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GOLDIE*-SUPPLEMENTED MODULES

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Abstract. Motivated by a relation on submodules of a module used by both A. W. Goldie and P. F. Smith, we say submodules X, Y of M are β^* *equivalent*, $X\beta^* Y$, if and only if $\frac{X+Y}{X}$ is small in $\frac{M}{X}$ and $\frac{X+Y}{Y}$ is small in $\frac{M}{Y}$. We show that the β^* relation is an equivalence relation and has good behaviour with respect to addition of submodules, homomorphisms and supplements. We apply these results to introduce the class of $\mathcal{G}^*$-supplemented modules and to investigate this class and the class of H-supplemented modules. These classes are located among various well-known classes of modules related to the class of lifting modules. Moreover these classes are used to explore an open question of S. H. Mohamed and B. J. Mueller. Examples are provided to illustrate and delimit the theory.

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1. Introduction. Throughout this paper, rings are associative with unity and modules are unital right R-modules, where R denotes such a ring and M denotes such a module. The motivation for this paper comes from two sources. The first is the open problem posed in [**6**, Open Problem #18, p. 107]: Is every H-supplemented module supplemented? Recall that a module M is H-*supplemented* if for every submodule A there is a direct summand D of M such that $A + X = M$ if and only if $D + X = M$ [**6**, p. 95]. In [**6**, Definition 4.4, p. 56], a module M is called *supplemented* if for any two submodules A and B with $A + B = M$, B contains a supplement of A. This definition

* This paper is dedicated to Patrick F. Smith on the occasion of his retirement.

of supplemented is equivalent to the more recent terminology of amply supplemented (see [**2**, p. 237] or [**8**, p. 54]).

The second source of motivation is provided by the concept of the beta equivalence relation defined on the set of submodules of a module. From [**1**], for submodules X, Y of M, $X\beta Y$ if and only if $X \cap Y \leq_e X$ and $X \cap Y \leq_e Y$. An equivalent form of this relation was used by Goldie on the right ideals of a ring in his seminal paper [**4**] and by Smith in [**7**]. Since this relation has proved fruitful in various applications such as characterizing when a pure subgroup of an Abelian group is a direct summand, it is natural to consider its dual relation. We say submodules X, Y *of* M are β^* *equivalent,* $X\beta^* Y$, if and only if $\frac{X+Y}{X}$ is small in $\frac{M}{X}$ and $\frac{X+Y}{Y}$ is small in $\frac{M}{Y}$.

We combine the above motivations by defining the following two types of modules:

(1) We say M is *Goldie**-*lifting,* $\mathcal{G}^*$-*lifting*, if and only if for each submodule X of M there exists a direct summand D of M such that $X\beta^* D$.

(2) We say M is *Goldie**-*supplementing,* $\mathcal{G}^*$-*supplementing,* if and only if for each submodule X of M there exists a supplement submodule S of M such that $X\beta^* S$.

In Section 2, we investigate the basic properties of the β^* relation. We show it is indeed an equivalence relation on the set of submodules of M, it is a congruence relative to addition in the lattice of submodules of M and it behaves well with respect to (weak) supplements and to homomorphic images.

In Section 3, our Theorem 3.5 generalizes and extends the main result of [**5**]. In Theorem 3.6, we compare the $\mathcal{G}^*$-lifting and $\mathcal{G}^*$-supplementing classes to various other well-known classes of modules that are related to the class of lifting modules. In particular, we show that the following implications hold between the various concepts:

(1) lifting $\Longrightarrow$ $\mathcal{G}^*$-lifting $\Longleftrightarrow$ H-supplemented $\Longrightarrow$ $\mathcal{G}^*$-supplemented $\Longrightarrow$ supplemented,

(2) lifting $\Longrightarrow$ amply supplemented $\Longrightarrow$ $\mathcal{G}^*$-supplemented,

(3) lifting $\Longleftrightarrow$ amply supplemented and strongly $\oplus$-supplemented,

(4) $\mathcal{G}^*$-lifting $\Leftarrow$ $\mathcal{G}^*$-supplemented and strongly $\oplus$-supplemented.

Theorem 3.6 motivates us to pose the following question:

Must a $\mathcal{G}^*$-supplemented module be amply supplemented?

From Theorem 3.6, a positive answer to our question implies a positive answer to the open question of Mohamed and Muller stated above in [**6**, Open Problem #18, p. 107]. A negative answer provides an interesting class of modules strictly between the classes of amply supplemented and supplemented modules. Moreover, we investigate the behaviour of the $\mathcal{G}^*$-supplemented condition with respect to direct sums and summands. Theorem 3.8 provides a structure theorem for a class of Noetherian modules.

Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$, $X \leq_s M$, $X \overset{cs}{\hookrightarrow} M$, $Rad(M)$ and $End(M)$ denote X is a submodule of M, X is a small submodule of M, X is cosmall in M, the Jacobson radical of M and the ring of endomorphisms of M, respectively.

Recall from [**2**] that a submodule $N \leq M$ is called a *supplement* (*weak supplement*) of a submodule L of M if $N + L = M$ and $N \cap L \leq_s N$ ($N \cap L \leq_s M$). The module M is called (*weakly*) *supplemented* if every submodule N of M has a (weak) supplement. M is called *lifting* if every submodule N of M contains a direct summand D of M such that $\frac{N}{D} \leq_s \frac{M}{D}$. A submodule N of M has *ample supplements* in M if for every

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$L \leq M$ with $M = N + L$, there is a supplement L' of N with $L' \leq L$. The module M is called *amply supplemented* if all submodules have ample supplements in M. M is called $\oplus$-*supplemented* if every submodule of M has a supplement that is a direct summand. A supplemented module M is called *strongly* $\oplus$-*supplemented* if every supplement submodule of M is a direct summand. From [**3**, p. 50], we say a submodule X of M is *projection invariant* if $eX \subseteq X$ for each $e = e^2 \in End(M_R)$. $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Q}$ denote the ring of integers, the ring of integers modulo n and the field of rational numbers, respectively. Other terminology and notation can be found in [**2**, **6**, **8**].

