfrak{Q}$ into itself.

Every inner automorphism of $\mathfrak{M}_{C(\mu)}$ can be extended to an inner automorphism of $\mathfrak{M}_{\mathfrak{Q}}$, so F, restricted to $\mathfrak{D}_\mu$, must admit every inner automorphism of $\mathfrak{M}_{C(\mu)}$. The fact that a function from $\mathfrak{M}_{C(\mu)}$ to $\mathfrak{M}_{C(\mu)}$ admits all the inner automorphisms of $\mathfrak{M}_{C(\mu)}$ was enough to lead Rinehart **(5)** to a characterization of intrinsic functions on $\mathfrak{M}_{C(\mu)}$. This characterization, adapted to the case at hand, is contained in Theorems 3.2 and 3.3, the proofs of which are given in **(5)**.

THEOREM 3.2. *An intrinsic function F on a domain $\mathfrak{D} \subseteq \mathfrak{M}_{\mathfrak{Q}}$ induces a unique single-valued function $f(\lambda, \sigma_1, \ldots, \sigma_{n-1})$ mapping a subset of each $\mathfrak{E}_{C(\mu)}$ (n-dimensional vector space over $C(\mu)$) into $C(\mu)$, μ being an arbitrary unit-vector quaternion. The function f is defined at any point*

$$P_\mu^{\ 0} = (\lambda_0, \sigma_1^{\ 0}, \ldots, \sigma_{n-1}^{\ 0}) \in \mathfrak{E}_{C(\mu)}$$

for which there exists a non-derogatory matrix $J_\mu \in \mathfrak{D}_\mu = \mathfrak{D} \cap \mathfrak{M}_{C(\mu)}$ with λ_0 as an eigenvalue and with characteristic polynomial

$$c(x) = x^n - \sigma_1^{\ 0} x^{n-1} + \ldots + (-1)^{n-1}\sigma_{n-1}^{\ 0} x + (-1)^n \sigma_n^{\ 0}.$$

The value of f at P_μ^0 is independent of the choice of the non-derogatory matrix $J_\mu \in \mathfrak{D}_\mu$ and is given by $f(P_\mu^0) = \lambda_0[F(J_\mu)] = L_{J\mu}(\lambda_0)$ where $L_{J\mu}(x)$ is a polynomial with coefficients in $C(\mu)$ such that $F(J_\mu) = L_{J\mu}(J_\mu)$ and $\lambda_0[A]$ denotes an eigenvalue of A. If P_ν^0 is obtained from P_μ^0 by replacing μ by ν, then

$$f(P_\nu^0) = f(P_\mu^0)|_{\mu=\nu}.$$

The induced function f, described in Theorem 3.2, will be called the stem function of F.

If $J_\mu \in \mathfrak{D}_\mu$ has repeated eigenvalues, then in any $\mathfrak{M}_{C(\mu)}$ neighbourhood of J_μ there are non-derogatory matrices with the same eigenvalues as J_μ **(5)**. If there is a derogatory $J_\mu \in \mathfrak{D}_\mu$, then, since $\mathfrak{D}_\mu$ is open, there is an $\mathfrak{M}_{C(\mu)}$ neighbourhood of J_μ contained in $\mathfrak{D}_\mu$, and hence there is a non-derogatory matrix $J_\mu' \in \mathfrak{D}_\mu$ with the same eigenvalues as J_μ. J_μ' can be used to define f at the points of $\mathfrak{E}_{C(\mu)}$ associated with J_μ.

THEOREM 3.3. *Let F be an intrinsic function on a domain $\mathfrak{D} \subseteq \mathfrak{M}_{\mathfrak{Q}}$. Consider $A \in \mathfrak{D}$ and let P_μ be a non-singular matrix of $\mathfrak{M}_{\mathfrak{Q}}$ such that*

$$P_\mu^{-1} A P_\mu = J_\mu \in \mathfrak{M}_{C(\mu)}.$$

Let $f(z, \sigma_1, \ldots, \sigma_{n-1})$ be the function from $\mathfrak{E}_{C(\mu)}$ to $C(\mu)$ induced by F (Theorem 3.2). If $f_{J\mu}(z)$ denotes the function of z only, $f(z, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu])$, then $F(J_\mu)$ must be given by the primary function value $f_{J\mu}(J_\mu)$, according to the customary definition of the extension of a function of a single complex variable to $\mathfrak{M}_{C(\mu)}$, if either

