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RATIONAL REPRESENTATIONS OF GL_2

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Abstract. Let F be an algebraically closed field of characteristic p. We fashion an infinite dimensional basic algebra $\underleftarrow{\mathcal{C}}_p(F)$, with a transparent combinatorial structure, which controls the rational representation theory of $GL_2(F)$.

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1. Introduction. In any first course on representation theory, the students will become familiar with the representations of the algebraic group $GL_2(\mathbb{C})$, or with those of some close relation of this group. It is therefore surprising that over a century after the birth of group representation theory, anything remains to be said about GL_2. However, development in the modular theory has been much slower than in characteristic zero and even in the smallest cases no full understanding has yet been reached. In this paper, we wish to pursuade the reader that there is structure underlying the rational representation theory of GL_2 over a field of positive characteristics, as simple as the structure appearing in characteristic zero, although quite different in nature.

Of course, even in positive characteristic, the usual first questions about GL_2-modules were answered long ago: irreducibles are parametrised by elements of the dominant region of the weight lattice, and have realisations as tensor products of Frobenius twists of socles of symmetric powers of the natural representation in small degrees, and powers of the determinant representation. However, the situation is more delicate than implied by these easy truths. There are homological interactions between irreducible modules, and for a deeper understanding one ought to contemplate the manner in which these interactions occur. This is the concern of our paper.

We shall be more precise. Let A be an algebra with a self-dual bimodule T. Let B be the algebra whose category of ungraded representations is equivalent to the category of graded representations of the trivial extension algebra of A by T. Let C be the trivial extension algebra of B by its dual. Modulo the infinite dimensionality of C, we have a map

$$\mathcal{C} \circlearrowright \{\text{algebras with a self-dual bimodule}\},$$

which takes an algebra A, with an A-A bimodule T, such that ${}_AT_A \cong {}_AT_A^*$, to a symmetric algebra C. The self-dual bimodule corresponding to C is the regular bimodule ${}_CC_C$.

For every $n \in \mathbb{N}$, there is a localisation

$$\mathcal{C}_n \circlearrowleft \{\text{algebras with a self-dual bimodule}\}$$

of $\mathcal{C}$. There is a canonical epimorphism

$$A \twoheadleftarrow \mathcal{C}_n(A).$$

Taking the inverse limit of the sequence

$$A \twoheadleftarrow \mathcal{C}_n(A) \twoheadleftarrow \mathcal{C}_n(\mathcal{C}_n(A)) \twoheadleftarrow \mathcal{C}_n(\mathcal{C}_n(\mathcal{C}_n(A))) \twoheadleftarrow \dots,$$

we obtain an algebra $\underleftarrow{\mathcal{C}}_n(A)$. Taking the union of the corresponding sequence of fully faithful embeddings of categories of finite dimensional modules

$$A \to \mathcal{C}_n(A)\text{-mod} \hookrightarrow \mathcal{C}_n(\mathcal{C}_n(A))\text{-mod} \hookrightarrow \mathcal{C}_n(\mathcal{C}_n(\mathcal{C}_n(A)))\text{-mod} \hookrightarrow \dots,$$

gives us a module category that we denote $\underleftarrow{\mathcal{C}}_n(A)$-mod.

Let $\mathcal{S}(2) = \bigoplus_{r \geq 0} \mathcal{S}(2, r)$ be the Schur algebra associated to GL_2, defined over an algebraically closed field F of characteristic $p > 0$ [**5**]. There is a sequence of natural surjections

$$\mathcal{S}(2, r) \twoheadleftarrow \mathcal{S}(2, r+2) \twoheadleftarrow \mathcal{S}(2, r+4) \twoheadleftarrow \mathcal{S}(2, r+6) \twoheadleftarrow \dots$$

Let $\mathcal{S}(2, \underline{r})$ be the inverse limit of this directed sequence of algebra epimorphisms, and $\mathcal{S}(2, \underline{r})$-mod be the category of modules over this algebra, which are finite dimensional modules for some $\mathcal{S}(2, r+2x)$. The category of rational representations of $GL_2(F)$ is equivalent to the direct sum of categories $\bigoplus_{r \in \mathbb{Z}} \mathcal{S}(2, \underline{r})$-mod.

In the sequel, we define a certain filtration on $\mathcal{S}(2, r)$, refining the radical filtration, and denote by $\mathcal{G}(2, r)$ the graded ring associated to this filtration. There is a compatible sequence of surjections

$$\mathcal{G}(2, r) \twoheadleftarrow \mathcal{G}(2, r+2) \twoheadleftarrow \mathcal{G}(2, r+4) \twoheadleftarrow \mathcal{G}(2, r+6) \twoheadleftarrow \dots$$

Let $\mathcal{G}(2, \underline{r})$ be the inverse limit of this directed sequence of algebra epimorphisms, and $\mathcal{G}(2, \underline{r})$-mod be the category of modules over this algebra, which are finite dimensional modules for some $\mathcal{G}(2, r+2x)$. Our main result is the following:

THEOREM 1. *Every block of* $\mathcal{G}(2, \underline{r})$-mod *is equivalent to* $\underleftarrow{\mathcal{C}}_p(F)$-mod.

Our proof of Theorem 1 is inductive. We apply the results of Erdmann, Henke and Koenig concerning $\mathcal{S}(2, r)$ ([**4**, **7**]) to prove that certain Ringel self-dual blocks of $\mathcal{G}(2, r)$ are equivalent to $\mathcal{C}_p^d(F)$, for some d. Since every block of $\mathcal{G}(2, r)$ is a quotient of such a Ringel self-dual block, the theorem follows.

In fact, we prove a rather stronger statement. Let S be a Ringel self-dual Schur algebra $\mathcal{S}(2, r)$. We demonstrate the existence of a filtration by ideals,

$$S \supset \mathcal{N} \supset \mathcal{N}^2 \supset 0,$$

whose associated graded ring is Morita equivalent to $\mathcal{C}_a(A) \oplus F^{\oplus m}$, where A is a smaller Ringel self-dual Schur algebra $\mathcal{S}(2, s)$, $2 \leq a \leq p$, and m is some multiplicity.

In an earlier chapter, we give careful definitions of B, C and $\mathcal{C}_p$, and prove that under favourable conditions they respect the quasi-heredity condition.

This is all very pleasing, but more is true. In fact $\mathcal{S}(2,r) \cong \mathcal{G}(2,r)$, for all r, and therefore $\mathcal{S}(2,\underline{r}) \cong \mathcal{G}(2,\underline{r})$. In other words, the following is true.

THEOREM 2. *Every block of rational representations of $GL_2(F)$ is equivalent to $\underleftarrow{\mathcal{C}}_p(F)$-mod.*

Theorem 2 is proved in a sequel to this paper [**9**].

2. Setup. Throughout this paper, F will be a field and A an F-algebra. Let $\mathcal{J}(A)$ denote the Jacobson radical of A.

We suppose that A is a locally finite dimensional algebra. In other words, there exists a set Λ, indexing a set of orthogonal idempotents $\{e_\lambda\}_{\lambda\in\Lambda}$, such that $A \cong \bigoplus_{\lambda,\mu\in\Lambda} e_\lambda A e_\mu$, and Ae_λ and $e_\lambda A$ are finite dimensional. We assume that $A/\mathcal{J}(A) = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a direct sum of matrix rings M_λ over F, where e_λ is the unit of M_λ. Thus, Λ is an indexing set for isomorphism classes of simple A-modules. By the idempotent decomposition, simple modules have projective covers and injective hulls, providing 1–1-correspondences between isomorphism classes of simples, projectives and injectives.

We denote by A-mod the category of finite dimensional left A-modules M such that $M = \oplus_\lambda e_\lambda M$, and by mod-$A$ the category of finite dimensional right A-modules M such that $M = \oplus_\lambda M e_\lambda$. Given a finite dimensional left/right module M, we write the dual of M as $M^* = \mathrm{Hom}_F(M,F)$, a right/left module. We write A-proj for the category of finite dimensional projective left A-modules. Given a collection $X \subset A$-mod, we denote by $\mathcal{F}(X)$ the category of modules filtered by objects in X.

Now let Λ be a poset which is interval-finite (i.e. for every $\mu \le \lambda \in \Lambda$ the set $\{\nu | \mu \le \nu \le \lambda\}$ is finite).

Recall that mod-A is the highest weight category in the sense of Cline, Parshall and Scott [**1**] if, for every $\lambda \in \Lambda$ there exists an irreducible right module $L^r(\lambda)$, a costandard right module $\nabla^r(\lambda)$, which embeds into the injective hull $I^r(\lambda)$ of $L^r(\lambda)$, such that the cokernel of this inclusion is filtered by $\nabla^r(\mu)$ for $\mu > \lambda$, and $\nabla^r(\lambda)/\operatorname{soc}\nabla^r(\lambda)$ consists of composition factors $L^r(\nu)$ for $\nu < \lambda$. Dualizing with respect to F, we find this is equivalent to the corresponding projective indecomposable left modules $P(\lambda) \in A$-mod having standard filtrations. So, for every $\lambda \in \Lambda$ there exists a standard module $\Delta(\lambda)$ and an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$, the kernel of which is filtered by modules $\Delta(\mu)$ for $\mu > \lambda$, and the kernel of the map $\Delta(\lambda) \twoheadrightarrow L^l(\lambda)$ consists of composition factors of the form $L(\nu)$ for $\nu < \lambda$.

