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ON p-LARGE SUBGROUPS OF p-TORSION GROUPS

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RESUMÉ. Les groupes p-torsion forment une classe de groupes abéliens mixtes dont les sous-groupes de p-base sont de torsion. Nous montrons ici que la généralisation naturelle à ces groupes de la notion de sous-groupe large développée pour les groupes primaires par R. S. Pierce, permet d'obtenir des résultats analogues. Ainsi nous caractérisons les sous-groupes p-larges d'un groupe p-torsion G en fonction des suites non-décroissantes d'entiers non-négatifs $u=(u_i)$ qui satisfont à la condition d'écart pour G. On obtient: un sous-groupe A du groupe p-torsion G est p-large si, et seulement si A est de la forme $G(u)$ pour une suite u telle que pour tout $x \in G$, la suite $(h(p^i x))$ est plus grande presque partout que la suite u.

Nous déterminons aussi, les sous-groupes p-large de $\hat{B}$, le complété p-adique d'une somme directe de groupes cycliques non bornés B, ainsi que ceux des sous-groupes p-purs totalement invariants de $\hat{B}$ engendrés par un élément.

An abelian group is said to be a p-torison group if its p-basic subgroups are p-primary groups for a fixed prime number p. The notion of p-large subgroups of an abelian group was introduced in [1]. It is a natural generalization of the concept of large subgroups of primary groups of R. S. Pierce [5]. A subgroup A of an abelian group G is said to be p-large in G if A is fully invariant in G and $A+B=G$, for every p-basic subgroup b of G. In the first section of this article we characterize the p-large subgroups of p-torsion groups in terms of certain sequences of non negative integers, and give some of their most important properties. the remaining sections deal with p-large subgroups of $\hat{B}$ the p-adic completion of an unbounded direct sum of cyclic primary groups, and the p-large subgroups of the smallest p-pure fully invariant subgroups of $\hat{B}$ containing a given element. All groups considered here are abelian and the notation and terminology, except for items explicitly designated, follow the usage established in [3].

1. **A characterization of p-large subgroups of p-torsion groups.** Many properties of large subgroups established in [5] extend with no difficulty to p-large subgroups of p-torsion groups. In particular we single out the following facts whose proofs are exactly the same as in [5] p. 219.

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LEMMA 1.1. *Let G be a p-torsion group and A and C be fully invariant subgroups of G. Then for every p-basic subgroup B of G we have:*

$$(A+B)\cap C=(A\cap C)+(B\cap C).$$

PROPOSITION 1.2. *Let G be a p-torsion group. We have the following properties:*

(i) *The set of p-large subgroups of G is closed under finite interections.*

(ii) *If A is a p-large subgroup of G then p^nA is a p-large subgroup of G for every positive integer n.*

(iii) *Every p-large subgroup of G contains $p^{\omega}G=\bigcap_{n=1}^{\infty}p^nG$.*

We recall the following definitions:

Let G be a group and $x\in G$. The pth-Ulm sequence of x in G is $H_p^G(x)=(h_p(x), h_p(px), \dots)$, where $h_p(x)$ is the p-height of x in G. Given a sequence $u=(u_0, u_1, \dots)$ of ordinal numbers we let:

$$G(u)=\{x\in G \mid H_p^G(x)\geq u\}.$$

$G(u)$ is a fully invariant subgroup of G.

LEMMA 1.3. *Let G be a p-torsion group, x and y be elements of G_p such that $H_p(x)\leqq H_p(y)$, and $\langle x\rangle\cap p^{\omega}G=\langle y\rangle\cap p^{\omega}G=0$. Then there exists an endomorphism f of G such that $f(x)=y$.*

Proof. As in the proof of Lemma 2.4 of [5] p. 223, there exist finite isomorphic subgroups X and Y of G_p which are pure in G_p and such that $x\in X$ and $y\in Y$. Clearly $H_p^X(x)\leq H_p^Y(y)$ and an obvious extension of Lemma 65.5 of [3] vol. II yields a homomorphism g from X to Y, such that $g(x)=y$. However X being finite and pure in G, it is a direct summand of G. The corresponding projection of G onto X followed by g gives an endomorphism of G that maps x onto y.