(I) *J_μ has distinct eigenvalues, or*

(II) *J_μ has repeated eigenvalues, J_μ is an interior point of $\mathfrak{D}_\mu$, F is continuous at J_μ, and $f_{J\mu}(z)$ is analytic in a z-neighbourhood of the repeated eigenvalues of J_μ.*

The $\sigma_i[J_\mu]$ are the coefficients in the characteristic polynomial of J_μ:

$$|xI - J_\mu| = x^n - \sigma_1[J_\mu]x^{n-1} + \sigma_2[J_\mu]x^{n-2} + \ldots + (-1)^{n-1}\sigma_{n-1}[J_\mu]x + (-1)^n \sigma_n[J_\mu].$$

The extension mentioned in Theorem 3.3 is that provided by the Lagrange-Hermite interpolation formula:

$$\text{(3.1)}\quad F(J_\mu) = f_{J\mu}(J_\mu) = \sum_{j=1}^{k} \left(\prod_{i \neq j} (J_\mu - \alpha_i I)^{s_i} \left[\sum_{m=0}^{s_i - 1} \frac{1}{m!} (J_\mu - \alpha_j I)^m H_{mj}(f_{J\mu}(z)) \right] \right),$$

where the $\alpha_i \in C(\mu)$ are the distinct eigenvalues of J_μ, with multiplicity s_i in the characteristic equation of J_μ, and where

$$\text{(3.2)}\qquad H_{mj}(f_{J\mu}(z)) = \frac{d^m}{dz^m} \left(f_{J\mu}(z) \prod_{p \neq j} (z - \alpha_p)^{-s_p} \right)_{z = \alpha_j}.$$

If $J_\nu \in \mathfrak{M}_{C(\nu)}$ is obtained from J_μ by replacing μ by ν, then $F(J_\nu)$ is obtainable from $F(J_\mu)$ by the same replacement since, if ρ is a unit-vector quaternion such that $\rho^{-1}\mu\rho = \nu$, then $\rho^{-1}J_\mu\rho = J_\nu$ and, by the intrinsic property of F, we must have

$$F(J_\nu) = F(\rho^{-1}J_\mu\rho) = \rho^{-1}F(J_\mu)\rho.$$

Since $F(J_\mu) \in \mathfrak{M}_{C(\mu)}$, the last term above is precisely $F(J_\mu)$ with μ replaced by ν.

Thus we see that the behaviour of F on any of the domains $\mathfrak{D}_\mu$ is determined by the behaviour of F on $\mathfrak{D}_{i_1}$. For the rest of this section we shall focus our attention on the domain $\mathfrak{D}_{i_1}$. For simplicity in notation let us set $J_{i_1} = J$ and $\mathfrak{D}_{i_1} = \mathfrak{D}_1$.

It is known **(3)** that $F(J)$, as given by (3.1), with $\mu = i_1$, is also given, according to the definition of Giorgi, by

$$\text{(3.3)}\qquad F(J) = f_J(J) = f_J(J_1) \dotplus \ldots \dotplus f_J(J_k),$$

where J is a direct sum, $J = J_1 \dotplus \ldots \dotplus J_k$, of Jordan blocks of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & \lambda_i \end{bmatrix}$$

with $\lambda_i = a_i + b_i i_1$, where a_i and b_i are real, and where

$$\text{(3.4)}\quad f_J(J_i) = \begin{bmatrix} f_J(\lambda_i) & f_J{}'(\lambda_i) & \frac{1}{2!} f_J{}''(\lambda_i) & \ldots & \frac{1}{(s_i - 1)!} f_J{}^{(s_i - 1)}(\lambda_i) \\ 0 & f_J(\lambda_i) & f_J{}'(\lambda_i) & \ldots & \frac{1}{(s_i - 2)!} f_J{}^{(s_i - 2)}(\lambda_i) \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & f_J{}'(\lambda_i) \\ 0 & 0 & 0 & \ldots & f_J(\lambda_i) \end{bmatrix}.$$