Let $J \subset \Lambda$ be a non-empty finitely generated ideal. The subcategory mod-$A[J]$ of objects, which only have composition factors $L(\nu)$ for $\nu \in J$ is the highest weight category, whenever mod-A is the highest weight category ([**1**], Theorem 3.5). Let $A^J = A/\sum_{\lambda\notin J} Ae_\lambda A$. Then A^J is a locally finite dimensional algebra and mod-$A[J] \cong$ mod-A^J.

Let $I \subset \Lambda$ be a non-empty finitely generated coideal and define $A_I := \bigoplus\limits_{\lambda,\mu\in I} e_\lambda A e_\mu$.

LEMMA 3. *If* mod-A *is the highest weight category, then* mod-A_I *is the highest weight category.*

Proof. We construct Δ-filtrations of projectives in A_I-mod. Projectives in A_I-mod are of the form $P_{A_I}(\lambda) := \mathrm{Hom}_A(\bigoplus\limits_{\mu\in I} Ae_\mu, P_A(\lambda))$. We define $\Delta_{A_I}(\lambda) :=$

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$\mathrm{Hom}_A(\bigoplus_{\mu\in I} Ae_\mu, \Delta_A(\lambda))$. Since $\mathrm{Hom}_A(\bigoplus_{\mu\in I} Ae_\mu, -)$ is exact, we obtain a filtration of $P_{A_I}(\lambda)$ respecting the necessary conditions on orders. □

Let us define $A_I^J := (A^J)_{I\cap J}$.

If $I \cap J$ is finite, then A_I^J is a finite dimensional quasi-hereditary algebra, whenever mod- A is the highest weight category ([**1**], Theorem 3.5).

PROPOSITION 4. mod- A *is the highest weight category if and only if* A_I^J *is quasi-hereditary for all finitely generated coideals* I *and finitely generated ideals* J *such that* $I \cap J$ *is finite.*

Proof. As noted above, the 'only if' statement is well known [**1**]. So, suppose A_I^J is quasi-hereditary for all suitable I and J. By a standard argument of Dlab [**2**], the existence of the highest weight structure on mod- A is equivalent to the surjective multiplication map

$$\frac{Ae_\lambda}{\sum_{\mu>\lambda} Ae_\mu Ae_\lambda} \otimes_F \frac{e_\lambda A}{\sum_{\mu>\lambda} e_\lambda Ae_\mu A} \longrightarrow \frac{\sum_{\mu\geq\lambda} Ae_\mu A}{\sum_{\mu>\lambda} Ae_\mu A},$$

being an isomorphism, for all $\lambda \in \Lambda$. But this can be checked on arbitrarily large finite truncations of Λ containing λ. □

COROLLARY 5. *For a locally finite-dimensional algebra* A*,* A -mod *is a highest weight category if and only if* mod- A *is a highest weight category.*

Proof. Follows immediately from Proposition 4 and the same statement for finite-dimensional algebras ([**10**], Section 4.3(b)). □

DEFINITION 6. A locally finite-dimensional algebra A is *quasi-hereditary* if A -mod and mod- A are the highest weight categories.

Note that by Corollary 5, we can now move freely between left and right modules, and standard and costandard filtrations and we have the usual duality relations between standard modules on one side and costandard modules on the other: $\Delta^r(\lambda) \cong \nabla(\lambda)^*$, $\nabla^r(\lambda) \cong \Delta(\lambda)^*$.

For the rest of this chapter, let A be a locally finite-dimensional quasi-hereditary algebra with poset Λ of weights, left standard modules $\Delta(\lambda)$, left costandard modules $\nabla(\lambda)$, right standard modules $\Delta^r(\lambda)$ and right costandard modules $\nabla^r(\lambda)$. The remaining propositions in this chapter are all proved by cutting down to a suitable finite-dimensional subquotient and applying Ringel's tilting theory for finite-dimensional quasi-hereditary algebras there [**11**]. We therefore omit the proofs.

DEFINITION 7. $T \in A$ -mod is called *tilting* if it is filtered by standard and costandard modules.

PROPOSITION 8. *There is a one-to-one correspondence between* Λ *and the set of indecomposable tilting modules in* A -mod.

We denote by $T(\lambda)$ the unique indecomposable tilting module such that $[T(\lambda) : L(\lambda)] = 1$, and $[T(\lambda) : L(\mu)] \neq 0$ implies $\mu \leq \lambda$.

DEFINITION 9. We say that A' is *Ringel dual* to A if there exist multiplicities $n_\lambda \in \mathbb{Z}_{\geq 1}$, such that $A' \cong End(T)^{op}$, where $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)^{n_\lambda}$.

If A, A' are Ringel dual, then $T = \bigoplus_{\lambda\in\Lambda} T(\lambda)^{n_\lambda}$ is an A-A' bimodule. In these circumstances, we call it a tilting bimodule.

For any subset Γ of Λ let Γ' equal Γ as a set, but with the opposite order. Thus, for an ideal $J \subseteq \Lambda$ we obtain a coideal $J' \subseteq \Lambda'$, for a coideal $I \subseteq \Lambda$ we obtain an ideal $I' \subseteq \Lambda'$.

Proposition 10. *A' is quasi-hereditary with poset Λ'.*

Proposition 11. *$A'' \cong A$.*

Proposition 12.
(i) $(A^J)' \cong A'_{J'}$.
(ii) $(A_I)' \cong (A')^{I'}$.

3. Algebraic constructions. Throughout this chapter A will be a finite-dimensional algebra, endowed with an A-A-bimodule T. We also assume T is *faithful*, meaning it is faithful as a left A-module.

Define $B_0 := \bigoplus_{i\in\mathbb{Z}} A_i$, where $A_i \cong A$ for all $i \in \mathbb{Z}$. Then B_0 is a locally finite dimensional algebra. We define $B_1 := \bigoplus_{i\in\mathbb{Z}} {}_iT_{i+1}$ as a B_0, B_0-bimodule, where each ${}_iT_{i+1}$ is isomorphic to T but with action of A_i on the left and of A_{i+1} on the right.

Let B be the trivial extension of B_0 by B_1; we can think of this as a matrix

$$B = \begin{pmatrix} \ddots & {}_{i-2}T_{i-1} & 0 & \cdots & & \\ 0 & A_{i-1} & {}_{i-1}T_i & 0 & \cdots & \\ \cdots & 0 & A_i & {}_iT_{i+1} & 0 & \cdots \\ & \cdots & 0 & A_{i+1} & {}_{i+1}T_{i+2} & 0 \\ & & & & A_{i+2} & \ddots \\ & & & & & \ddots \end{pmatrix},$$

where the A_i is on the leading diagonal. The algebra B is locally finite dimensional. Let $e_i = 1_{A_i}$. Let

$$B^* = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_F(Be_i, F),$$

a B-B-bimodule. Let C be the trivial extension of B by B^*. Then C is a locally finite dimensional algebra. As a trivial extension of an algebra by its dual, C is a symmetric algebra, thanks to a canonical isomorphism $C \cong C^*$ of C-C-bimodules, where $C^* = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_F(Ce_i, F)$; this isomorphism identifies B^* with B^* via the identity and B with $(B^*)^*$ via the canonical map.

Let C^n denote the quotient $C/\sum_{i>n} Ce_iC$ of C. Let C_1^n denote the subalgebra $\sum_{i,j\geq 1} e_iC^ne_j$ of C^n.

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LEMMA 13. *The algebra C_1^n is $\mathbb{Z}$-graded, concentrated in degrees* 0, 1 *and* 2. *In descending vertical order, its components in degrees* 0, 1 *and* 2 *are*

$$\bigoplus_{1\leq i\leq n} A_i,$$

$$\bigoplus_{1\leq i\leq n-1} ({}_iT_{i+1} \oplus {}_iT_{i+1}{}^*),$$

$$\bigoplus_{1\leq i\leq n-1} A_i^*.$$

Let $D_\infty = \langle \sigma, \tau \,|\, \sigma^2 = 1, \sigma\tau\sigma = \tau^{-1}\rangle$ denote the infinite dihedral group.

LEMMA 14. *Suppose that $T \cong T^*$, as A-A-bimodules. Then D_∞ acts as automorphisms of C. The space*

$$T_1^n = \bigoplus_{\substack{0\leq j\leq n-1\\ 1\leq i\leq n}} e_iCe_j$$

has the structure of a faithful self-dual C_1^n-C_1^n-bimodule.

Proof. We define an action of D_∞ on C as follows: The involution σ sends A_i to A_{-i} via the identity, A_i^* to A_{-i}^* via the identity, ${}_{i-1}T_i$ to $({}_{-(i-1)}T_{-i})^*$ via the isomorphism $T \cong T^*$, and analogously $({}_{i+1}T_i)^*$ to ${}_{-(i+1)}T_{-i}$. Thanks to the assumption that ${}_AT_A^* \cong {}_AT_A$, we see that this is indeed an algebra isomorphism. The translation τ acts as the automorphism of C given by shifting indices by 1, e.g. $\tau(e_i) = e_{i+1}$.

Of course, C itself is a C-C-bimodule, but what about the truncation T_1^n ? The idempotents e_i, for $i > n$ act as zero on Ce_j, for $j < n$. Therefore, C_1^n acts naturally on the left of T_1^n. After twisting the right action of C on itself by the automorphism $\sigma \circ \tau^{-n}$, we can similarly observe a right action of C_1^n on T_1^n.