2. The β^* Relation. In this section, we develop the basic properties of the β^* relation on the set of submodules of M. These properties will be used in Section 3.

Definition 2.1. We define the relation 'β^*' on the set of submodules of M by $X\beta^* Y$ if and only if $\frac{X+Y}{X} \leq_s \frac{M}{X}$ and $\frac{X+Y}{Y} \leq_s \frac{M}{Y}$.

Lemma 2.2. *β^* is an equivalence relation.*

Proof. The reflexive and symmetric properties are clear. For transitivity, assume $X\beta^* Y$ and $Y\beta^* Z$. So

$$\frac{X+Y}{X} \leq_s \frac{M}{X} \text{ and } \frac{X+Y}{Y} \leq_s \frac{M}{Y}$$

and

$$\frac{Y+Z}{Y} \leq_s \frac{M}{Y} \text{ and } \frac{Y+Z}{Z} \leq_s \frac{M}{Z}.$$

Assume $\frac{B}{X} \leq \frac{M}{X}$ such that $\frac{X+Z}{X} + \frac{B}{X} = \frac{M}{X}$. Then

$$\frac{X+Z+B}{X} = \frac{Z+B}{X} = \frac{M}{X},$$

so $Z + B = M$. Hence

$$\frac{M}{Y} = \frac{Z+Y+B}{Y} = \frac{Z+Y}{Y} + \frac{Y+B}{Y}.$$

Since $\frac{Z+Y}{Y} \leq_s \frac{M}{Y}$, $\frac{Y+B}{Y} = \frac{M}{Y}$. Hence $Y + B = M$. Then

$$\frac{M}{X} = \frac{Y+B}{X} = \frac{X+Y}{X} + \frac{B}{X}.$$

Since $\frac{X+Y}{X} \leq_s \frac{M}{X}$, $\frac{B}{X} = \frac{M}{X}$. Therefore $B = M$, so $\frac{X+Z}{X} \leq_s \frac{M}{X}$. Similarly, $\frac{X+Z}{Z} \leq_s \frac{M}{Z}$. $\square$

Observe that the zero submodule is β^* equivalent to any small submodule. Also, note that two submodules may be isomorphic but not β^* equivalent. For example, let F be a field and $R = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$, $X = \left[\begin{smallmatrix} 0 & F \\ 0 & 0 \end{smallmatrix}\right]$ and $Y = \left[\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right]$. Then X is R-isomorphic to Y, but X is not β^* equivalent to Y as can be seen by taking $A = \left[\begin{smallmatrix} F & F \\ 0 & 0 \end{smallmatrix}\right]$ in (iii) of the following result. Moreover, if $M = \mathbb{Z}_{\mathbb{Z}}$ then $m\mathbb{Z}\beta^* n\mathbb{Z}$ if and only if m and n are divisible by the same primes.

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THEOREM 2.3. *Let $X, Y \leq M$. The following are equivalent:*

(i) $X\beta^* Y$.

(ii) $X \overset{cs}{\hookrightarrow} X + Y$ *and* $Y \overset{cs}{\hookrightarrow} X + Y$.

(iii) *For each* $A \leq M_R$ *such that* $X + Y + A = M$ *then* $X + A = M$ *and* $Y + A = M$.

(iv) *If* $K \leq M$ *with* $X + K = M$ *then* $Y + K = M$, *and if* $H \leq M$ *with* $Y + H = M$ *then* $X + H = M$.

Proof. (i) $\Longrightarrow$ (ii) Clear.
(ii) $\Longrightarrow$ (iii) Let $A \leq M$ such that $X + Y + A = M$. Then

$$\frac{X+Y}{Y} + \frac{Y+A}{Y} = \frac{M}{Y} \Longrightarrow \frac{Y+A}{Y} = \frac{M}{Y} \Longrightarrow Y + A = M.$$

Similarly, $X + A = M$.
(iii) $\Longleftrightarrow$ (iv) Let $K \leq M$ such that $X + K = M$. Then $X + K + Y = M$. By (iii), $Y + K = M$. Let $H \leq M$ such that $Y + H = M$. Then $X + Y + H = M$. By (iii), $X + H = M$.
Conversely, assume $X + Y + A = M$. Then $X + (Y + A) = M$. So $Y + (Y + A) = M$. Hence $Y + A = M$. Similarly, $X + A = M$.
(iii) $\Longrightarrow$ (i) Let $\frac{B}{Y} \leq \frac{M}{Y}$ such that $\frac{X+Y}{Y} + \frac{B}{Y} = \frac{M}{Y}$. Then $X + Y + B = M$. Hence $Y + B = B = M$ (since $Y \subseteq B$), so $\frac{X+Y}{Y} \leq_s \frac{M}{Y}$. Similarly, $\frac{X+Y}{X} \leq_s \frac{M}{X}$. □

COROLLARY 2.4. *Let $X, Y \leq M$ such that $X \subseteq Y + B$ and $Y \subseteq X + A$, where $A, B \leq_s M$. Then $X\beta^* Y$.*

Proof. Let $X + Y + K = M$, for some $K \leq M$. Then $(Y + B) + Y + K = M$. So $Y + B + K = M$. Hence $Y + K = M$. Similarly, $X + K = M$. □

Note that there are modules M with $K, X, Y \leq M$ such that $M = X + K = Y + K$, but X is not β^* related to Y. Take $R = M = \mathbb{Z}$, $K = 3\mathbb{Z}$, $X = 2\mathbb{Z}$ and $Y = 5\mathbb{Z}$.