LEMMA 1.4. *Let G be a group, A a subgroup of G such that $p^{\omega}G=p^{\omega}A$ and let $x\in G_p$. Then there exists $a\in A$, such that: $H_p^G(x-a)=H_p^G(x)$, and $\langle x-a\rangle\cap p^{\omega}G=0$.*

Proof. Let $0(x)=p^k$ and let $H_p^G(x)=(n_0, n_1, \dots)$. Now $n_{k+i}=\infty$ $i=0, 1, \dots$. If all $n_j=\infty$, $j=0, 1, \dots$, then $x\in p^{\omega}G$, and since $p^{\omega}G=p^{\omega}A$, we let $a=x$. Similarly if $n_j\neq\infty$, $0\leq j<k$, then $\langle x\rangle\cap p^{\omega}G=0$, and we let $a=0$. Thus the lemma is true for the trivial cases. Suppose then that $n_t=\infty$, with $0<t<k$, and t is the smallest index for which $n_t=\infty$. Clearly $p^tx\in p^{\omega}G=p^{\omega}A$. Choose $b\in A$, such that: $p^tx=p^{r+1}b$, where $r>n_{t-1}$, and let $a=p^{r-t+1}b$. It is easy to verify that for $0\leq i<t$, $h_p(p^ia)\geq r-t+i+1>n_i$. It follows that: $h_p(p^i(x-a))=h_p(p^ix)=n_i$, for $0\leq i<t$, and $h_p(p^i(x-a))=\infty$, for $i\geq t$. Therefore $H_p^G(x-a)=H_p^G(x)$. Clearly $\langle x-a\rangle\cap p^{\omega}G=0$.

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LEMMA 1.5. *Let G be a p-torsion group and A a p-large subgroup of G. Then A_p is a large subgroup of G_p.*

Proof. The only difficulty is in showing that A_p is fully invariant in G_p. Let $x \in A_p$ and let g be an endomorphism of G_p. Clearly $p^\omega A = p^\omega G$ since $p^n A$ is p-large for every n and $p^\omega G$ is contained in every p-large subgroup of G. Thus we can apply Lemma 1.4 to x and to $g(x)$ to obtain a and $a' \in A$ such that: $\langle x-a\rangle \cap p^\omega G = 0 = \langle g(x)-a'\rangle \cap p^\omega G$, and $H_p^G(x-a) = H_p^G(x) \le H_p^G(g(x)) = H_p^G(g(x)-a')$. Thus, by Lemma 1.3 there exists an endomorphism f of G such that $f(x-a) = g(x)-a'$. However, $x-a \in A$ and A is fully invariant in G. Therefore $g(x)-a' \in A$, and it follows that $g(x) \in A \cap G_p = A_p$.

PROPOSITION 1.6. *Let A be a p-large subgroup of a p-torsion group G. Then there exists a strictly increasing sequence of non-negative integers u such that: $A = G(u)$.*

Proof. Since A_p is a large subgroup of G_p by Lemma 1.5, we can use the characterization of large subgroups in [5] or in [3] Theorem 67.2 vol. II. Thus there exists a strictly increasing sequence of non-negative integers u such that: $A_p = G_p(u)$. Now $A + G_p = G$ and A/A_p is p-divisible. Therefore $H_p^G(a) \ge u$, for every $a \in A$, and $A \subset G(u)$. But $G_p(u) = G(u) \cap G_p = A_p$. Furthermore $G/A_p = A/A_p \oplus G_p/A_p$, therefore $G(u)/A_p = A/A_p$, and $A = G(u)$.

REMARK. This proposition is a reworking of Proposition 4.4 in [1] where the difficulty of showing that A_p is indeed fully invariant in G_p was overlooked.

For the remainder of this section we let G be a p-torsion group and $u = (u_0, u_1, \ldots)$ a strictly increasing sequence of non-negative integers. The following is an easy consequence of Proposition 1.6.

COROLLARY 1.7. *$G(u)$ is p-large in G if and only if $G = G(u) + G_p$.*

In order to describe which sequences u determine p-large subgroups of a p-torsion group G we need the following concept:

DEFINITION 1.8. Let $v = (v_0, v_1, \ldots)$ be an increasing sequence of non-negative integers or ∞. We say that v is larger than u almost everywhere if there exists a non-negative integer k such that: $v_i \ge u_i$, for all $i > k$. We write $v \gtrdot u$. For a group G and a sequence u we let:

$$\underset{\cdot}{G}(u) = \{x \in G \mid H_p^G(x) \gtrdot u\}.$$

It is easy to show that $\underset{\cdot}{G}(u)$ is a fully invariant subgroup of G containing $G_p + G(u)$. Note that $\underset{\cdot}{G}(u) = \{x \in G \mid p^n x \in G(u)$ for some $n\}$.

THEOREM 1.9. *A subgroup A of p-torsion group G is a p-large subgroup of G if and only if $A = G(u)$ for some strictly increasing sequence of non-negative* *integers such that for every* $x \in G$, $H_p^G(x) \gtrapprox u$. *In other words if and only if* $A = G(u)$ *and* $G = \underset{\cdot}{G}(u)$.