Now

$$\left(\bigoplus_{\substack{0\leq j\leq n-1\\ 1\leq i\leq n}} e_iCe_j\right)^* \cong \bigoplus_{\substack{0\leq j\leq n-1\\ 1\leq i\leq n}} e_jCe_i = \sigma\tau^{-n}\big(T_1^n\big)$$

via the canonical symmetric algebra structure on C, so T_1^n admits the structure of a C_1^n-C_1^n-bimodule, and is self-dual. To check that T_1^n is faithful, it is enough to check that each component of C_1^n in the decomposition of Lemma 13 acts faithfully on the left of T_1^n. Each component acts as if by multiplication in C_1^n, with the exception of the component ${}_{n-1}T_n$. To check this acts faithfully, consider its action on $({}_{n-1}T_n)^*$. Left multiplication by an element $t \in {}_{n-1}T_n$ gives a map

$$\alpha_t : ({}_{n-1}T_n)^* \to A_{n-1}^*,$$

which is the dual of the map

$$\alpha_t^* : A_{n-1} \to {}_{n-1}T_n$$

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given by right multiplication by t. Since α_t^* is non-zero, α_t is non-zero, implying the action of ${}_{n-1}T_n$ is faithful as required. □

DEFINITION 15. Let

$$\mathcal{C}_n \circlearrowright \{\text{algebras with a self-dual faithful bimodule}\}$$

be the map which takes the pair (A, T) to the pair (C_1^n, T_1^n).

In this definition, by a self-dual A-A-bimodule, we mean a bimodule ${}_AT_A$, endowed with an explicit isomorphism $T \cong T^*$. When employing the above definition, we sometimes forget the self-dual bimodules, and simply write $\mathcal{C}_n(A)$ for the algebra C_1^n.

Assume now that A is a quasi-hereditary algebra with poset Λ of weights. Let $\Lambda_B^1 = \amalg_{i\in\mathbb{Z}}\Lambda[i]$, with the same ordering as in Λ within each $\Lambda[i]$, and $\lambda[i] < \mu[j]$ for $i \neq j$ if and only if $i > j \in \mathbb{Z}$. Let $\Lambda_B^2 = \amalg_{i\in\mathbb{Z}}\Lambda[i]$, with the same ordering as in Λ within each $\Lambda[i]$, and $\lambda[i] < \mu[j]$ for $i \neq j$ if and only if $i < j \in \mathbb{Z}$.

The partially ordered sets Λ_B^1, Λ_B^2 index the irreducible B_0-modules. The algebra B_0 is obviously locally finite-dimensional and quasi-hereditary with respect to the posets Λ_B^1 and Λ_B^2.

For the remainder of this chapter, we assume that A is Ringel self-dual, and that T is a tilting bimodule for A, such that $T_A \cong ({}_AT)^*$. Thus, $T_A \in \mathcal{F}(\Delta^r) \cap \mathcal{F}(\nabla^r)$.

THEOREM 16. *B is quasi-hereditary with respect to the poset Λ_B^1, with standard and costandard modules*

$$\Delta_B^1(\lambda[i]) = \Delta_{B_0}(\lambda[i]) \qquad \textit{and} \qquad \nabla_B^1(\lambda[i]) = \mathrm{Hom}_{B_0}(B, \nabla_{B_0}(\lambda[i])).$$

B is quasi-hereditary with respect to Λ_B^2, with standard and costandard modules

$$\Delta_B^2(\lambda[i]) = B \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \qquad \textit{and} \qquad \nabla_B^2(\lambda[i]) = \nabla_{B_0}(\lambda[i]).$$

B is Ringel self-dual and Ringel duality exchanges the two quasi-hereditary structures on B.

Proof. First observe that indeed B is locally finite-dimensional and Λ_B^1 and Λ_B^2 index simple modules since B_1 forms a nilpotent ideal in B.

(1) $\Delta_B^1(\lambda[i])$ has a simple top and the radical consists of composition factors with smaller indices.

Obvious from B_0,

(2) B-proj $\subset \mathcal{F}(\Delta_B^1)$ with order relations as required.

We show that ${}_BBe_i \in \mathcal{F}(\Delta_B^1)$ for all $i \in \mathbb{Z}$. But ${}_BBe_i$ has a filtration with a submodule ${}_{i-1}T_i$ as the first composition factor of the filtration and A_i as the second. As left B-module, ${}_{i-1}T_i$ is filtered by $\Delta_{B_0}(\lambda[i-1])$ and A_i is filtered by $\Delta_{B_0}(\lambda[i])$ with $\lambda \in \Lambda$. Since for A_i the filtration by Δ_{B_0}s is in the right order (on every direct summand) and $\Delta_{B_0}(\lambda[i-1]) > \Delta_{B_0}(\mu[i])$ for all $\lambda, \mu \in \Lambda$, the filtration respects the necessary inequalities on labels.

(3) $\Delta_B^2(\lambda[i])$ has a simple top and the radical consists of composition factors with smaller indices.

$\Delta_B^2(\lambda[i])$ has a submodule isomorphic to $B_1 \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \cong {}_{i-1}T_i \underset{A_i}{\otimes} \Delta_{A_i}(\lambda)$, the quotient by which is isomorphic to $B_0 \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \cong \Delta_{B_0}(\lambda[i])$. The latter has simple head, and all other composition factors have smaller indices by the quasi-hereditary structure

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of B_0. The former has composition factors with labels in $\Lambda[i-1]$, which, since in this ordering $i-1 < i$, are smaller as desired. Furthermore, B_1 is a nilpotent ideal in B, thus the above submodule does not contribute to the head of the module.

(4) B-proj $\subset \mathcal{F}(\Delta^2_B)$ with order relations as required.

$B_{B_0} \cong (B_0)_{B_0} \oplus (B_1)_{B_0}$ and $(B_0)_{B_0}$ is projective, hence flat. We claim that $(B_1)_{B_0} \underset{B_0}{\otimes} -$ is exact on $\mathcal{F}(\Delta_{B_0})$. To prove this, it suffices to check that ${}_{i-1}T_i \underset{A_i}{\otimes} -$ is exact on $\mathcal{F}(\Delta_{A_i})$. So let $M \in \mathcal{F}(\Delta_{A_i})$ and consider ${}_{i-1}T_i \underset{A_i}{\otimes} M$. This being in $\mathcal{F}(\Delta_{A_{i-1}})$, is equivalent to $({}_{i-1}T_i \underset{A_i}{\otimes} M)^*$ being in $\mathcal{F}(\nabla^r_{A_{i-1}})$. Now $({}_{i-1}T_i \underset{A_i}{\otimes} M)^* = \mathrm{Hom}_F({}_{i-1}T_i \underset{A_i}{\otimes} M, F) \cong \mathrm{Hom}_{\text{mod-}A_i}(T_i, M^*)$. But $M \in \mathcal{F}(\Delta_{A_i})$ implies $M^* \in \mathcal{F}(\nabla^r_{A_i})$ and, by the assumption that $T_A \cong ({}_AT)^*$, T_A is also a tilting module for mod-A. Therefore, $\mathrm{Hom}_{\text{mod-}A_i}(T_i, -)$ is exact on $\mathcal{F}(\nabla^r_{A_i})$ by **[3]**, A4 (1), and thus $({}_{i-1}T_i \underset{A_i}{\otimes} M)^* \in \mathcal{F}(\nabla^r_{A_{i-1}})$. So $B \underset{B_0}{\otimes} -$ is exact on $\mathcal{F}(\Delta_{B_0})$, and ${}_BB \in \mathcal{F}(\Delta^2_B)$. The required ordering conditions follow immediately from those for B_0.

This finishes the proof of B having two quasi-hereditary structures.

Similarly, we find that for the right module categories, with respect to Λ^1_B, we have

$$\Delta^{1,r}_B(\lambda[i]) = \Delta^r_{B_0}(\lambda[i]) \underset{B_0}{\otimes} B,$$

and with respect to Λ^2_B,

$$\Delta^{2,r}_B(\lambda[i]) = \Delta^r_{B_0}(\lambda[i]).$$

By duality, we now see that

$$\begin{aligned}\nabla^1_B(\lambda[i]) &= (\Delta^{1,r}(\lambda[i]))^* = \mathrm{Hom}_F(\Delta^r_{B_0}(\lambda[i]) \underset{B_0}{\otimes} B, F)\\ &\cong \mathrm{Hom}_{B_0}(B, (\Delta^r_{B_0}(\lambda[i]))^*) \cong \mathrm{Hom}_{B_0}(B, \nabla_{B_0}(\lambda[i])),\\ \nabla^{1,r}(\lambda[i]) &= (\Delta^1(\lambda[i]))^* = (\Delta_{B_0}(\lambda[i]))^* = \nabla^r_{B_0}(\lambda[i]),\\ \nabla^2_B(\lambda[i]) &= (\Delta^{2,r}(\lambda[i]))^* = (\Delta^r_{B_0}(\lambda[i]))^* = \nabla_{B_0}(\lambda[i]),\end{aligned}$$

and

$$\begin{aligned}\nabla^{2,r}(\lambda[i]) &= (\Delta^2(\lambda[i]))^* = \mathrm{Hom}_F(B \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]), F)\\ &\cong \mathrm{Hom}_{\text{mod-}B_0}(B, (\Delta_{B_0}(\lambda[i]))^*) \cong \mathrm{Hom}_{B_0}(B, \nabla^r_{B_0}(\lambda[i])).\end{aligned}$$

To prove the Ringel self-duality of B, we need the following lemma.