Since F is intrinsic we must have, as mentioned at the beginning of this section, $F(J^*) = F(J)^*$. $F(J)$ can be computed as above, yielding the matrix described by (3.3) and (3.4). There is always an inner automorphism of $\mathfrak{M}_{C(i_1)}$ which maps J into J^T, so, we must have by the intrinsic property of F, $F(J^T) = F(J)^T$. Thus we can use $F(J^*) = F(J^{*T})^T = F(\bar{J})^T$, and Giorgi's definition **(3)** to compute $F(J^*)$. Specifically we have

$$F(\bar{J}) = f_{\bar{J}}(\bar{J}_1) \dotplus \ldots \dotplus f_{\bar{J}}(\bar{J}_k),$$

where

$$f_{\bar{J}}(z) = f(z, \sigma_1[\bar{J}], \ldots, \sigma_{n-1}[\bar{J}]) = f(z, \overline{\sigma_1[J]}, \ldots, \overline{\sigma_{n-1}[J]}),$$

and where

$$f_{\bar{J}}(\bar{J}_i) = \begin{bmatrix} f_{\bar{J}}(\bar{\lambda}_i) & f_{\bar{J}}'(\bar{\lambda}_i) & \frac{1}{2!}f_{\bar{J}}''(\bar{\lambda}_i) & \ldots & \frac{1}{(s_i-1)!}f_{\bar{J}}^{(s_i-1)}(\bar{\lambda}_i) \\ 0 & f_{\bar{J}}(\bar{\lambda}_i) & f_{\bar{J}}'(\bar{\lambda}_i) & \ldots & \frac{1}{(s_i-2)!}f_{\bar{J}}^{(s_i-2)}(\bar{\lambda}_i) \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & f_{\bar{J}}'(\bar{\lambda}_i) \\ 0 & 0 & 0 & \ldots & f_{\bar{J}}(\bar{\lambda}_i) \end{bmatrix}.$$

An element-by-element comparison of $F(J)^*$ and $F(J^*)$ now yields

$$\overline{f_J(\lambda_i)} = f_{\bar{J}}(\bar{\lambda}_i)$$

or

$$\overline{f(z, \sigma_1[J], \ldots, \sigma_{n-1}[J])} = f(z, \overline{\sigma_1[J]}, \ldots, \overline{\sigma_{n-1}[J]}) \tag{3.5}$$

at the eigenvalues of J. A similar relation is readily seen to hold for the z-derivatives of f at the repeated eigenvalues of J.

A function f, from C to C, is intrinsic if and only if $f(\bar{z}) = \overline{f(z)}$ **(4)**, so a reasonable extension of this terminology is given by the following definition.

DEFINITION 3.1. *A function from a domain $\mathfrak{B}$ of $\mathfrak{E}_{C(i_1)}$ to $C(i_1)$ will be called intrinsic at the point $(z, \sigma_1, \ldots, \sigma_{n-1}) \in \mathfrak{B}$ if*

$$(\bar{z}, \bar{\sigma}_1, \ldots, \overline{\sigma_{n-1}}) \in \mathfrak{B} \quad \text{and}\ \overline{f(z, \sigma_1, \ldots, \sigma_{n-1})} = f(\bar{z}, \bar{\sigma}_1, \ldots \overline{\sigma_{n-1}}).$$

Now consider any $A \in \mathfrak{D}$ and let P be such that $P^{-1}AP = J$ in $\mathfrak{M}_{C(i_1)}$. Now,

$$\begin{aligned} F(A) &= PF(J)P^{-1} = Pf_J(J)P^{-1} \\ &= P\sum_{j=1}^{k}\left(\prod_{i\neq j}(J-\alpha_i I)^{s_i}\left[\sum_{m=0}^{s_j-1}\frac{1}{m!}(J-\alpha_j I)^m H_{mj}(f_J(z))\right]\right)P^{-1}, \end{aligned}$$

or

$$F(A) = \sum_{j=1}^{k}\left(\prod_{i\neq j}(A - P(\alpha_i I)P^{-1})^{s_i}\left[\sum_{m=0}^{s_j-1}\frac{1}{m!}(A - P(\alpha_j I)P^{-1})^m \times P(H_{mj}(f_J(z))I)P^{-1}\right]\right), \tag{3.6}$$

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where the α_i are the distinct eigenvalues of J with multiplicities s_i in the characteristic equation of J, and $H_{mj}(f_J(z))$ is given by (3.2).

The matrices $P(\alpha_i I)P^{-1}$ which appear in (3.6) are not in general equal to the scalar matrices $\alpha_i I$ since, if $A \notin \mathfrak{D}_1$, the elements of P are not all from $C(i_1)$, and hence do not commute with non-real α_i. The same remark applies to the matrices $P(H_{mj}(f_J(z))I)P^{-1}$.