Proof. Let A be a p-large subgroup of G. From Proposition 1.6 there exists a sequence u such that $A = G(u)$. From the remark after Definition 1.8 and Corollary 1.7:

$$\underset{\cdot}{G}(u) \supset G_p + G(u) = G.$$

Therefore $\underset{\cdot}{G}(u) = G$. Conversely, suppose $A = G(u)$ and $\underset{\cdot}{G}(u) = G$. Let $x \in G$, then $H_p^G(x) \gtrapprox u$. This means that there exists k such that: $h_p(p^i x) \geq u_i$, for all $i > k$. In particular: $p^{k+1}x = p^m y$, for some $y \in G$, and $m \geq u_{k+1}$. Let $z = p^{m-k-1}y$. We claim that $z \in G(u)$. Indeed, for $i > k$, $p^i z = p^i x$, therefore $h_p^G(p^i z) = h_p^G(p^i x) \geqq u_i$. But for $0 \leq i \leq k$, $h_p^G(p^i z) \geq m - k - 1 + i \geq u_{k+1} - (k+1) + i$. Now, u being a strictly increasing sequence of non-negative integers, we have: $u_{j+1} \geq u_j + 1$, $j = 0, 1, \ldots$. Thus, by adding term to term the $k - i + 1$ inequalities for $i \leq j \leq k$, we obtain $u_{k+1} \geq u_i + k - i + 1$. Therefore $h_p^G(p^i z) \geq u_i$. It follows that $z \in G(u)$, $x - z \in G_p$, and $x = z + (x - z) \in G(u) + G_p$. This shows that $G = G(u) + G_p$. From Corollary 1.7, $G(u) = A$ is p-large in G.

This characterization can be used to extend to p-torsion groups and p-large subgroups other properties of large subgroups of primary groups. Thus we have:

COROLLARY 1.10. *Let F be a p-pure subgroup of a p-torsion group G and let A be a p-large subgroup of G. Then* $A \cap F$ *is a p-large subgroup of F.*

Another consequence of the preceding results in the following:

PROPOSITION 1.11. *Let G be a p-torsion group, B a p-basic subgroup of G and A a p-large subgroup of G. Then* $B \cap A$ *is a p-basic subgroup of A and G/A is a direct sum of cyclic p-groups.*

Proof. $B \cap A = B \cap A_p$, and since A_p is large in G_p, $B \cap A$ is a p-basic subgroup of A_p. But A_p is pure in A and A/A_p is p-divisible. Therefore $B \cap A$ is a p-basic subgroup of A. Now G/A is isomorphic to $B/B \cap A$ and by Corollary 1.10, $B \cap A$ is fully invariant in B. It follows that $B/B \cap A$ is a direct sum of cyclic p-groups.

We conclude this section with one more property of p-large subgroups corresponding to Theorem 2.13 of [5]. The proof is the same as in [5].

PROPOSITION 1.12. *Let a be a p-large subgroup of a p-torsion group G and let L be a p-large subgroup of A. Then L is a p-large subgroup of G.*

REMARK. As in the case of primary groups, a fully invariant subgroup A of a p-torsion group is p-large in G if and only if for some p-basic subgroup B which is not a summand of G, $A + B = G$.

2. **The p-large subgroups of $\hat{B}$.** Let $\hat{B}$ be the completion with respect to the p-adiac topology of an unbounded direct sum of cyclic p-groups $B=\oplus B_n$ where B_n is the direct sum of cyclic groups of order p^n. Denote by $\bar{B}$ the p-primary subgroup of $\hat{B}$. It is well-known that $\hat{B}/\bar{B}$ is a torsion free divisible group. Thus $\hat{B}$ is a reduced p-torsion group and by Proposition 1.6 every p-large subgroup of $\hat{B}$ is of the form $\hat{B}(u)$ for some strictly increasing sequence of non-negative integers u. We show here that only sequences with finitely many gaps give rise to p-large subgroups of $\hat{B}$. We need a few preliminaries.

DEFINITION 2.1. Let $u=(u_0, u_1, \ldots)$ be a strictly increasing sequence of non negative integers. We say that u has a gap at i if $u_i+1<u_{i+1}$. We say that u has finitely many gaps if there exists k such that $u_k+i=u_{k+i}$ for every $i=1, 2, \ldots$, If no such k exists we say that u has infinitely many gaps. We say that u satisfies the gap condition with respect to a p-torsion group G if the u_i-th Ulm-Kaplansky invariant of G is non-zero each time there is a gap at i.