LEMMA 17. $\Delta^2_B(\lambda[i]) \cong \nabla^1_B(\lambda'[i-1])$.

Proof of the lemma. We know that $(\nabla^1_B(\lambda'[i-1]))^* \cong \Delta^{1,r}_B(\lambda'[i-1])$, so it suffices to show that there exists a non-degenerate bilinear form $\langle\ ,\ \rangle : \Delta^{1,r}_B(\lambda'[i-1]) \times \Delta^2_B(\lambda[i]) \longrightarrow F$ with the property that $\langle x, by\rangle = \langle xb, y\rangle$ for

$$x \in \Delta^{1,r}_B(\lambda'[i-1]), y \in \Delta^2_B(\lambda[i]), b \in B.$$

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This is equivalent to having a non-zero linear map

$$\Delta_B^{1,r}(\lambda'[i-1]) \otimes_F \Delta_B^2(\lambda[i]) \longrightarrow F,$$

which factors over

$$\begin{aligned}\Delta_B^{1,r}(\lambda'[i-1])) \otimes_B \Delta_B^2(\lambda[i]) &= \Delta_{B_0}^r(\lambda'[i-1]) \underset{B_0}{\otimes} B \underset{B}{\otimes} B \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \\ &\cong \Delta_{B_0}^r(\lambda'[i-1]) \underset{B_0}{\otimes} B \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \\ &\cong \Delta_{B_0}^r(\lambda'[i-1]) \underset{B_0}{\otimes} B_0 \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \\ &\quad \oplus \Delta_{B_0}^r(\lambda'[i-1]) \underset{B_0}{\otimes} B_1 \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]).\end{aligned}$$

But $\Delta_{B_0}^r(\lambda'[i-1]) \underset{B_0}{\otimes} B_0 \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \cong \Delta_{B_0}^r(\lambda'[i-1])e_{i-1} \underset{B_0}{\otimes} e_i\Delta_{B_0}(\lambda[i]) \cong 0$ and we claim that $\Delta_{B_0}^r(\lambda'[i-1]) \underset{B_0}{\otimes} B_1 \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i])$ is isomorphic to $\nabla_{B_0}^r(\lambda[i]) \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i])$. Indeed,

$$\begin{aligned}\left({}_{i-1}T_i \underset{A_i}{\otimes} \Delta_{A_i}(\lambda)\right)^* &= \operatorname{Hom}_F\left({}_{i-1}T_i \underset{A_i}{\otimes} \Delta_{A_i}(\lambda), F\right) \cong \operatorname{Hom}_{\text{mod-}A_i}({}_{i-1}T_i, (\Delta_{A_i}(\lambda))^*) \\ &\cong \operatorname{Hom}_{\text{mod-}A_i}\left({}_{i-1}T_i, \nabla_{A_i}^r(\lambda)\right) \cong \Delta_{A_{i-1}}^r(\lambda'),\end{aligned}$$

whence ${}_{i-1}T_i \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) = {}_{i-1}T_i \underset{A_i}{\otimes} \Delta_{A_i}(\lambda) \cong \nabla_{A_i}(\lambda') = \nabla_{B_0}(\lambda'[i-1])$. But $\nabla_{B_0}^r(\lambda[i])$ and $\Delta_{B_0}(\lambda[i])$ are dual to one another, which implies that

$$\nabla_{B_0}^r(\lambda[i]) \underset{B_0}{\otimes} \Delta_{B_0}(\lambda[i]) \cong F.$$

Thus

$$\Delta_B^{1,r}(\lambda'[i-1])) \otimes_B \Delta_B^2(\lambda[i]) \cong F,$$

as required. This completes the proof of the lemma. □

Proof of theorem, continued. By the lemma B-proj $\subset \mathcal{F}(\nabla_B^1) = \mathcal{F}(\Delta_B^2)$, but we also have B-proj $\subset \mathcal{F}(\Delta_B^1)$, hence projective modules are tilting modules in the first highest weight structure on B-mod. But clearly

$$\bigoplus_{\lambda[i],\mu[j]\in\Lambda_B} \operatorname{Hom}_B(P(\lambda[i]), P(\mu[j])) \cong B,$$

so B is indeed Ringel self-dual. Denoting the new standard modules by $\tilde{\Delta}_B$, we obtain

$$\tilde{\Delta}_B^1(\lambda[i]) = \operatorname{Hom}_B\left(B, \nabla_B^1(\lambda[i])\right) \cong \nabla_B^1(\lambda[i]) \cong \Delta_B^2\left(\lambda'[i+1]\right).$$

By the right analogue of the lemma we see that (right) projectives are tilting modules for the second highest weight structure on mod-B, and by the same computation as above, we obtain

$$\tilde{\Delta}_B^{2,r}(\lambda[i]) = \operatorname{Hom}_B\left(B, \nabla_B^{2,r}(\lambda[i])\right) \cong \nabla_B^{2,r}(\lambda[i]) \cong \Delta_B^{1,r}\left(\lambda'[i-1]\right).$$

Dualizing we see that

$$\tilde{\nabla}_B^{1,r}(\lambda[i]) \cong \big(\tilde{\Delta}_B^1(\lambda[i])\big)^* \cong \big(\Delta_B^2(\lambda'[i+1])\big)^* \cong \nabla_B^{2,r}\big(\lambda'[i+1]\big)$$

and

$$\tilde{\nabla}_B^2(\lambda[i]) \cong \big(\tilde{\Delta}_B^{2,r}(\lambda[i])\big)^* \cong \big(\Delta_B^{1,r}(\lambda'[i-1])\big)^* \cong \nabla_B^1\big(\lambda'[i-1]\big).$$

Since Δ's and ∇'s determine each other, this completes the proof of the theorem.

Let $\Lambda_C^1 = \Lambda_B^1$, and $\Lambda_C^2 = \Lambda_B^2$.

THEOREM 18. *C is quasi-hereditary with poset Λ_C^1, as well as with poset Λ_C^2. We have*

$$\nabla_C^1(\lambda[i]) = \nabla_B^1(\lambda[i]), \qquad \Delta_C^2(\lambda[i]) = \Delta_B^2(\lambda[i]).$$

Furthermore, C is Ringel self-dual, and Ringel duality exchanges the two highest weight structures on C.

Proof. The equality of the indexing sets for simple modules follows from the nilpotency of B^* in C. Now, Ce_i has a filtration with submodule $B^*e_i \cong (e_iB)^*$ and quotient Be_i. The latter has a filtration by $\Delta_B^2(\lambda[i])$, where $\lambda \in \Lambda$, with the necessary properties of Theorem 16. The former has a filtration by $(\Delta_B^{1,r}(\lambda[i]))^* \cong \nabla_B^1(\lambda[i]) \cong \Delta_B^2(\lambda'[i+1])$. So, as $i+1 > i$ we have a filtration respecting the necessary inequalities on labels.

The fact that C is symmetric follows from the general statement that the trivial extension of an algebra by its dual is symmetric.

Ringel self-duality follows immediately from symmetry, since projectives have a Δ-filtration, but as they are the same as injectives, also a ∇-filtration, thus projectives are tilting modules, implying Ringel self-duality. □

Set $J_n := \bigcup_{j\leq n} \Lambda[j]$ and $I_k := \bigcup_{i\geq k} \Lambda[i]$ and adopt the notational convention $C^n := C^{J_n}$, $C_k := C_{I_k}$, and $C_k^n := C_{I_k}^{J_n}$. These definitions agree with the definition of C_1^n given previously.

Let us now assume that ${}_AT_A^* \cong {}_AT_A$ as a bimodule. Recall that in these circumstances, $D_\infty =< \sigma, \tau >$ acts on C. Note that in the Ringel duality in Theorem 18, $C' = \tau^{-1}(C)$, since the projective $P(\lambda[i])$ has a submodule $\Delta_C(\lambda'[i+1])$, implying $P(\lambda[i]) \cong T_C(\lambda[i+1])$ and $P_{C'}(\lambda[i]) = \mathrm{Hom}_C(\bigoplus_{\substack{j\in\mathbb{Z}\\ \lambda\in\Lambda}} P(\lambda[j]), P(\lambda[i-1]))$.