PROPOSITION 2.5. *If $X, Y, K \subseteq M$ such that $M = X + K = Y + K$, $Y \cap K \subseteq X \cap K$ and $Y \overset{cs}{\hookrightarrow} X + Y$ (i.e. $\frac{X+Y}{Y} \leq_s \frac{M}{Y}$), then $X \overset{cs}{\hookrightarrow} X + Y$ (i.e. $\frac{X+Y}{X} \leq_s \frac{M}{X}$), so $X\beta^* Y$.*

Proof. There exist canonical isomorphisms $\frac{K}{Y\cap K} \overset{\theta}{\overset{\leftarrow}{\simeq}} \frac{Y+K}{Y} = \frac{M}{Y}$, $\frac{K}{X\cap K} \overset{\psi}{\overset{\rightarrow}{\simeq}} \frac{X+K}{X} = \frac{M}{X}$ and $f : \frac{K}{Y\cap K} \overset{epi}{\longrightarrow} \frac{K}{X\cap K}$, defined by $f(k + Y \cap K) = k + X \cap K$. Define $h : \frac{M}{Y} \longrightarrow \frac{M}{X}$ by $h = \psi f \theta$. Let $m \in M$. Then $m = y + k$, for some $y \in Y$ and $k \in K$. Hence $\theta(m + Y) = \theta(y + k + Y) = k + Y \cap K$, $f(k + Y \cap K) = k + X \cap K$ and $\psi(k + X \cap K) = k + X = k + x + X$ for any $x \in X$. Thus $h(m + Y) = k + x + X$.

We claim that $h(\frac{X+Y}{Y}) = \frac{X+Y}{X}$. To see that $h(\frac{X+Y}{Y}) \subseteq \frac{X+Y}{X}$, let $x + y + Y \in \frac{X+Y}{Y}$. Note that there exist $y_1 + k = x + y$, where $y_1 \in Y$ and $k \in K$. Hence $h(x + y + Y) = x + y - y_1 + X \in \frac{X+Y}{X}$. Now assume $x_2 + y_2 + X \in \frac{X+Y}{X}$. There exists $x_3 \in X$ and $k_3 \in K$ such that $y_2 = x_3 + k_3$. Hence $x_2 + y_2 + X = k_3 + X$. Now $k_3 = -x_3 + y_2 \in X + Y$. So

$$h(k_3 + Y) = \psi f \theta(k_3 + Y) = \psi f(k_3 + Y \cap K) = \psi(k_3 + X \cap K) = k_3 + X = x_2 + y_2 + X.$$

Thus $h(\frac{X+Y}{Y}) = \frac{X+Y}{X}$. Since $\frac{X+Y}{Y} \leq_s \frac{M}{Y}$ then $\frac{X+Y}{X} \leq_s \frac{M}{X}$, by [**2**, p. 11, 2.2(5)]. □

THEOREM 2.6. *Let $X, Y \leq M$ such that $X\beta^* Y$. Then*

(i) $X \leq_s M$ *if and only if* $Y \leq_s M$.

(ii) X has a (weak) supplement C in M if and only if C is a (weak) supplement for Y.

Proof. (i) ($\Longrightarrow$) Assume $X \leq_s M$. Let $K \leq M$ such that $Y + K = M$. Then $X + Y + K = M$. By Theorem 2.3, $X + K = M$. Since $X \leq_s M$, $K = M$. Thus $Y \leq_s M$.
($\Longleftarrow$) The converse is true because β^* is symmetric (Lemma 2.2).
(ii) Assume C is a supplement for X. Then $M = X + C = X + Y + C$. By Theorem 2.3, $Y + C = M$. Assume $K \subseteq C$ and $Y + K = M$. Then $X + Y + K = M$. By Theorem 2.3, $X + K = M$. By the minimality of C, $K = C$. Thus C is a supplement for Y. The converse is true because β^* is symmetric (Lemma 2.2). Therefore X has a supplement C if and only if C is a supplement for Y.

Now assume C is a weak supplement for X. Then $X + C = M$ and $X \cap C \leq_s M$. By Theorem 2.3, $Y + C = M$. We need to show that $Y \cap C \leq_s M$. Let $K \leq M$ such that $Y \cap C + K = M$. Since $Y \cap C \subseteq Y$, $Y + K = M$ and $C + K = M$. By Theorem 2.3, $X + K = M$. Since $Y \cap C \subseteq C$, the modular law yields that $C = C \cap M = (Y \cap C) + (C \cap K)$. Then

$$M = Y + C = Y + Y \cap C + C \cap K = Y + C \cap K.$$

Hence $X + Y + C \cap K = M$. By Theorem 2.3, $X + C \cap K = M$. So

$$K = K \cap M = K \cap (C \cap K + X) = (C \cap K) + X \cap K,$$

by the modular law. Now

$$M = C + K = (Y \cap C) + (C \cap K) + (X \cap K) \subseteq C + X \cap K \subseteq M.$$

Hence $M = C + (X \cap K)$. By the modular law,

$$X = X \cap M = X \cap \big((X \cap K) + C\big) = X \cap K + X \cap C.$$

Thus

$$M = X + K = X \cap C + X \cap K + K = X \cap K + K = K,$$

since $X \cap C \leq_s M$. Therefore $Y \cap C \leq_s M$. The converse holds by the symmetry of the β^* relation. □

Corollary 2.7. *Let $X, Y \subseteq M$ such that $X \subseteq Y$ and X has a weak supplement C in M. Then $X\beta^* Y$ if and only if $Y \cap C \leq_s M$.*

Proof. Assume $X\beta^* Y$. By Theorem 2.6, C is a weak supplement of Y. Hence $Y \cap C \leq_s M$.