LEMMA 2.2. *If u has finitely many gaps then for any group G, $G(u)$ contains p^nG for some $n\geq 0$, and thus, $G(u)$ is p-large in G.*

Proof. By hypothesis, there exists k such that $u_k+i=u_{k+i}$, $i=1, 2, \ldots$. Let $n=u_k$ and suppose $x\in p^nG$. Then $x=p^ny$, and $h_p(p^ix)\geq u_k+i=u_{k+i}\geq u_i$, for every $i=0, 1, \ldots$. Therefore $x\in G(u)$ and $p^nG\subset G(u)$.

NOTATION 2.3. For a sequence u and an element x of G we let

$$L(u, x)=\{i \mid h_p^G(p^ix)<u_i\}.$$

Clearly, $H_p^G(x)\gtrapprox u$ if and only if $L(u, x)$ is finite. Theorem 1.9 can be restated in terms of $L(u, x)$; namely for a p-torsion group G and a sequence u, $G(u)$ is p-large in G if and only if $L(u, x)$ is finite for every $x\in G$.

LEMMA 2.4. Let u be a strictly increasing sequence of non-negative integers with infinitely many gaps. Then there exists $x\in\hat{B}$ such that $L(u, x)$ is infinite.

Proof. Write $B=\oplus_{i=1}^{\infty} B_i$ where $B_i=\oplus Z(p^i)$, $i\geq 1$. Let $n1$ be an integer such that $u_{n1}+1<u_{n1+1}$ and choose $m1\geq u_{n1}+2$ such that $B_{m1}\neq 0$. Choose in B_{m1} an element a_{m1} such that $h_p(a_{m1})=u_{n1}-n1$. Now we repeat this operation for an integer $n2$, such that $n2\geq e(a_{m1})$, $u_{n2}+1<u_{n2+1}$ and $m_2\geq u_{n2}+2$ with $B_{m2}\neq 0$ and we choose in B_{m2} an element a_{m2} such that $h_p(a_{m2})=u_{n2}-n_2$. We continue this process and obtain a sequence a_{mi} of elements such that: $h_p(a_{mi})<h_p(a_{m(i+1)})<\cdots$ and $n1+2\leq e(a_{m1})\leq n2<n2+2\leq e(a_{m2})\leq\cdots$. Now let $x=(a_j)\in\prod_{j=1}^{\infty} B_j$ where $a_j=0$ unless $j=ni$ for some i, in which case $a_j=a_{ni}$. This $x\in\hat{B}$ since $h_p(a_n)$ increases as n increases. Furthermore $L(u, x)$ is infinite. In fact $h(p^{ni}x)=h(p^{ni}a_{mi})=u_{ni}$ and

$$h_p(p^{ni+1}x)=u_{ni}+1<u_{ni+1} \quad \text{for all} \quad i=1, 2, \ldots.$$

The preceding lemma could also be derived from Proposition 3.5 of (4).

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Lemma 2.2 and Lemma 2.4 and the remark before Lemma 2.4, immediately yield the following:

PROPOSITION 2.5. *$\hat{B}(u)$ is p-large in $\hat{B}$ if and only if u has finitely many gaps.*

THEOREM 2.6. *The p-large subgroups of $\hat{B}$ are precisely the fully invaraiant subgroups of $\hat{B}$ which contain $p^n\hat{B}$ for some $n \geq 0$. They are in one to one correspondence with the strictly increasing sequences of non-negative integers with finitely many gaps which satisfy the gap-condition.*

First Proof. Follows immediately from Proposition 2.5 and Lemma 2.2. The unicity of the sequence follows from Theorem 67.1, p. 12 in [3], vol. II.

Second Proof. Let A be a p-large subgroup of $\hat{B}$. Then from Proposition 1.11 $\hat{B}/A$ is a direct sum of cyclic groups. But $\hat{B}$ is an algebraically compact reduced group. From Corollary 39.2 in [3], vol. I, since $(\hat{B}/A)^1 = 0$, A and $\hat{B}/A$ are algebraically compact reduced groups. From Corollary 40.3 in [3], vol. I, $\hat{B}/A$ is bounded. Therefore A contains $p^n\hat{B}$ for some n.

3. p-**pure fully invariant subgroups of** $\hat{B}$. We consider here the simplest kind of p-pure fully invariant subgroups of $\hat{B}$ and we determine completely their p-large subgroups. Given $x \in \hat{B}$, let $h = H_p(x)$ and

$$\hat{B}_x = \{y \in \hat{B} \mid p^n y \in \quad -B(h) \text{ for some } n \geq 0\} = \hat{B}(h)$$

$\hat{B}_x$ is a p-pure fully invariant subgroup of $\hat{B}$ containing $\bar{B}$. It is in fact the smallest p-pure fully invariant subgroup of $\hat{B}$ containing x.