THEOREM 19. *C_1^n for $n \geq 1$ is Ringel self-dual, and the tilting bimodule $T_{C_1^n}$ is a self-dual bimodule.*

Proof. By Proposition 12, $(C_1^n)' \cong (C')_{J_n'}^{I_1'}$ with the ordering $i > i+1$ on $\mathbb{Z}$. Therefore,

$$(C_1^n)' \cong (\tau^{-1}C)_{J_n'}^{I_1'} \cong C_{J_{n-1}'}^{I_0'} \overset{\sigma}{\cong} C_{-(n-1)}^0 \overset{\tau^n}{\cong} C_1^n.$$

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The tilting module $T_{C_1^n}$ satisfies

$$\begin{aligned} T_{C_1^n} &= \bigoplus_{\substack{j\le n\\ \lambda\in\Lambda}} \mathrm{Hom}_C\left(\bigoplus_{i\ge 1} Ce_i, T_C(\lambda[j])\right)\\ &\cong \bigoplus_{\substack{j\le n\\ \lambda\in\Lambda}} \mathrm{Hom}_C\left(\bigoplus_{i\ge 1} Ce_i, P_C(\lambda[j-1])\right)\\ &\cong \bigoplus_{\substack{j\le n-1\\ \lambda\in\Lambda}} \mathrm{Hom}_C\left(\bigoplus_{i\ge 1} Ce_i, P_C(\lambda[j])\right)\\ &= \bigoplus_{\substack{0\le j\le n-1\\ 1\le i\le n}} \mathrm{Hom}_C(Ce_i, Ce_j)\\ &\cong \bigoplus_{\substack{0\le j\le n-1\\ 1\le i\le n}} e_iCe_j. \end{aligned}$$

The first equality comes from the fact that factoring out a heredity ideal doesn't change the tilting modules for the remaining labels and that the tilting module for a heredity subalgebra is the tilting module multiplied by the idempotent. The fourth equality takes into account that we only have non-zero maps from Ce_i to itself or to $C1_{A_{i\pm 1}}$. Now

$$\left({}_{C_1^n}\Big(\bigoplus_{\substack{0\le j\le n-1\\ 1\le i\le n}} e_iCe_j\Big)_{(C_1^n)'}\right)^* = {}_{(C_1^n)'}\left(\bigoplus_{\substack{0\le j\le n-1\\ 1\le i\le n}} e_jCe_i\right)_{C_1^n}$$

(by self-duality of C), but to view this as a $(C_1^n, (C_1^n)')$-bimodule we have to twist with $\sigma\circ\tau^{-n}$ on the left and its inverse on the right, which yields $\bigoplus_{\substack{0\le j\le n-1\\ 1\le i\le n}} e_iCe_j$ as desired. □

COROLLARY 20. *The map $\mathcal{C}_n$ restricts to a map*

$$\mathcal{C}_n \circlearrowleft \{\textit{quasi-hereditary algebras with a self-dual tilting bimodule}\}.$$

4. Schur algebras. Let M denote the algebra of $n\times n$ matrices over F. Recall that the Schur algebra $\mathcal{S}(n,r)$ is defined to be the subalgebra $(M^{\otimes r})^{\Sigma_r}$ of fixed points under the action of the symmetric group Σ_r on $M^{\otimes r}$. The category of representations of $\mathcal{S}(n,r)$ can be identified with the category of polynomial representations of $GL_n(F)$, of degree r [**5**].

Let $\Lambda^+(n,r)$, the set of partitions of r with n parts or fewer, given the dominance ordering. The algebra $\mathcal{S}(n,r)$ is quasi-hereditary with respect to the poset $\Lambda^+(n,r)$. We write $\xi_\lambda\in\mathcal{S}(n,r)$ for Green's idempotents in $\mathcal{S}(n,r)$, for $\lambda\in\Lambda(n,r)$ (the definition of these objects, consult Green's monograph [**5**]).

In this paper, we are only concerned with $\mathcal{S}(2, r)$, but it will be useful to recall some facts about Ringel duality, which hold for general n.

LEMMA 21. (Donkin [3], Section 4.1). *Let $n \geq r$. Then $\bigwedge^r(M)$ is a tilting $\mathcal{S}(n, r)$-$\mathcal{S}(n, r)$-bimodule.*

When $n \geq r$, let $\mathcal{S}'(n, r) = \mathcal{S}(n, r)$. When $n < r$, let

$$\mathcal{S}'(n, r) \cong \mathcal{S}(r, r) / \sum_{\lambda \notin \Lambda(n,r)} \mathcal{S}(r, r)\xi_\lambda \mathcal{S}(r, r).$$

The algebras $\mathcal{S}(n, r)$, $\mathcal{S}'(n, r)$ are Ringel dual.

The Schur algebra possesses a natural anti-automorphism inherited from the transpose operator on M. We also call this anti-automorphism the transpose operator, and denote by s^T the twist of an element s by the transpose operator. Since $\xi_\lambda^T = \xi_\lambda$ for all λ, the transpose operator descends to an anti-automorphism of $\mathcal{S}'(n, r)$.

If A is an algebra, endowed with an anti-automorphism x, then given any left/right A-module M, we define the right/left A-module M^{op} to be that obtained by twisting the action of A on M by x. If A_1 and A_2 are algebras, endowed with anti-automorphisms x_1 and x_2, then given an A_1-A_2-bimodule M, we define the A_2-A_1-bimodule M^{op} to be that obtained by twisting the actions of A_1 and A_2 on M by x_1 and x_2.

LEMMA 22. *Let ${}_{\mathcal{S}(n,r)}T_{\mathcal{S}'(n,r)}$ be a tilting bimodule. Then $T^{op} \cong T^*$, as $\mathcal{S}'(n, r)$-$\mathcal{S}(n, r)$-bimodules, where T^{op} is obtained after twisting T by the transpose anti-automorphisms of $\mathcal{S}'(n, r)$ and $\mathcal{S}(n, r)$.*

Proof. In case $n \geq r$, we have $T \cong \bigwedge^r(M)$. However, it is well known that $\bigwedge^r(M)$ is self-dual, which is to say $\bigwedge^r(M)^{op} \cong \bigwedge^r(M)^*$.

The case $n < r$ follows by truncation from the case $n = r$. Indeed, in this case, we have $T = (\sum_{\lambda \in \Lambda(n,r)} \xi_\lambda) \bigwedge^r(M)$. Since $\xi_\lambda^T = \xi_\lambda$, this bimodule is also self-dual. □

We now restrict our study to the case $n = 2$. Suppose F is a field of characteristic $p > 0$. Let $\mathcal{S} = \mathcal{S}(2, r)$ be the Schur algebra over F, where $r = ap^k - 2$ or $r = ap^k - 3$ for some $k \geq 1$ and $2 \leq a \leq p$. Along with the cases $r < p^2$ and $r = ap^k - 1$, these are exactly the Schur algebras, which are Ringel self-dual ([4], Theorem 27). Furthermore, $\mathcal{S}(2, ap^k - 1)$ is Morita equivalent to $\mathcal{S}(2, ap^k - 3) \oplus F$ ([4], Corollary 2).

If r is odd, our index set Λ for the quasi-hereditary structure of $\mathcal{S}$ consists of all odd natural numbers up to r; if r is even, it consists of all even natural numbers up to r, including 0. Here, we identify an element j of Λ in this index set with the two-part partition $(r - j, j) \in \Lambda^+(2, r)$.

The following definitions assume p as odd. If r is odd, let $A = \mathcal{S}(2, p^k - 2)$ and if r is even, let $A = \mathcal{S}(2, p^k - 3)$. We define subsets I_j, for $1 \leq j \leq a$, of Λ as follows:

	r odd
j odd	$I_j = \{\lambda \in \Lambda \mid (j-1)p^k + 1 \leq \lambda \leq jp^k - 2\}$
j even	$I_j = \{\lambda \in \Lambda \mid (j-1)p^k \leq \lambda \leq jp^k - 3\}$
	r even
j odd	$I_j = \{\lambda \in \Lambda \mid (j-1)p^k \leq \lambda \leq jp^k - 3\}$
j even	$I_j = \{\lambda \in \Lambda \mid (j-1)p^k + 1 \leq \lambda \leq jp^k - 2\}$

.

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In case $p = 2$ (and thus necessarily $a = 2$), let $A = \mathcal{S}(2, 2^k - 3)$ if r is odd, and $A = \mathcal{S}(2, 2^k - 2)$ if r is even. We define subsets I_j, for $j = 1, 2$, of Λ as follows:

	r odd
$j = 1$	$I_j = \{1, 3, \ldots, 2^{k-1} - 3\}$
$j = 2$	$I_j = \{ 2^{k-1} + 1, 2^{k-1} + 3, \ldots, 2^k - 3\}$
	r even
$j = 1$	$I_j = \{0, 2, \ldots, 2^{k-1} - 2\}$
$j = 2$	$I_j = \{2^{k-1}, 2^{k-1} + 2, \ldots, 2^k - 2\}$

Let us define $I_0 := \Lambda \setminus (\bigcup_{1 \le j \le a} I_j)$.

For $1 \le j \le a$, set $b_j := \min\{I_j\}$, $r_j := \max\{I_j\}$.

We choose orthogonal idempotents $\{e_\lambda\}_{\lambda \in \Lambda}$ in $\mathcal{S}$, such that $\mathcal{S} \cong \bigoplus_{\lambda, \mu \in \Lambda} e_\lambda \mathcal{S} e_\mu$, and $\mathcal{S}/\mathcal{J}(\mathcal{S}) = \bigoplus_{\lambda \in \Lambda} M_\lambda$ is a direct sum of matrix rings M_λ over F, where e_λ is the unit of M_λ.

Let $f_j := \sum_{\lambda \in I_j} e_\lambda$, where $e_\lambda \in \mathcal{S}$ is the primitive idempotent corresponding to the projective $P(\lambda)$. Let $\varepsilon_j = \sum_{i \ge j} f_i$.

By the work of Henke and Koenig, there are idempotents $\eta_j \in \mathcal{S}$ (denoted ξ_l^o in [7]), and explicit isomorphisms $\Phi_j : A \to \eta_j \mathcal{S} \eta_j / \eta_j \mathcal{S} \eta_{j+1} \mathcal{S} \eta_j$, for $1 \le j \le a$ ([7], Theorem 3.3).