Conversely, assume $Y \cap C \leq_s M$. Let $K_R \leq M_R$ such that $X + Y + K = M$. Since $X \subseteq Y$, $Y + K = M$. Since $X + C = M$ and $X \subseteq Y$, the modular law yields $Y = X + Y \cap C$. Then

$$M = Y + K = X + Y \cap C + K = X + K,$$

since $Y \cap C \leq_s M$. By Theorem 2.3, $X\beta^* Y$.

Note that from the hypothesis and the modular law, $Y = X + Y \cap C$ and $X \cap C \leq_s M$. □

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Observe that for a minimal right ideal X of R there is a direct summand D in R such that $X\beta^*D$. To see this, note that either $X = eR$ for some idempotent (in which case X is a direct summand) or $X^2 = 0$ (in which case $X\beta^*0$).

COROLLARY 2.8. *Let $M = C \oplus D$ and $L, S, X \leq M$, where S is a weak supplement of L.*

(i) If $X\beta^*S$ then $\frac{X}{X\cap L} \cong \frac{S}{S\cap L}$. If S is a supplement, there is a small cover $f : S \longrightarrow \frac{X}{X\cap L}$. If X is projective, then there is an epimorphism $h : X \longrightarrow S$. Whereas, if S is projective, then there is a projective cover $g : S \longrightarrow X$.

(ii) If $X\beta^*D$, then $\frac{X}{X\cap C} \cong D$, $X \cap C \leq_s C$ and there is a small cover $h : \frac{M}{D} \longrightarrow \frac{M}{X}$. If M is $\mathcal{G}^*$-lifting, then for each X there is a direct summand D such that $X\beta^*D$, an epimorphism $f : X \longrightarrow D$ and a small cover $h : \frac{M}{D} \longrightarrow \frac{M}{X}$.

(iii) Assume $X \subseteq D$. Then $X\beta^*D$ if and only if $X = D$. In particular, $X\beta^*M$ if and only if $X = M$.

(iv) Assume $D \subseteq X$. Then $X\beta^*D$ if and only if $X \cap C \leq_s M$.

Proof. (i) Since $X\beta^*S$, Theorem 2.3 yields $\frac{X}{X\cap L} \cong \frac{X+L}{L} = \frac{M}{L} = \frac{S+L}{L} \cong \frac{S}{S\cap L}$. The remainder of the proof of this part follows from properties of a projective module.

(ii) This part is a consequence of (i) and Theorem 2.6 which yields that C is a supplement of X. Hence $X \cap C \leq_s C$. Then there is a small cover $k : C \longrightarrow \frac{C}{X\cap C}$. Consequently, there is a small cover $h : \frac{M}{D} \longrightarrow \frac{M}{X}$.

(iii) ($\Longrightarrow$) Assume $X\beta^*D$. Then $X + D + C = M$. By Theorem 2.3, $X \oplus C = M$. Thus $X = D$.

($\Longleftarrow$) Since β^* is reflexive (Lemma 2.2), $X = D \Longrightarrow X\beta^*D$.

(iv) This part is a consequence of Corollary 2.7. □

Note that in Corollary 2.8(ii), if $Rad(M) = 0$, then $X \cap C = 0$ so $M = C \oplus X$.

PROPOSITION 2.9. *Let $f : M \longrightarrow N$ be an epimorphism. Then*

(i) If $X, Y \leq M$ such that $X\beta^*Y$, then $f(X)\beta^*f(Y)$.

(ii) If $X, Y \leq N$ such that $X\beta^*Y$, then $f^{-1}(X)\beta^*f^{-1}(Y)$.

(iii) If f is a small cover, $X \leq M$ and $K \leq N$ such that $f(X)\beta^*K$, then $X\beta^*f^{-1}(K)$.

Proof. (i) Assume that $f(X) + f(Y) + K = N$. Then $X + Y + f^{-1}(K) = M$. To see this, let $m \in M$. There exists $x \in X$, $y \in Y$ and $k \in K$ such that $f(m) = f(x) + f(y) + k$. Hence $f(m - x - y) = k$. So $m - x - y \in f^{-1}(K)$. Thus $m \in X + Y + f^{-1}(K)$. So $X + Y + f^{-1}(K) = M$. Hence $M = X + f^{-1}(K) = Y + f^{-1}(K)$. Consequently, $N = f(X) + K = f(Y) + K$. Therefore, from Theorem 2.3, $f(X)\beta^*f(Y)$.

(ii) Let $f^{-1}(X) + f^{-1}(Y) + H = M$. Then $X + Y + f(H) = N$, so, by Theorem 2.3, $X + f(H) = N = Y + f(H)$. Let $m \in M$. Then $f(m) = x + f(h)$ for some $x \in X$ and $h \in H$. Hence $f(m - h) = x$. So $m - h \in f^{-1}(X)$. Thus $m \in f^{-1}(X) + H$. Hence $M = f^{-1}(X) + H$. Similarly, $M = f^{-1}(Y) + H$. Therefore, from Theorem 2.3, $f^{-1}(X)\beta^*f^{-1}(Y)$.

(iii) Assume that $M = X + f^{-1}(K) + A$. Then $N = f(X) + K + f(A)$. Since $f(X)\beta^*K$, Theorem 2.3 yields $N = f(A) + f(X)$ and $N = f(A) + K$. Hence $N = f(A + X) = f(M)$ and $N = f(M) = f(A) + K$. Let $m \in M$. Since $N = f(A) + f(X)$, there exist $a \in A$, $x \in X$ such that $f(m) = f(a) + f(x)$. Hence $f(m - a - x) = 0$. Then $m - a - x \in Kerf$ and $m \in A + X + Kerf$. So $M = A + X + Kerf$. Since $Kerf \leq_s M$, $M = A + X$.