PROPOSITION 3.1. *For every strictly increasing sequence of non-negative integers u with infinitely many gaps there exists a p-pure fully invariant subgroup G of a $\hat{B}$ such that $G(u)$ is p-large in G.*

Proof. Let $u = (u_i)_{i=0}^{\infty}$ be the given sequence. Derive from u the sequence $h = (h_i)_{i=0}^{\infty}$, such that: $h_k = u_{2k}$, for every $k = 0, 1, \ldots$. The sequence h has a gap at each $i = 0, 1, \ldots$. Take $B = \oplus B_n$ such that for every k, $B_{h_k+1} \neq 0$. Then, by Proposition 3.5 of [4], there exists $x \in \hat{B}$ such that $H_p(x) = h$. Let $G = \hat{B}_x$. We show now that $L(u, y)$ is finite for all $y \in G$. Since $p^n y \in \hat{B}(h)$, there exists an endomorphism of $\hat{B}$ which maps x onto $p^n y$, thus $h_p(p^{n+i}y) \geq h_p(p^i x) = u_{2i}$, but $h_p(p^{n+i}y) = h_{n+i}$ therefore $h_{n+i} \geq u_{2i} \geq u_{n+i}$, for all $i \geq n$ and $L(u, y)$ is finite. Therefore, $G(u)$ is p-large in G.

We consider now a group $G = \hat{B}_x$ and characterize the sequences u for which $G(u)$ is a p-large subgroup of G in terms of the height sequence of x in $\hat{B}$. We bear in mind that $G(u)$ is p-large in G if and only if $L(u, y)$ is finite for every $y \in G$.

LEMMA 3.2. *Let $G = \hat{B}_x$, and let u be a strictly increasing sequence of non-negative integers with infinitely many gaps satisfying the gap condition with*

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respect to $\hat{B}$. Let $H_p(x) = h = (h_i)_{i=0}^{\infty}$, $u = (u_i)_{i=0}^{\infty}$. If $G(u)$ is p-large in G then for every $m \geq 0$, there exists $j_m \geq 0$, such that $h_i \geq u_{i+m}$, whenever $i \geq j_m$.

Proof. We know that $L(u, x)$ is finite. Thus for $m = 0$ there exists j_0 such that $i > j_0$ implies $h_i \geq u_i$. Now $p^m G(u)$ is also p-large in G. Define $v = (v_i)$, where $v_i = u_{i+m}$. We claim that $p^m G(u) = G(v)$. Clearly $p^m G(u) \subset G(v)$. In order to show the reverse inclusion we note first that $G(u) = G \cap \hat{B}(u)$ since G is p-pure in $\hat{B}$. Since u satisfies the gap condition, there exists $z \in \hat{B}$, such that:

$$\hat{B}(u) = Ez = \{\varphi(z) \mid \varphi \in \text{End } \hat{B}\} \quad \text{and} \quad H_p(z) = u.$$

Thus: $H_p(p^m z) = v$, and $\hat{B}(v) = Ep^m z = p^m \hat{B}(u)$. Clearly:

$$G(v) = G \cap \hat{B}(v) = G \cap p^m \hat{B}(u) \subset p^m G \cap p^m \hat{B}(u) \subset p^m (G \cap \hat{B}(u)) = p^m G(u).$$

Now we apply the same reasoning to $G(v)$ as to $G(u)$ and find that there exists j_m such that: $i \geq j_m$ implies $h_i \geq v_i = u_{i+m}$.

THEOREM 3.3. *Let G, H, u be as in Lemma 3.2. Then $G(u)$ is p-large in G if and only if: for every $m \geq 0$, there exists j_m such that $i \geq j_m$ implies $h_i \geq u_{i+m}$. Moreover every p-large subgroup of G is of the form $G(u)$ for a unique such sequence u.*

Proof. The necessity follows from Lemma 3.2. For the sufficiency, we need only show that $L(u, y)$ is finite for every $y \in G$. Now, there exists m such that $p^m y \in \hat{B}(h)$; therefore, $h_{i+m}(y) = h_p(p^{i+m} y) \geq h_i$, and for $i \geq j_m : h_{i+m}(y) \geq h_i \geq u_{i+m}$. Therefore $L(u, y)$ is finite for every $y \in G$. It follows that $G(u)$ is p-large in G. The unicity follows from Theorem 67.1, p. 12, in vol. II of [3].

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