We now assume that the idempotents e_λ are chosen in such a way that $e_\lambda \eta_j = \eta_j e_\lambda$, for $\lambda \in \Lambda$, $1 \le j \le a$. It therefore follows that the idempotents η_j commute with f_i, ϵ_i as well, and we have $\varepsilon_k \eta_j = \eta_k$, for $1 \le j \le k \le a$.

Let $\mathcal{R}_j := f_j \mathcal{S} f_j / f_j \mathcal{S} f_{j+1} \mathcal{S} f_j$, for $1 \le j \le a$.

LEMMA 23. *The algebra $\mathcal{R}_j$ is Morita equivalent to A, for $1 \le j \le a$. We have $f_j \mathcal{S} f_i = 0$ unless $j - 1 \le i \le j + 1$, and*

$$\mathcal{S} = \bigoplus_{j=1}^{a} f_j \mathcal{S} f_j \oplus \bigoplus_{j=1}^{a-1} (f_j \mathcal{S} f_{j+1} + f_{j+1} \mathcal{S} f_j) \oplus \bigoplus_{\lambda \in I_0} e_\lambda \mathcal{S} e_\lambda. \tag{1}$$

Proof. From the decomposition matrix of $\mathcal{S}$ [6], we see that $f_j \mathcal{S} f_i = 0$ unless $j - 1 \le i \le j + 1$ and that for $\lambda \in I_0$, $e_\lambda \mathcal{S} e_\mu = e_\mu \mathcal{S} e_\lambda = 0$ unless $\mu = \lambda$, when it is isomorphic to F. Hence,

$$\mathcal{R}_j = f_j \mathcal{S} f_j / f_j \mathcal{S} f_{j+1} \mathcal{S} f_j \cong \varepsilon_j \mathcal{S} \varepsilon_j / \varepsilon_j \mathcal{S} \varepsilon_{j+1} \mathcal{S} \varepsilon_j.$$

This algebra is Morita equivalent to $\eta_j \mathcal{S} \eta_j / \eta_j \mathcal{S} \eta_{j+1} \mathcal{S} \eta_j$, which is isomorphic to A. This completes the proof of the lemma. □

REMARK 24. It will be important to us that the Henke–Koenig isomorphism Φ_j between A and $\eta_j \mathcal{S} \eta_j / \eta_j \mathcal{S} \eta_{j+1} \mathcal{S} \eta_j$ is compatible with the transpose operators on $\mathcal{S}$, A. To be more explicit, $\eta_j^T = \eta_j$, and $\Phi_j(a^T) = \Phi_j(a)^T$, for $a \in A$.

LEMMA 25. *We have $f_a \mathcal{S} f_{a-1} \mathcal{S} f_a = 0$.*

Proof. This is a reformulation of [4], Proposition 25. Indeed, according to this proposition, $\mathcal{S} f_{a-1} \mathcal{S} f_a$ is the submodule of $\mathcal{S} f_a$ consisting of all composition factors of

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the form $L(\lambda)$, $\lambda \in I_{a-1}$, implying

$$f_a\mathcal{S}f_{a-1}\mathcal{S}f_a \cong \mathrm{Hom}_{\mathcal{S}}(\mathcal{S}f_a, \mathcal{S}f_{a-1}\mathcal{S}f_a) = 0. \qquad \square$$

LEMMA 26. *$X_j = f_j\mathcal{S}f_{j+1}$ is an $\mathcal{R}_j$-$\mathcal{R}_{j+1}$-tilting bimodule.*

Proof. By Lemmas 23 and 25, the $f_j\mathcal{S}f_j$-$f_{j+1}\mathcal{S}f_{j+1}$-bimodule X_j is in fact an $\mathcal{R}_j$-$\mathcal{R}_{j+1}$-bimodule. It remains to show that ${}_{\mathcal{R}_j}X_j$ is a full tilting module, and $End_{\mathcal{R}_j}(X_j) = \mathcal{R}_{j+1}$.

By the same argument as in Lemma 23, we can reduce to the case where $a = 2$ by considering all modules for the subalgebra $\varepsilon_j\mathcal{S}\varepsilon_j/\varepsilon_j\mathcal{S}\varepsilon_{j+2}\mathcal{S}\varepsilon_j$. So let $\mathcal{S} = \mathcal{S}(2, r)$, where $r \in \{2p^k - 2, 2p^k - 3\}$ and use the notation from above. We need to show that $f_1\mathcal{S}f_2 \in \mathcal{S}/\mathcal{S}\varepsilon_2\mathcal{S}$-mod is a tilting module. But by [**4**], Proposition 25, $\mathcal{S}f_1\mathcal{S}f_2 \subseteq \mathcal{S}f_2$ is the submodule consisting of all composition factors of the form $L(\lambda)$ for $\lambda \in I_1$ and is isomorphic to the full tilting module for $\mathcal{S}(2, \max\{I_1\})$. But by the first of these facts the action factors over $\mathcal{R}_1 = \mathcal{S}/\mathcal{S}\varepsilon_2\mathcal{S} \cong \mathcal{S}(2, r_1)$, so that it is a full tilting module for this algebra.

Now we have a canonical map from $\mathcal{R}_2 = f_2\mathcal{S}f_2$ to $\mathrm{End}_{\mathcal{R}_1}(f_1\mathcal{S}f_2)$. Given the fact that A is Ringel self-dual, we know that $\mathcal{R}_2$, A and $\mathrm{End}_{\mathcal{R}_1}(f_1\mathcal{S}f_2)$ are Morita equivalent, thus $\mathcal{R}_2$ and $\mathrm{End}_{\mathcal{R}_1}(f_1\mathcal{S}f_2)$ are isomorphic. It therefore suffices to prove injectivity of this map. So, suppose it has a nontrivial kernel. This is equivalent to the existence of an endomorphism ϕ of $\mathcal{S}f_2$, annihilating all composition factors of the form $L(\lambda)$ for $\lambda \in I_1$ (namely $\mathcal{S}f_1\mathcal{S}f_2$). But all composition factors of the socle of $\mathcal{S}f_2$ are of the form $L(\lambda)$ for $\lambda \in I_1$, by [**4**], Lemma 3, and thus $\mathrm{im}\,\phi \cap \mathrm{soc}\,\mathcal{S}f_2 = 0$ forcing ϕ to be zero. $\square$

REMARK 27. Note that it follows from the proof of the lemma that $f_j\mathcal{S}f_{j+1}\mathcal{S}f_j$ is the annihilator of $f_j\mathcal{S}f_{j+1}$ in $f_j\mathcal{S}f_j$. Since by Remark 23 $f_{j-1}\mathcal{S}f_j\mathcal{S}f_{j+1} \subseteq f_{j-1}\mathcal{S}f_{j+1} = 0$, it follows that $f_j\mathcal{S}f_{j-1}\mathcal{S}f_j \subseteq f_j\mathcal{S}f_{j+1}\mathcal{S}f_j$.

Let $\bar{X}_j = f_{j+1}\mathcal{S}f_j$. By Lemmas 23 and 25, $\bar{X}_j$ is an $\mathcal{R}_{j+1}$-$\mathcal{R}_j$-bimodule.

Let X_j^{op} be the $\mathcal{R}_{j+1}$-$\mathcal{R}_j$-bimodule obtained by passing ${}_{\mathcal{R}_j}X_{j\mathcal{R}_{j+1}}$ via the established Morita equivalences to the category of A-A-bimodules, twisting on both sides by the transpose automorphism of A, and then passing via Morita equivalence to the category of $\mathcal{R}_{j+1}$-$\mathcal{R}_j$-bimodules.

LEMMA 28. *There is an isomorphism of $\mathcal{R}_{j+1}$-$\mathcal{R}_j$-bimodules, $\bar{X}_j \cong X_j^{op}$.*

Proof. We have

$$X_j = f_j\mathcal{S}f_{j+1} \cong \varepsilon_j\varepsilon_{j+1}/\varepsilon_{j+1}\mathcal{S}\varepsilon_{j+1}.$$

This passes, via Morita equivalence, to the A-A-bimodule

$$\begin{aligned}\eta_j\mathcal{S}f_j \underset{f_j\mathcal{S}f_j}{\otimes} f_j\mathcal{S}f_{j+1} \underset{f_{j+1}\mathcal{S}f_{j+1}}{\otimes} f_{j+1}\mathcal{S}\eta_{j+1} \\ \cong \eta_jf_j\mathcal{S}f_{j+1}\eta_{j+1} \cong \eta_j\mathcal{S}\eta_{j+1}/\eta_j\varepsilon_{j+1}\mathcal{S}\eta_{j+1}.\end{aligned}$$

Since twisting by the transpose operator exchanges the irreducible modules $L(\lambda)$, $L^r(\lambda)$, the projective $\mathcal{S}$-modules $\mathcal{S}f_j$ and $\mathcal{S}f_j^T$ are isomorphic. We therefore have

$$\begin{aligned}\bar{X}_j &= f_{j+1}\mathcal{S}f_j = f_{j+1}\mathcal{S}f_{j+1}^T\mathcal{S}f_j^T\mathcal{S}f_j \\ &\cong f_{j+1}\mathcal{S}f_{j+1}^T \underset{\varepsilon_{j+1}^T\mathcal{S}\varepsilon_{j+1}^T}{\otimes} (\varepsilon_{j+1}^T\mathcal{S}\varepsilon_j^T/\varepsilon_{j+1}^T\mathcal{S}\varepsilon_{j+1}^T) \underset{\varepsilon_j^T\mathcal{S}\varepsilon_j^T}{\otimes} f_j^T\mathcal{S}f_j.\end{aligned}$$