Since $N = f(A) + K$, there exist $a \in A$, $k \in K$ such that $f(m) = f(a) + k$. Since f is an epimorphism, there exists $y \in M$ such that $f(y) = k$. Then $f(m - a - y) = 0$. Hence

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$m - a - y \in Kerf$, thus $m \in Kerf + A + f^{-1}(K)$. So $M = Kerf + A + f^{-1}(K)$. Since $Kerf \leq_s M$, $M = A + f^{-1}(K)$. □

PROPOSITION 2.10. *Let $X \leq M$ and K a maximal submodule of M.*

(i) Let $C_1, C_2 \leq M$ such that $C_1 + C_2 = M$, $C_2 \neq M$ and $X\beta^* C_1$. Then $X \nsubseteq C_2$.

(ii) If $X\beta^* Y$ and $X \subseteq K$, then $Y \subseteq K$.

(iii) If $X\beta^* K$, then $X \subseteq K$. Hence, if $X\beta^* Y$ then $X \subseteq Rad(M)$ if and only if $Y \subseteq Rad(M)$.

(iv) If $X\beta^* K$ and $X + W = M$ with $X \cap W \leq_s M$, then $K = X + (K \cap W)$ and $K \cap W \leq_s M$.

Proof. (i) Assume $X \subseteq C_2$. Then $C_1 + X + C_2 = M$. Hence $C_1 + C_2 = M$ and $X + C_2 = M$. But $X \subseteq C_2 \Longrightarrow C_2 = M$, a contradiction.

(ii) Assume $Y \nsubseteq K$. Then $Y + K = M$ and $Y + K + X = M$. Hence $K + X = M$. But $X \subseteq K \Longrightarrow K = M$, a contradiction.

(iii) This part follows from (ii) using the symmetry of β^* with K and X replacing X and Y, respectively.

(iv) This part follows from part (iii) and Theorem 2.6. □

Note that from Lemma 2.10 (i), if $X, Y \leq M$ such that $X \subsetneq M$ and $X\beta^* Y$, then $X + Y \neq M$.

PROPOSITION 2.11. *Let $X_1, X_2, Y_1, Y_2 \leq M$ such that $X_1\beta^* Y_1$ and $X_2\beta^* Y_2$. Then $(X_1 + X_2)\beta^*(Y_1 + Y_2)$ and $(X_1 + Y_2)\beta^*(Y_1 + X_2)$. In particular, $X_1 + X_2 = M$ if and only if $Y_1 + Y_2 = M$ and $X_1 + Y_2 = M$ if and only if $Y_1 + X_2 = M$.*

Proof. Let $K \leq M$ such that $X_1 + X_2 + Y_1 + Y_2 + K = M$. Then $X_2 + Y_1 + Y_2 + K = M$ and $X_1 + X_2 + Y_2 + K = M$, because $X_1\beta^* Y_1$. Moreover $Y_1 + Y_2 + K = M$ and $X_1 + X_2 + K = M$, because $X_2\beta^* Y_2$. From Theorem 2.3, $(X_1 + X_2)\beta^*(Y_1 + Y_2)$. Using Lemma 2.2, we obtain $(X_1 + Y_2)\beta^*(Y_1 + X_2)$. By Corollary 2.8 (iii), $X_1 + X_2 = M$ if and only if $Y_1 + Y_2 = M$ and $X_1 + Y_2 = M$ if and only if $Y_1 + X_2 = M$. □

COROLLARY 2.12. *Let $X, Y \leq M$ and $J \leq_s M$. Then $X\beta^* Y$ if and only if $X\beta^*(Y + J)$.*

Proof. ($\Rightarrow$) This implication follows from Proposition 2.11 and the fact that $0\beta^* J$. ($\Leftarrow$) As above, $Y\beta^*(Y + J)$. Now the implication follows from the transitivity of the β^* relation. □

COROLLARY 2.13. *Let $X, Y_1, \ldots, Y_n \leq M$. If $X\beta^* Y_i$ for each i, then $X\beta^* Y$, where $Y = \Sigma_{i=1}^{n} Y_i$.*

DEFINITION 2.14. Let $X \leq M$. Then $X_{\beta^*} := \Sigma\{N \leq M \mid N\beta^* X\}$.

Observe that $Rad(M) = 0_{\beta^*}$ and that if $X \leq K$ then $X^*_\beta \leq K$, where K is a maximal submodule of M by Proposition 2.10. Moreover, if $X\beta^* Y$, then $X_{\beta^*} = Y_{\beta^*}$. Finally, if M is Noetherian, Corollary 2.13 yields that $X\beta^* X_{\beta^*}$. However, this is not true in general by the following example.

EXAMPLE 2.15. Proposition 2.11 can be extended to finite but not infinite sums. Let $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Then $\mathbb{Q} = Rad(\mathbb{Q}) = \sum_{m \in \mathbb{Z}^+} \frac{1}{m}\mathbb{Z}$ but each $\frac{1}{m}\mathbb{Z} \leq_s \mathbb{Q}$ hence $\frac{1}{m}\mathbb{Z}\beta^* 0$ for each m. If Proposition 2.11 was true for even countably infinite sums then $\mathbb{Q}\beta^* 0$, a contradiction since $\mathbb{Q}$ is not small in $\mathbb{Q}$.

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PROPOSITION 2.16. *Let M be a Noetherian module which is weakly supplemented (e.g. $\mathcal{G}^*$-supplemented) and $X \leq M$. Then $X_{\beta^*} = X + Rad(M)$.*

Proof. By Corollary 2.13, $X\beta^* X_{\beta^*}$. The result is now a consequence of Corollary 2.7 and Corollary 2.12. $\square$

Observe that from Proposition 2.9, if $f : M \longrightarrow N$ is an epimorphism and $X \leq M$, then $f(X_{\beta^*}) \leq [f(X)]_{\beta^*}$. Moreover, if f is a small cover, then $f(X_{\beta^*}) = [f(X)]_{\beta^*}$.