THEOREM 3.4. *If F is an intrinsic function on $D \subseteq \mathfrak{M}_{\mathfrak{Q}}$, and if $A \in \mathfrak{D}$, then $F(A)$ is given by formula* (3.6) *provided the restrictions of Theorem* 3.3 *are satisfied. The stem function f is an intrinsic function from $\mathfrak{E}_{C(i_1)}$ to $C(i_1)$ at the points $(\lambda_i, \sigma_1[J], \ldots, \sigma_{n-1}[J])$, where the λ_i, $i = 1, 2, \ldots, k$, are the eigenvalues of J. If λ_i is a repeated eigenvalue of multiplicity s_i, then the first $s_i - 1$ z-derivatives of f are intrinsic at $(\lambda_i, \sigma_1[J], \ldots, \sigma_{n-1}[J])$.*

4. The *n*-ary functions on $\mathfrak{M}_{\mathfrak{Q}}$. Conversely, let us now consider a function $f(z, \sigma_1, \ldots, \sigma_{n-1})$ from $\mathfrak{E}_C$ to C whose domain (open set) $\mathfrak{B}$ is such that $\overline{\mathfrak{B}} = \mathfrak{B}$. We shall investigate an extension of f to $\mathfrak{M}_{\mathfrak{Q}}$. For any unit-vector quaternion μ we can define f on $\mathfrak{E}_{C(\mu)}$ by means of the isomorphism between C and $C(\mu)$; that is, if f is defined at $P_i^0 \in \mathfrak{B}$, then f is defined at the point $P_\mu^0 = P_i^0|_{i=\mu} \in \mathfrak{E}_{C(\mu)}$, the value being $f(P_i^0)|_{i=\mu} = f(P_\mu^0)$. In **(5)**, such functions, with suitable differentiability requirements, are extended to intrinsic functions on $\mathfrak{M}_{C(\mu)}$ by defining the value $f(J_\mu, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu])$ to be the value of the primary function extension to $\mathfrak{M}_{C(\mu)}$ of the stem function

$$f_{J\mu}(z) = f(z, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu]).$$

Let $\alpha_1, \ldots, \alpha_k$ be the distinct eigenvalues of J_μ with respective multiplicities $s_1, \ldots, s_k$ in the characteristic equation of J_μ. Specifically we have

$$(4.1) \quad f_{J\mu}(J_\mu) = \sum_{j=1}^{k} \left(\prod_{i \neq j} (J_\mu - \alpha_i I)^{s_i} \left[\sum_{m=0}^{s_j - 1} \frac{1}{m!} (J_\mu - \alpha_j I)^m H_{mj}(f_{J\mu}(z)) \right] \right),$$

where $H_{mj}(f_{J\mu}(z))$ is given by (3.2).

Let $\mathfrak{D}_\mu$ represent the domain of definition of (4.1). $J_\mu \in \mathfrak{D}_\mu$ if all the points

$$(\alpha_i, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu]), \qquad i = 1, 2, \ldots, k,$$

are in $\mathfrak{B}_\mu$, with $f_{J\mu}(z)$ analytic at α_i, if $s_i > 1$, $i = 1, 2, \ldots, k$.

Rinehart **(5)** has called the function, from $\mathfrak{M}_{C(\mu)}$ to $\mathfrak{M}_{C(\mu)}$, defined by (4.1) an *n*-ary function on $\mathfrak{M}_{C(\mu)}$ with stem function $f(z, \sigma_1, \ldots, \sigma_{n-1})$.

Let us further assume that, for each $J_\mu \in \mathfrak{D}_\mu$, $f(z, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu])$ is an intrinsic function from $\mathfrak{E}_{C(\mu)}$ to $C(\mu)$ at the eigenvalues of J_μ (see Definition 3.1). If $(\lambda_i, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu])$ is a point of analyticity of $f_{J\mu}(z)$, then it follows from the Taylor series that the z-derivatives of f are all intrinsic functions from $\mathfrak{E}_{C(\mu)}$ to $C(\mu)$ at the point $(\lambda_i, \sigma_1[J_\mu], \ldots, \sigma_{n-1}[J_\mu])$.