This passes, via Morita equivalence, to the A-A-bimodule

$$\eta_{j+1}\mathcal{S}f_{j+1}^T \underset{f_{j+1}^T\mathcal{S}f_{j+1}^T}{\otimes} f_{j+1}^T\mathcal{S}f_j^T \underset{f_j^T\mathcal{S}f_j^T}{\otimes} f_j^T\mathcal{S}\eta_{j+1}$$
$$\cong \eta_{j+1}f_{j+1}^T\mathcal{S}f_j^T\eta_j \cong \eta_{j+1}\mathcal{S}\eta_j/\eta_{j+1}\mathcal{S}\varepsilon_{j+1}^T\eta_j.$$

Applying the transpose anti-automorphism to $\mathcal{S}$, we exchange the bimodules $\eta_j\mathcal{S}\eta_{j+1}/\eta_j\varepsilon_{j+1}\mathcal{S}\eta_{j+1}$ and $\eta_{j+1}\mathcal{S}\eta_j/\eta_{j+1}\mathcal{S}\varepsilon_{j+1}^T\eta_j$, the left and right actions being twisted by the transpose operator. However, the transpose operator is compatible with the Henke–Koenig isomorphisms, and therefore an equivalent statement is that passing to the opposite exchanges X_j and $\bar{X}_j$. We therefore have $\bar{X}_j \cong X_j^{op}$, as required. □

Let us define

$$\mathcal{N} := \sum_{j=1}^{a-1}(f_j\mathcal{S}f_{j+1} + f_{j+1}\mathcal{S}f_j + f_j\mathcal{S}f_{j+1}\mathcal{S}f_j),$$
$$\mathcal{N}_2 := \sum_{j=1}^{a-1} f_j\mathcal{S}f_{j+1}\mathcal{S}f_j.$$

PROPOSITION 29. *We have a filtration of $\mathcal{S}$ by ideals,*

$$\mathcal{S} \supset \mathcal{N} \supset \mathcal{N}^2 \supset 0. \tag{2}$$

Furthermore, $\mathcal{N}^2 = \mathcal{N}_2$, and $\mathcal{N}^3 = 0$. We have isomorphisms of $\mathcal{S}$-$\mathcal{S}$-bimodules,

$$\mathcal{S}/\mathcal{N} \cong \bigoplus_{1\le j\le a}\mathcal{R}_j \oplus \bigoplus_{\lambda\in I_0} e_\lambda\mathcal{S}e_\lambda,$$
$$\mathcal{N}/\mathcal{N}^2 \cong \bigoplus_{1\le j\le a-1}(X_j \oplus X_j^*),$$
$$\mathcal{N}^2 \cong \bigoplus_{1\le j\le a-1}\mathcal{R}_j^*.$$

Proof. The first statement as well as $\mathcal{N}^2 = \mathcal{N}_2$ and $\mathcal{N}^3 = 0$, are easily verified using Lemmas 23 and 25 and Remark 27. From (1) we see that

$$\mathcal{S}/\mathcal{N} \cong \bigoplus_{1\le j\le a} f_j\mathcal{S}f_j/(f_j\mathcal{S}f_{j+1}\mathcal{S}f_j)$$
$$\cong \bigoplus_{1\le j\le a}\mathcal{R}_j \oplus \bigoplus_{\lambda\in I_0} e_\lambda\mathcal{S}e_\lambda,$$

and by Lemmas 22, 26 and 28,

$$\mathcal{N}/\mathcal{N}^2 \cong \bigoplus_{1\le j\le a-1}(f_j\mathcal{S}f_{j+1} + f_{j+1}\mathcal{S}f_j)$$
$$\cong \bigoplus_{1\le j\le a-1}(X_j + X_j^{op}) \cong \bigoplus_{1\le j\le a-1}(X_j + X_j^*).$$

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Now all that is left to show is that $f_j\mathcal{S}f_{j+1}\mathcal{S}f_j \cong \mathcal{R}_j^*$. To see this, note that by repeatedly applying Remark 23

$$f_j\mathcal{S}f_{j+1}\mathcal{S}f_j = f_j\mathcal{S}\varepsilon_{r_{j+1}}\mathcal{S}f_j \cong f_j\mathcal{S}\varepsilon_{r_{j+1}} \underset{\varepsilon_{r_{j+1}}\mathcal{S}\varepsilon_{r_{j+1}}}{\otimes} \varepsilon_{r_{j+1}}\mathcal{S}f_j$$
$$\cong f_j\mathcal{S}f_{j+1} \underset{\mathcal{R}_{j+1}}{\otimes} f_{j+1}\mathcal{S}f_j.$$

But

$$\mathrm{Hom}_F\left(f_j\mathcal{S}f_{j+1} \underset{\mathcal{R}_{j+1}}{\otimes} f_{j+1}\mathcal{S}f_j, F\right) \cong \mathrm{Hom}_{\text{mod-}\mathcal{R}_{j+1}}\left(X_j, X_j^{op*}\right)$$
$$\cong \mathrm{End}_{\text{mod-}\mathcal{R}_{j+1}}(X_j) \cong \mathcal{R}_j,$$

thus $f_j\mathcal{S}f_{j+1}\mathcal{S}f_j \cong \mathcal{R}_j^*$, as claimed. □

Let $C_1^a = \mathcal{C}_1^a(A)$ be the algebra obtained by applying the construction $\mathcal{C}_1^a$ of the previous chapter to the algebra A, and its self-dual bimodule T.

THEOREM 30. *The graded algebra $\mathcal{S}_{gr}$ associated to the filtration $\mathcal{S} \supset \mathcal{N} \supset \mathcal{N}^2 \supset 0$ is Morita equivalent to $C_1^a \oplus F^{\oplus I_0}$.*

Proof. By Proposition 29, we know that $\mathcal{S}_{gr}$ is Morita equivalent to $\tilde{C}_1^a \oplus F^{\oplus I_0}$, where $\tilde{C}_1^a$ is $\mathbb{Z}$-graded, concentrated in degrees 0, 1 and 2. In descending vertical order, the components of $\tilde{C}_1^a$ in degrees 0, 1 and 2 are,

$$\bigoplus_{1\le i\le a} \tilde{A}_i,$$
$$\bigoplus_{1\le i\le a-1} ({}_i\tilde{T}_{i+1} \oplus {}_i\tilde{T}_{i+1}^*),$$
$$\bigoplus_{1\le i\le a-1} \tilde{A}_i^*,$$

where $\tilde{A}_i$ is isomorphic to A, and ${}_i\tilde{T}_{i+1}$ is a tilting $\tilde{A}_i$-$\tilde{A}_{i+1}$-bimodule. Twisting the isomorphisms $\tilde{A}_i \cong A$ by automorphisms of A if necessary, we may assume that ${}_i\tilde{T}_{i+1} \cong {}_AT_A$. We proceed to piece together an algebra isomorphism between $\tilde{C}_1^a$ and C_1^a itself.

We know from the proof of the previous proposition that multiplication $f_j\mathcal{S}f_{j+1} \underset{F}{\otimes} f_{j+1}\mathcal{S}f_j \twoheadrightarrow \mathcal{R}_j^*$ is surjective, for $1 \le j \le a-1$. Therefore, multiplication ${}_j\tilde{T}_{j+1} \underset{F}{\otimes} {}_j\tilde{T}_{j+1}^* \twoheadrightarrow \tilde{A}_j^*$ is also surjective.

Since we have a canonical isomorphism ${}_jT_{j+1} \underset{A_{j+1}}{\otimes} {}_jT_{j+1}^* \cong A_j^*$, we consequently obtain an isomorphism $\tilde{A}_j^* \cong A^*$ of A-A-bimodules.

We now claim that multiplication ${}_j\tilde{T}_{j+1}^* \underset{F}{\otimes} {}_j\tilde{T}_{j+1} \twoheadrightarrow \tilde{A}_{j+1}^*$ is also surjective, for $1 \le j \le a-2$. Equivalently, we claim that multiplication $f_{j+1}\mathcal{S}f_j \underset{F}{\otimes} f_j\mathcal{S}f_{j+1} \to \mathcal{R}_{j+1}^*$ is surjective. Indeed, this multiplication is inherited from the left module structure on the maximal submodule M of $\mathcal{S}f_{j+1}$ whose composition factors $L(\lambda)$ respect $\lambda \in I_{j+1}$. The submodule M has a filtration with submodule $\mathcal{R}_{j+1}^*$ and quotient $f_j\mathcal{S}f_{j+1}$. Note that $\mathcal{S}f_j$ is a tilting module ([**4**], Corollary 21, Lemma 24) and therefore self-dual. Therefore M^{op*} is the maximal quotient of $\mathcal{S}f_{j+1}$, all of whose composition factors $L(\lambda)$ respect

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$\lambda \in I_{j+1}$. M^{op*} has a filtration with submodule $f_j\mathcal{S}f_{j+1}$ and quotient isomorphic to $\mathcal{R}_{j+1}$. However, we precisely know the structure of this module. For instance, the product $f_j\mathcal{S}f_{j+1} \otimes_F \mathcal{R}_{j+1} \to f_j\mathcal{S}f_{j+1}$ corresponds to the right action of $T \otimes A \to T$. Since the product on M is dual to that on M, the map $f_{j+1}\mathcal{S}f_j \otimes_F f_j\mathcal{S}f_{j+1} \twoheadrightarrow \tilde{A}^*_{j+1}$ is dual to the map $A \hookrightarrow T \otimes T^*$, and is therefore surjective, as required.