PROPOSITION 2.17. *Let $S \leq M$ and $I \leq R$. If $X \leq R$ such that $I^n \leq X \leq I$ for some positive integer n, then $SI\beta^* SX$. In particular, $I\beta^* X$.*

Proof. Clearly, the statement is true for $n = 1$. Assume that $n > 1$. Let $B \leq M$ such that $SI + B = M$. Then $SI^2 + BI = MI, \ldots, SI^n + BI^{n-1} = MI^{n-1}$. Hence

$$M = SI + B \leq MI + B = SI^2 + BI + B = SI^2 + B \leq MI^2 + B \leq \cdots \leq SI^n + B \leq M.$$

Thus $SI^n + B = M$. So $SX + B = M$. By Theorem 2.3, $SI\beta^* SX$. $\square$

EXAMPLE 2.18. Let $M = R = \mathbb{Z}$ and $K = p\mathbb{Z}$, for some prime p. Take $X = p^2\mathbb{Z}$. Then $X_{\beta^*} = K \neq X = X + Rad(M)$. Therefore the condition that X has a weak supplement is not superfluous in Propositions 2.10 (iv) and 2.16.

3. $\mathcal{G}^*$-lifting and $\mathcal{G}^*$-supplemented. In this section, we use the β^* equivalence relation to define the class of $\mathcal{G}^*$-lifting modules and the class of $\mathcal{G}^*$-supplemented modules. Some basic properties including behaviour with respect to direct sums and direct summands are developed for these classes. We locate these classes of modules between the class of lifting modules and the class of supplemented modules. Moreover, we indicate a connection between these modules and an Open Problem of Mohamed and Mšller [**6**, Open Problem #18, p. 107].

DEFINITION 3.1. (i) We say M is 'Goldie*-lifting, $\mathcal{G}^*$-lifting', if and only if for each $X \leq M$ there exists a direct summand D of M such that $X\beta^* D$.

(ii) We say M is 'Goldie*-supplemented, $\mathcal{G}^*$-supplemented', if and only if for each $X \leq M$ there exists a supplement submodule S of M such that $X\beta^* S$.

THEOREM 3.2. *M is $\mathcal{G}^*$-supplemented ($\mathcal{G}^*$-lifting) if and only if for each $X \leq M$ there exists a supplement S (direct summand D) and a small submodule H of M such that*

$$X + H = S + H = X + S$$
$$(X + H = D + H = X + D).$$

Proof. Assume that M is $\mathcal{G}^*$-supplemented. There exists a supplement S such that $X\beta^* S$. Hence there exists $W \leq M$ such that $S + W = M$ and $S \cap W \leq_s S$. By Proposition 2.11, $X\beta^*(X + S)$ and $S\beta^*(X + S)$. From Theorem 2.6, W is a weak supplement for S, X, and $X + S$. By the modular law,

$$X + H = S + H = X + S,$$

where $H = (X + S) \cap W \leq_s M$.
The converse follows from Corollary 2.4. The proof is similar for $\mathcal{G}^*$-lifting. $\square$

COROLLARY 3.3. *(i) If for each $X \leq M$ there exists a supplement S and $H \leq_s M$ such that $X = S + H$, then M is $\mathcal{G}^*$-supplemented. The converse holds if M is also distributive.*

(ii) Let M be $\mathcal{G}^*$-supplemented and $X \leq M$ such that $Rad(M) \leq X$. Then $X = S + H$, where S is a supplement and $H \leq_s M$.

Proof. (i) From Theorem 3.2 the hypothesis implies that M is $\mathcal{G}^*$-supplemented. Assume that M is $\mathcal{G}^*$-supplemented and distributive. Let $X \leq M$. Then there are $S, L \leq M$ such that $X\beta^*S$, $S + L = M$ and $S \cap L \leq_s S$. By Theorem 2.3, $X + L = M$. So $S = S \cap (X + L) = S \cap X + S \cap L = S \cap X$. Hence $S \leq X$. From Theorem 2.6, L is a weak supplement of X, so $X \cap L \leq_s M$. Thus $X = X \cap (S + L) = S + H$, where $H = X \cap L$.

(ii) This part follows from Theorem 3.2. □

COROLLARY 3.4. *Assume $Rad(M)$ is small in M (e.g. M is finitely generated). Then M is $\mathcal{G}^*$-supplemented if and only if for each $X \leq M$ there exists a supplement submodule S of M such that $S + Rad(M) = X + Rad(M)$.*

Proof. This result is a consequence of Theorem 3.2 and Corollary 2.4. □

The following theorem generalizes and extends Theorem 3.16 of [**5**]. To see this observe that our result holds if every submodule of M is projection invariant (i.e. $X \leq M$ and $e = e^2 \in End(M_R)$, then $eX \subseteq X$).

THEOREM 3.5. *Assume that $M = A \oplus B$, where $A = aM$, $B = bM$, $\{a, b\}$ is a set of orthogonal idempotents of $End(M_R)$, and $U = aU + bU$ for each $U \leq M$ (e.g. each U is fully invariant). Then M is $\mathcal{G}^*$-supplemented ($\mathcal{G}^*$-lifting) if and only if A and B are $\mathcal{G}^*$-supplemented ($\mathcal{G}^*$-lifting).*

Proof. ($\Rightarrow$) Let $X \leq A$. Then there exist $S, L \leq M$ such that $S + L = M$, $S \cap L \leq_s S$ and $X\beta^*S$. We claim that $X\beta^*aS$ as submodules of A. To see this, suppose that $X + aS + K = A$, for some $K \leq A$. Then $X + S + K + L = M$. By Theorem 2.3, $X + K + L = M$. Then $X + K + aL = A$ and $bL = B$. Thus $bS \leq bL$. So $aS + L = M$. Since S is a supplement of L in M, $S = aS$. Therefore $X\beta^*aS$.