There is an obvious 1–1 correspondence between the points of $\mathfrak{D}_\mu$ and the points of $\mathfrak{D}_\nu$, where ν is any other unit-vector quaternion. It is clear from the

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way that f was defined on $\mathfrak{C}_{C(\mu)}$ and on $\mathfrak{C}_{C(\nu)}$ that $f_{J\nu}(J_\nu)$ is obtainable from $f_{J\mu}(J_\mu)$ by simply replacing μ by ν. That is, if $\rho^{-1}\mu\rho = \nu$, then

$$\rho^{-1}f_{J\mu}(J_\mu)\rho = f_{J\nu}(J_\nu).$$

Now let $\mathfrak{D} \subseteq \mathfrak{M}_{\mathfrak{Q}}$ be the set of all $A \in \mathfrak{M}_{\mathfrak{Q}}$ which are similar to an element of $\mathfrak{D}_1$. Since all the sets $\mathfrak{D}_\mu$ are essentially the same, we would get the same $\mathfrak{D}$ if we chose a different unit-vector quaternion than i_1. Note that $\mathfrak{D}$ properly contains $\bigcup_\mu \mathfrak{D}_\mu$.

Let us now observe that, if $P^{-1}AP = J \in \mathfrak{M}_{C(i_1)}$ and $\rho^{-1}i_1\rho = \mu$, μ any unit-vector quaternion, then $(P\rho)^{-1}A(P\rho) = J_\mu$. Let us also note that, for $f_J(J)$ given by (4.1),

$$Pf_J(J)P^{-1} = P\rho\rho^{-1}f_J(J)\rho\rho^{-1}P^{-1} = (P\rho)f_{J\mu}(J_\mu)(P\rho)^{-1}.$$

Thus, we can extend f to a function F on $\mathfrak{D}$, independently of the choice of the complex field $C(\mu)$, by defining

$$F(A) = f(A, \sigma_1[A], \ldots, \sigma_{n-1}[A]) = Pf_J(J)P^{-1}. \tag{4.2}$$

$F(A)$, as defined by (4.1) and (42.), will be called an *n-ary function* on $\mathfrak{M}_{\mathfrak{Q}}$ with stem function $f(z, \sigma_1, \ldots, \sigma_{n-1})$ provided the above restrictions, regarding the intrinsic nature and the analyticity of f, are satisfied.

Let Ω be any automorphism of $\mathfrak{M}_{\mathfrak{Q}}$; as remarked following Theorem 2.2, Ω must be an inner automorphism of $\mathfrak{M}_{\mathfrak{Q}}$. Let $W \in \mathfrak{M}_{\mathfrak{Q}}$ be such that $\Omega Z = WZW^{-1}$ for all Z in $\mathfrak{M}_{\mathfrak{Q}}$.

From (4.2) we have $\Omega F(A) = (\Omega P)(\Omega f_J(J))(\Omega P^{-1})$. Since $P^{-1}AP = J$ we have

$$(\Omega P^{-1})(\Omega A)(\Omega P) = \Omega J = WJW^{-1},$$

or

$$W^{-1}(\Omega P)^{-1}(\Omega A)(\Omega P)W = [(\Omega P)W]^{-1}(\Omega A)[(\Omega P)W] = J.$$

Now, computing $F(\Omega A)$ from (4.2), we have

$$\begin{aligned} F(\Omega A) &= [(\Omega P)W]f_J(J)[(\Omega P)W]^{-1} \\ &= (\Omega P)Wf_J(J)W^{-1}(\Omega P)^{-1} = (\Omega P)(\Omega f_J(J))(\Omega P^{-1}) = \Omega F(A). \end{aligned}$$

Thus F admits all the automorphisms of $\mathfrak{M}_{\mathfrak{Q}}$.

We can show that our extension is independent of the choice of P. If $P^{-1}AP = J = R^{-1}AR$, then $J = [R^{-1}P]^{-1}J[R^{-1}P]$ and from the argument above we have

$$F(J) = F([R^{-1}P]^{-1}J[R^{-1}P]) = [R^{-1}P]^{-1}F(J)[R^{-1}P].$$

It follows that $RF(J)R^{-1} = PF(J)P^{-1}$ and thus that our extension is unambiguous.