We have now proven that $f_i\mathcal{S}f_{i-1}\mathcal{S}f_i = f_i\mathcal{S}f_{i+1}\mathcal{S}f_i$, for $2 \le i \le a-1$. We therefore have isomorphisms

$$\tilde{A}^*_i \cong {}_i\tilde{T}_{i-1} \underset{\tilde{A}_{i-1}}{\otimes} {}_{i-1}\tilde{T}_i \cong {}_i\tilde{T}_{i+1} \underset{\tilde{A}_{i+1}}{\otimes} {}_{i+1}\tilde{T}_i \cong \tilde{A}^*_i,$$

of $\tilde{A}_i$-$\tilde{A}_i$-bimodules. Let us denote this chain of isomorphisms ϕ_i. We have

$$\mathrm{Hom}_{A\otimes A^{op}}(A^*, A^*) \cong \mathrm{Hom}_{A\otimes A^{op}}(A, A) \cong Z(A),$$

and thus ϕ_i is multiplication by a central element in $\tilde{A}_i$. Multiplying the bimodules ${}_i\tilde{T}_{i+1}$ by these central elements, if necessary, we can assume that in fact $\phi_i = 1$, for $1 \le i \le a-1$.

It is now clear that the sum of our bimodule isomorphisms

$$\tilde{A}_i \cong A_i, \quad {}_i\tilde{T}_{i+1} \cong {}_iT_{i+1}, \quad {}_i\tilde{T}^*_{i+1} \cong {}_iT^*_{i+1}, \quad \tilde{A}^*_i \cong A^*_i$$

defines an algebra isomorphism from $\tilde{C}^a_1$ to C^a_1, as required. □

5. GL_2**.** In this chapter, we give precise statements for Theorems 1 and 2, together with a justification of Theorem 1.

The determinant representation of $GL_n(F)$ is a polynomial representation of degree n. Therefore, tensoring with the determinant representation defines an exact functor from the category of polynomial $GL_n(F)$ representations of degree r to the category of polynomial GL_n representations of degree $r+n$, carrying simple modules to simple modules. Correspondingly, the Schur algebra $\mathcal{S}(n, r)$ can be realised as a quotient of $\mathcal{S}(n, r+n)$ by an idempotent ideal $\mathcal{S}(n, r+n)i\mathcal{S}(n, r+n)$. We denote by $\mathcal{S}(n, \underline{r})$ the inverse limit of the sequence of algebra epimorphisms

$$\mathcal{S}(n, r) \twoheadleftarrow \mathcal{S}(n, r+n) \twoheadleftarrow \mathcal{S}(n, r+2n) \twoheadleftarrow \dots$$

The centre Z of $GL_n(F)$ is isomorphic to $F^\times$, and its group of rational characters is therefore isomorphic to $\mathbb{Z}$. The category of rational representations of $GL_n(F)$ on which Z acts by the character $r \in \mathbb{Z}$ is naturally equivalent to $\mathcal{S}(n, \underline{r})$-mod. The category of rational representations of $GL_n(F)$ is therefore isomorphic to the module category of $\bigoplus_{r\in\mathbb{Z}} \mathcal{S}(n, \underline{r})$.

For any finite dimensional algebra A, the algebra $\mathcal{C}_n(A)$ has an ideal

$$\bigoplus_{1\le i\le n-1} (A_{i+1} \oplus {}_iT_{i+1} \oplus {}_iT^*_{i+1} \oplus A^*_i),$$

the quotient by which is $A_1 \cong A$. In this way, we obtain a sequence of algebra epimorphisms,

$$A \twoheadleftarrow \mathcal{C}_n(A) \twoheadleftarrow \mathcal{C}_n(\mathcal{C}_n(A)) \twoheadleftarrow \dots$$

We denote by $\underleftarrow{\mathcal{C}}_n(A)$ the inverse limit of this sequence of maps. The statement of Theorem 2 is now completely precise.

THEOREM 2. Every block of rational representations of $GL_2(F)$ is equivalent to $\underleftarrow{\mathcal{C}}_p(F)$-mod.

An equivalent statement is that every block of $\mathcal{S}(2, \underline{r})$ is Morita equivalent to $\underleftarrow{\mathcal{C}}_p(F)$. Another is that $\mathcal{S} \cong \mathcal{S}_{gr}$, in the notation of the last chapter. The proof of these equivalent statements is to be found in the sequel to this paper [**9**].

We now give some corollaries of our work in Chapter 4. Let $\mathcal{S}, \mathcal{N}, A, T$ be as defined there, and let U be an $\mathcal{S}$-$\mathcal{S}$-tilting bimodule.

LEMMA 31. *We have* $\mathcal{N}U = U\mathcal{N}$*, and* $\mathcal{N}\mathcal{S}^* = \mathcal{S}^*\mathcal{N}$.

Proof. A tilting bimodule for $\mathcal{S}$ is given by $U = (\bigoplus_{i=1}^{p-1} \mathcal{S}f_i) \oplus T$. Thus,

$$\mathcal{N}U = \bigoplus_{1 \leq i,j \leq p-1} f_j\mathcal{N}f_i = U\mathcal{N}.$$

We have $\mathcal{S}^* \cong (\bigoplus_{i=1}^{p-1} \mathcal{S}f_i) \oplus \mathcal{S}f_p^*$. Making this identification, we find

$$\mathcal{N}\mathcal{S}^* \cong \left(\bigoplus_{1 \leq i,j \leq p-1} f_j\mathcal{N}f_i \right) \oplus A_p^* \cong \mathcal{S}^*\mathcal{N}. \qquad \square$$

COROLLARY 32. *The space*

$$U_{gr} = \bigoplus_{i=0,1,2} \mathcal{N}^i U/\mathcal{N}^{i+1} U$$

is a $\mathcal{S}_{gr}$-$\mathcal{S}_{gr}$*-tilting bimodule. The space*

$$(\mathcal{S}^*)_{gr} = \bigoplus_{i=0,1,2} \mathcal{N}^i \mathcal{S}^*/\mathcal{N}^{i+1} \mathcal{S}^*$$

is a $\mathcal{S}_{gr}$-$\mathcal{S}_{gr}$*-bimodule, isomorphic to* $(\mathcal{S}_{gr})^*$.

By Theorem 30, $\mathcal{S}_{gr}$ is Morita equivalent to $\mathcal{C}_p(A) \oplus F^{\oplus I_0}$, where A is another Ringel self-dual Schur algebra. By induction we obtain the following:

COROLLARY 33. *There is a filtration of* $\mathcal{S}$ *by ideals, refining the radical filtration, whose associated graded ring* $\mathcal{G}$ *is Morita equivalent to a direct sum of algebras of the form* $\underleftarrow{\mathcal{C}}_p^d(F)$*, for* $d \in \mathbb{Z}_+$.

Given $r \in \mathbb{Z}_+$, we choose $d \geq r$, such that $\mathcal{S} = \mathcal{S}(2, d)$ is Ringel self-dual, and $d = r \pmod 2$. We have $\mathcal{S}(2, r) \cong \mathcal{S}/\mathcal{S}j\mathcal{S}$, for some idempotent j. We define $\mathcal{G}(2, r)$ to be $\mathcal{G}/\mathcal{G}j\mathcal{G}$, where $\mathcal{G}$ is the graded ring associated to $\mathcal{S}$ by Corollary 33. The algebra $\mathcal{G}(2, r)$ is independent of choice of d, and we have algebra epimorphisms

$$\mathcal{G}(2, r) \twoheadleftarrow \mathcal{G}(2, r+2) \twoheadleftarrow \mathcal{G}(2, r+4) \twoheadleftarrow \dots$$

between graded rings $\mathcal{G}(2, r) = \mathcal{G}(\mathcal{S}(2, r))$ of Schur algebras.

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The statement of Theorem 1 is now completely precise. Its truth is clear from Corollary 33.

THEOREM 1. Every block of $\mathcal{G}(2, \underline{r})$-mod is Morita equivalent to $\underleftarrow{\mathcal{C}}_p(F)$-mod.

6. Epilogue. We end with some remarks.

The problem of finding gradings on modular representation categories is rather a general one, related to the celebrated conjecture of G. Lusztig concerning irreducible characters of algebraic groups (see [**8**]). For example, one expects blocks of Schur algebras $\mathcal{S}(n, n)$ to have a grading refining the radical filtration, at least when the weight of the block is less than p (for the definition of a weight, and more context for this conjecture, see [**12**]).

We have conjectured that blocks of Schur algebras $\mathcal{S}(n, n)$ are all derived equivalent to certain subquotients of a symmetric quasi-hereditary algebra, the Schiver double $\mathcal{D}_{A_\infty}$ (see [**12**, **13**]). The most obvious barrier to a proof of this is the difficulty of finding a suitable grading on the Rock blocks. Theorem 2 can be thought of as a simple analogue of the Schiver double conjecture, the algebra $\mathcal{C}_p(A)$ playing a similar role in this paper, to that played by the algebra $\mathcal{D}_{A_\infty}$ in the theory of Rock blocks. Indeed, the development of Theorem 2 was made with a view towards understanding better the Schiver double conjecture.

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