Now $aS + aL = A$ and $aL \cap aS \leq L \cap S \leq_s S = aS$. Hence aS is a supplement in A. Therefore A is $\mathcal{G}^*$-supplemented. Similarly, B is $\mathcal{G}^*$-supplemented. The proof for M being $\mathcal{G}^*$-lifting is similar.

($\Leftarrow$) Let $U \leq M$, $U_1 = aU$ and $U_2 = bU$. There exist $L_1, S_1 \leq A$ such that $U_1\beta^*S_1$, $L_1 + S_1 = A$ and $L_1 \cap S_1 \leq_s S_1$. Likewise, there exist $L_2, S_2 \leq B$ such that $U_2\beta^*S_2$, $L_2 + S_2 = B$ and $L_2 \cap S_2 \leq_s S_2$. By Proposition 2.11, $U\beta^*(S_1 + S_2)$. Moreover, $S_1 + S_2 + L_1 + L_2 = M$ and $(S_1 + S_2) \cap (L_1 + L_2) = (S_1 \cap L_1) + (S_2 \cap L_2)$. Assume that $(S_1 + S_2) \cap (L_1 + L_2) + K = S_1 + S_2$, for some $K \leq M$. Then $(S_1 \cap L_1) + (S_2 \cap L_2) + aK + bK = S_1 + S_2$. Hence $(S_1 \cap L_1) + aK = S_1$. But $S_1 \cap L_1 \leq_s S_1$. So $aK = S_1$. Similarly, $bK = S_2$. Hence $S_1 + S_2$ is a supplement in M. Therefore M is $\mathcal{G}^*$-supplemented. The proof for A and B being $\mathcal{G}^*$-lifting is similar. □

Recall from [**6**, p. 95] that M is called *H-supplemented* if for every submodule A there is a direct summand D such that $A + X = M$ if and only if $D + X = M$.

THEOREM 3.6. *Let M be a module and consider the following conditions:*

(a) *M is lifting.*
(b) *M is $\mathcal{G}^*$-lifting.*

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*(c) M is H-supplemented [**6**, p. 95].*

(d) M is $\mathcal{G}^*$-supplemented.

(e) M is supplemented.

Then $(a) \Longrightarrow (b) \Longleftrightarrow (c) \Longrightarrow (d) \Longrightarrow (e)$.

Proof. (a) $\Longrightarrow$ (b). This implication follows from [**2**, p. 266, 22.3 (a)$\Leftrightarrow$(b)] and Corollary 2.8 (iv).

(b) $\Longleftrightarrow$ (c). This equivalence follows from Theorem 2.3.

(b) $\Longrightarrow$ (d). This implication follows from the fact that every direct summand is a supplement.

(d) $\Longrightarrow$ (e). Let X be a submodule of M. Then $X\beta^* S$, where S is a supplement. So there exists $W \leq M$ such that S is a supplement of W. There exists a supplement T such that $W\beta^* T$. Hence S is a supplement of T, by Theorem 2.6. From [**2**, p. 234, 20.4(9)], T is a supplement of S. By Theorem 2.6, T is a supplement of X. Therefore M is supplemented. □

PROPOSITION 3.7. *(i) Let M be $\mathcal{G}^*$-supplemented and K a maximal submodule of M. Then $K = S + (K \cap T)$, where S is a supplement of T, T is a supplement of K, T is a local module, $K\beta^* S$ and K is $\mathcal{G}^*$-supplemented.*

(ii) Let M be $\mathcal{G}^*$-lifting and K a maximal submodule of M. There exist $C, D \leq M$ such that $M = C \oplus D$, $K\beta^* D$, $K = D \oplus (K \cap C)$, C is a local module and K is $\mathcal{G}^*$-lifting.

Proof. (i) Since M is $\mathcal{G}^*$-supplemented there exists a supplement S such that $K\beta^* S$. By Theorem 3.6, K has a supplement T. From Theorem 2.6, T is a supplement of S. By [**2**, p. 234, 20.4(9)], S is a supplement of T. Proposition 2.10 yields that $K = S + (K \cap T)$. Then $K \cap T$ is a maximal submodule of T. Let $t \in T$ such that $t \notin K$. Then $K + tR = M$. Since T is a supplement of K, $T = tR$. Therefore T is a local module. To see that K is $\mathcal{G}^*$-supplemented, let $X \leq K$. There exist $S_1, L_1 \leq M$ such that $X\beta^* S_1$, $S_1 + L_1 = M$ and $S_1 \cap L_1 \leq_s S_1$. By Proposition 2.10, $S_1 \leq K$. By the modular law, $K = S_1 + (K \cap L_1)$. But $S_1 \cap (K \cap L_1) \leq S_1 \cap L_1 \leq_s S_1$. So S_1 is a supplement in K. Therefore K is $\mathcal{G}^*$-supplemented.

(ii) Since M is $\mathcal{G}^*$-lifting there exist $C, D \leq M$ such that $M = C \oplus D$ and $K\beta^* D$. From Theorem 2.6, C is a supplement of K. By Proposition 2.10, $K = D \oplus (K \cap C)$. As in part (i), C is a local module. To see that K is $\mathcal{G}^*$-lifting, let $X \leq K$. There exist $C_1, D_1 \leq M$ such that $M = C_1 \oplus D_1$ and $X\beta^* D_1$. By Proposition 2.10, $D_1 \leq K$. Thus $K = D_1 \oplus (K \cap C_1)$. Therefore K is $\mathcal{G}^*$-lifting. □

Using the previous result, we obtain a structure theorem for $\mathcal{G}^*$-lifting Noetherian modules.