To show that F admits all anti-automorphisms of $\mathfrak{M}_{\mathfrak{Q}}$, it suffices to show that $F(A^*) = F(A)^*$. $P^{-1}AP = J \in \mathfrak{D}_1$, implies that $P^*A^*P^{*-1} = J^* \in \mathfrak{D}_1$, so that, by (4.1) and (4.2),

$$F(A^*) = P^{*-1} \sum_{j=1}^{k} \left(\prod_{i \neq j} (J^* - \bar{\alpha}_i I)^{s_i} \left[\sum_{m=0}^{s_j - 1} \frac{1}{m!} (J^* - \bar{\alpha}_j I)^m H'_{mj} \right] \right) P^*,$$

where

$$H'_{mj} = \frac{d^m}{dz^m} \left(f_{J^*}(z) \prod_{p \neq j} (z - \bar{\alpha}_p)^{-s_p} \right)_{z = \bar{\alpha}_j}.$$

The intrinsic requirements imposed above on f are adequate to ensure that $H'_{mj} = \bar{H}_{mj}$, so we have $F(A^*) = F(A)^*$.

We have shown that F admits all automorphisms and anti-automorphisms of $\mathfrak{M}_{\mathfrak{Q}}$ and have thus established the following theorem.

THEOREM 4.1. *An n-ary function on $\mathfrak{M}_{\mathfrak{Q}}$ is intrinsic.*

Theorems 4.1 and 4.2 (a restatement of Theorem 3.4 in n-ary function language) constitute our principal results.

THEOREM 4.2. *An intrinsic function on $\mathfrak{M}_{\mathfrak{Q}}$ subject to the conditions of Theorem* 3.3 *is an n-ary function on $\mathfrak{M}_{\mathfrak{Q}}$.*

Let us now consider an example of a binary function on $\mathfrak{M}_{\mathfrak{Q}} = \mathfrak{M}_{\mathfrak{Q}}^2$; thus $n = 2$. Take $f(z, \sigma_1) = \cos(\sigma_1 z)$. It is easy to check that $\overline{f(z, \sigma_1)} = f(\bar{z}, \overline{\sigma_1})$ and also that $\overline{f_1(z, \sigma_1)} = f_1(\bar{z}, \overline{\sigma_1})$, where $f_1(z, \sigma_1)$ is the first z-derivative of f. Since $f(z, \sigma_1)$ is an entire function, $f(z, \sigma_1)$ can be extended to an intrinsic function F, on all of $\mathfrak{M}_{\mathfrak{Q}}$, as described above.

Consider

$$J = \begin{bmatrix} i_1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathfrak{M}_{C(i_1)} \subset \mathfrak{M}_{\mathfrak{Q}};$$

$$\begin{aligned} F(J) &= f(J, \sigma_1[J]) = f(J, 1 + i_1) = \cos[(1 + i_1)J] \\ &= \begin{bmatrix} \cos(1 + i_1)i_1 & 0 \\ 0 & \cos(1 + i_1) \end{bmatrix} = \begin{bmatrix} \cos(i_1 - 1) & 0 \\ 0 & \cos(i_1 + 1) \end{bmatrix}. \end{aligned}$$

Since

$$\begin{aligned} \cos(i_1 + 1) &= \cos 1 \cos i_1 - \sin 1 \sin i_1 \\ &= \cos 1 \cosh 1 - i_1 \sin 1 \sinh 1 \approx (0.54)(1.54) - i_1(0.84)(1.18) \end{aligned}$$

is not real, it is clearly impossible to find a real polynomial in J yielding this non-real complex number in the 2, 2 position.

This example shows that it is impossible in general to represent an intrinsic function value on $\mathfrak{M}_{\mathfrak{Q}}$ as a real polynomial in the argument value. This is the first example of an intrinsic function on a simple algebra not having the property that the functional value is a polynomial in the argument value with coefficients from the ground field. For the algebra of complex matrices, considered as an algebra over the real field, our example shows that an intrinsic function on a normal simple algebra need not have the above property.†

†I am indebted to the referee for this observation.

5. Primary functions on $\mathfrak{M}_{\mathfrak{Q}}$. We now consider whether the extension techniques of Section 4 are consistent with the theory of primary functions on $\mathfrak{M}_{\mathfrak{Q}}$ as discussed in **(4)**.

Let $f(z)$ be an intrinsic function of the complex variable z ($f(\bar{z}) = \overline{f(z)}$) and suppose that f is sufficiently differentiable for f to be definable on an intrinsic domain $\mathfrak{D} \subseteq \mathfrak{M}_{\mathfrak{Q}}$. Let f_P represent the primary function on $\mathfrak{M}_{\mathfrak{Q}}$ obtained from f by the method discussed in **(4)**, and let f_N represent the n-ary function on $\mathfrak{M}_{\mathfrak{Q}}$ obtained from f by the method of Section 4, where $f(z)$ is to be thought of as a special function of the type $g(z, \sigma_1, \ldots, \sigma_{n-1})$.

We shall now check to see if, for any $A \in \mathfrak{D}$, $f_P(A) = f_N(A)$. Let $R \in \mathfrak{M}_{\mathfrak{Q}}$ be such that $R^{-1}AR = J \in \mathfrak{M}_{C(i_1)}$. Since f_P is intrinsic **(4)**, $f_P(A) = Rf_P(J)R^{-1}$. It was shown in Section 4 that $f_N(A) = Rf_N(J)R^{-1}$, so it will suffice to consider $f_N(J)$ and $f_P(J)$.

Let

$$c(x) = |xI - J| = \prod_{j=1}^{k} (x - \lambda_i)^{t_i}$$

be the characteristic polynomial of J, and denote by $p_1(z)$ the polynomial

$$p_1(z) = \sum_{j=1}^{k} \left(\prod_{i \neq j} (z - \lambda_i)^{t_i} \left[\sum_{m=0}^{t_j - 1} \frac{1}{m!} (z - \lambda_j)^m H_{mj} \right] \right),$$

where

$$H_{mj} = \frac{d^m}{dz^m} \left(f(z) \prod_{p \neq j} (z - \lambda_p)^{-t_p} \right)_{z = \lambda_j}.$$

Let

$$m(x) = \prod_{i=1}^{k} (x - \lambda_i)^{m_i}$$

be the minimum polynomial of J over the complex field $C(i_1)$, and let $p_2(z)$ be the polynomial computed in the same way as $p_1(z)$ except that each t_i is replaced by m_i. It is well known **(3)** that $p_2(J) = p_1(J) = f_N(J)$.

Let

$$r(x) = \prod_{i=1}^{m} (x - \alpha_i)^{s_i}$$

be the minimum polynomial of J over the real field. $r(x)$ is also the minimum polynomial of the matrix

$$\tilde{J} = \begin{bmatrix} J & 0 \\ 0 & \bar{J} \end{bmatrix} \in \mathfrak{M}^{2n}_{C(i_1)}.$$

Let $p_3(z)$ denote the real polynomial

$$p_3(z) = \sum_{j=1}^{m} \left(\prod_{i \neq j} (z - \alpha_i)^{s_i} \left[\sum_{m=0}^{s_j - 1} \frac{1}{m!} (z - \alpha_j)^m K_{mj} \right] \right),$$

where

$$K_{mj} = \frac{d^m}{dz^m}\left(f(z) \prod_{p \neq j} (z - \alpha_p)^{-s_p}\right)_{z=\alpha_j}.$$

Thus $p_3(J) = f_P(J)$ **(4)**.

Let us now think of $\tilde{J}$ as an element of an algebra over the complex field $C(i_1)$ and let us compute $f(\tilde{J})$. Using Bucheim's definition of $f(\tilde{J})$ **(3)**, we have

$$f(\tilde{J}) = p_3(\tilde{J}) = \begin{bmatrix} p_3(J) & 0 \\ 0 & p_3(\bar{J}) \end{bmatrix} = \begin{bmatrix} p_3(J) & 0 \\ 0 & \overline{p_3(J)} \end{bmatrix}.$$

Using Giorgi's definition of $f(\tilde{J})$ **(3)**, which is known to be equivalent to Bucheim's definition **(3)**, we have

$$f(\tilde{J}) = \begin{bmatrix} f(J) & 0 \\ 0 & f(\bar{J}) \end{bmatrix} = \begin{bmatrix} p_1(J) & 0 \\ 0 & p_1(\bar{J}) \end{bmatrix}.$$

The second equality above uses the fact that $p_1(J) = p_2(J)$ and the equivalence of Bucheim's and Giorgi's definitions of $f(J)$, where J is thought of as an element of an algebra over $C(i_1)$. Thus $f_P(J) = p_3(J) = p_1(J) = f_N(J)$, and we have shown that the extension described in Section 4 yields, for the special case just considered, results consistent with the theory of primary functions on $\mathfrak{M}_{\mathfrak{Q}}$ **(4)**.

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