THEOREM 3.8. *Let M be a Noetherian module such that each submodule is projection invariant. If M is $\mathcal{G}^*$-lifting, then M is a finite direct sum of local modules.*

Proof. Since M is Noetherian, it is a finite direct sum of indecomposable Noetherian modules. By [**5**, Corollary 2.4.], each indecomposable direct summand of M is $\mathcal{G}^*$-lifting. From Proposition 3.7 (ii), each indecomposable direct summand of M is local. □

EXAMPLE 3.9. In this example, we show that the one-way implications of Theorem 3.6 cannot be reversed.

(i) Let $R = \mathbb{Z}_8$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8$. By [**6**, p. 97]., M is $\mathcal{G}^*$-lifting (i.e. H-supplemented) but not lifting.

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(ii) Let R be a commutative local ring which has two incomparable ideals I and J. Let $M = R/I \oplus R/J$. By [**6**, p. 97, Lemma A.4(1)], M is amply supplemented hence $\mathcal{G}^*$-supplemented by Proposition 3.11; but M is not $\mathcal{G}^*$-lifting by [**6**, p. 97, Lemma A.4(3)] and Theorem 3.6. For a concrete example, let F be a field and

$$T = F[x]/ < x^4 >= \left\{ a\overline{1} + b\overline{x} + c\overline{x}^2 + d\overline{x}^3 \mid a, b, c, d \in F \text{ and } \overline{x} = x+ < x^4 > \right\}.$$

Let $R = \{a\overline{1} + c\overline{x}^2 + d\overline{x}^3 \in T\}$. Thus R is a subring of T. Moreover R is a commutative local Kasch Ring. The ideals of R are: 0, R, $F\overline{x}^2$, $F\overline{x}^3$, $F\overline{x}^2 + F\overline{x}^3$. Note that $F\overline{x}^2 + F\overline{x}^3$ is maximal and $F\overline{x}^2 \cap F\overline{x}^3 = 0$. Then $M = R/F\overline{x}^2 \oplus R/F\overline{x}^3$ is $\oplus$-supplemented and amply supplemented but not $\mathcal{G}^*$-lifting [**6**, p. 97].

(iii) (See [**2**, p. 279, Example 23.7]) Let K be the quotient field of a discrete valuation domain R which is not complete. Let $M = K \oplus K$. Then M is supplemented but not $\mathcal{G}^*$-supplemented. To see this, assume that M is $\mathcal{G}^*$-supplemented. Let $X \leq M$. Then $X\beta^*S$, where S is a supplement in M. By [**2**, p. 233, 20.2], S is coclosed in M. From [**2**, p. 279, Example 23.7], S is a direct summand of M. Thus M is $\mathcal{G}^*$-lifting. By Theorem 3.6 and [**6**, p. 97–98], M is not $\mathcal{G}^*$-lifting, a contradiction.

COROLLARY 3.10. *(i) M is a lifting module if and only if M is amply supplemented and strongly $\oplus$-supplemented.*

(ii) If M is $\mathcal{G}^*$-lifting, then M is $\mathcal{G}^*$-supplemented and $\oplus$-supplemented.

(iii) If M is $\mathcal{G}^*$-supplemented and strongly $\oplus$-supplemented, then M is $\mathcal{G}^*$-lifting.

Proof. (i) This part is in [**2**, p. 266].

(ii) This part follows from Theorem 3.6 and [**6**, pp. 95–96].

(iii) Let $X \leq M$. Then there exists a supplement submodule S of M such that $X\beta^*S$. Since M is strongly $\oplus$-supplemented S is a direct summand of M. Hence M is $\mathcal{G}^*$-lifting. □

PROPOSITION 3.11. *Let M be a module. Consider the following conditions:*

(i) M is amply supplemented.

(ii) For each $X \leq M$ there exists a supplement S and $L \leq M$ such that $M = S + L = X + L$, $S \cap L \subseteq X \cap L$ and $S \overset{cs}{\hookrightarrow} X + S$.

(iii) M is $\mathcal{G}^*$-supplemented.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

Proof. (i) $\Longrightarrow$ (ii) Assume X is small. Take $S = 0$ and $L = M$. So now assume X is not small. Since M is weakly supplemented, there exists L such that $X + L = M$ and $X \cap L \leq_s M$. Also, there exists a supplement S of L such that $S \subseteq X$. Hence $S + L = M$ and $S \cap L \subseteq X \cap L$ and $S \cap L \leq_s S$. Thus L is a weak supplement for both S and X. By Corollary 2.7, $X\beta^*S$. From Theorem 2.3, $S \overset{cs}{\hookrightarrow} X + S$.

(ii) $\Longrightarrow$ (iii) By Proposition 2.5, $X\beta^*S$. Thus M is $\mathcal{G}^*$-supplemented. □

Note that if M is amply supplemented then its submodules which have the same supplements in M are β^* equivalent, by [**2**, p. 254, Exercise 8].

PROPOSITION 3.12. *If M is a quasi-projective module then the following conditions are equivalent:*

(i) M is supplemented,

(ii) M is $\mathcal{G}^*$-supplemented,

(iii) M is amply supplemented,

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(iv) *M is lifting,*
(v) *M is $\mathcal{G}^*$ lifting,*
(vi) *M is semiperfect.*

Proof. This result follows Theorem 3.6, Proposition 3.11 and [**2**, 27.8 and 27.21]. □

Open questions and problems

(1) Must a $\mathcal{G}^*$-supplemented module be amply supplemented?
(2) If M is finitely generated and each maximal submodule is β^* equivalent to a supplement submodule, is M $\mathcal{G}^*$-supplemented?
(3) Characterize those rings for which every (cyclic, finitely generated) module is $\mathcal{G}^*$-lifting.
(4) Characterize those rings for which every (cyclic, finitely generated) module is $\mathcal{G}^*$-supplementing.

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