THE CENTERS OF SEMI-SIMPLE ALGEBRAS OVER A COMMUTATIVE RING

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Recently A. Hattori introduced in [7], [8], [9] the notion of simple algebras over a commutative ring. Especially, in [9], he examined, as a fundamental problem on simple algebras, whether a directly indecomposable semi-simple algebra is simple or not and gave affirmative answers to this in some particular cases. In this note we shall first prove, as a complete answer to this, that any directly indecomposable semi-simple algebra over a Noetherian ring is simple.

Hattori proposed in [8] several problems on semi-simple algebras. Secondly we shall give some informations on the problem ([8], Problem 4) whether a central semi-simple algebra is separable or not. Furthermore we shall show a class of p-trivial simple algebras which is different from that in [11] ([8], Problem 11), and finally we shall give the commutor theory of simple subalgebras in a central separable algebra in the complete form.

Throughout this note we denote by R a commutative ring and by Λ a not always commutative ring.

A semi-simple R-algebra Λ is said to be a *simple R-algebra* (cf. [9], [11]), if there exists a left Λ-module E satisfying the following conditions:

i) E is a finitely generated projective Λ-module.

ii) E is Λ-indecomposable.

iii) E is Λ-completely faithful.

By [5], (6. 1), if Λ is finitely generated over its center, we can replace iii) by the following

iii′) E is Λ-faithful.

1. Decomposability of a semi-simple algebra to simple algebras.

We begin with

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LEMMA 1.1. *Let Λ be an R-algebra which is a finitely generated R-module and P a finitely generated projective left (right) Λ-module, Then*

(1) *$\mathfrak{T}_\Lambda(P)$ is a finitely generated R-module.*(*)

(2) *For any multiplicative system S ($\not\ni 0$) of R, we have*

$$\mathfrak{T}_{\Lambda_s}(P_s)=\{\mathfrak{T}_\Lambda(P)\}_s .$$

Proof. As it is easy, we omit it.

PROPOSITION 1.2. *Let Λ be a ring with center C. Suppose that Λ is a finitely generated C-module and that, for any maximal ideal $\mathfrak{m}$ of C, $\Lambda/\mathfrak{m}\Lambda$ is a primary ring. Then the following statements are equivalent:*

(1) *C is directly indecomposable.*

(2) *Any finitely generated projective non-zero left (right) Λ-module is Λ-completely faithful.*

(3) *There exists a finitely generated, projective, completely faithful, indecomposable left (right) Λ-module.*

Proof. The implication (3) $\Longrightarrow$ (1) is obvious. Suppose that C is directly indecomposable. Let P be a finitely generated projective non-zero left (right) Λ-module. By (1.1) and [5], (1.2), $\mathfrak{T}_\Lambda(P)$ is an idempotent two-sided ideal of Λ which is a finitely generated C-module. Let $\mathfrak{M}$ be a maximal two-sided ideal of Λ, and put $\mathfrak{m}=\mathfrak{M}\cap C$. Then $\mathfrak{m}$ is a maximal ideal of C and we have $\mathfrak{M}^l\subseteq\mathfrak{m}\Lambda$ for some positive integer l. If $\mathfrak{T}_\Lambda(P)\subseteq\mathfrak{M}$, then we have $\mathfrak{T}_\Lambda(P)\subseteq\mathfrak{m}\Lambda$ (since $\mathfrak{T}_\Lambda(P)$ is an idempotent ideal of Λ). According to (1.1), we have $\mathfrak{T}_{\Lambda_\mathfrak{m}}(P_\mathfrak{m})=\{\mathfrak{T}_\Lambda(P)\}_\mathfrak{m}\subseteq\mathfrak{m}\Lambda_\mathfrak{m}$. Hence we have $\mathfrak{m}P_\mathfrak{m}=P_\mathfrak{m}$ and so we obtain $P_\mathfrak{m}=0$. As $\mathfrak{T}_\Lambda(P)$ is C-finitely generated, we have $s\mathfrak{T}_\Lambda(P)=0$ for some s in $C-\mathfrak{m}$. Now we put $\mathfrak{c}=\mathrm{Ann}_C\,\mathfrak{T}_\Lambda(P)$. Then, if $\mathfrak{T}_\Lambda(P)\subseteq\mathfrak{m}\Lambda$, we have $\mathfrak{c}\not\subseteq\mathfrak{m}\Lambda$. Hence a two-sided ideal $\mathfrak{c}\Lambda+\mathfrak{T}_\Lambda(P)$ is not contained in any maximal two-sided ideal of Λ. Therefore we have $\Lambda=\mathfrak{c}\Lambda+\mathfrak{T}_\Lambda(P)$. As $\mathfrak{c}\mathfrak{T}_\Lambda(P)=0$, Λ is the direct sum of two-sided ideals $\mathfrak{c}\Lambda$ and $\mathfrak{T}_\Lambda(P)$. Since C is directly indecomposable and $\mathfrak{T}_\Lambda(P)\neq 0$, we have $\mathfrak{T}_\Lambda(P)=\Lambda$. This proves (1) $\Longrightarrow$ (2).

Suppose (2), and let e be a non-trivial idempotent of Λ. Then Λe is Λ-completely faithful and therefore we have $\Lambda e\Lambda=\Lambda$. Hence e is not contained in any $\mathfrak{m}\Lambda$. Therefore the image of e in $\Lambda/\mathfrak{m}\Lambda$ is also a non-trivial idempotent of $\Lambda/\mathfrak{m}\Lambda$. Since $\Lambda/\mathfrak{m}\Lambda$ is an Artinian primary ring, there are a finite number of

(*) $\mathfrak{T}_\Lambda(P)$ denotes the trace ideal of P (cf. [1]).

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orthogonal primitive idempotents in $\Lambda/\mathfrak{m}\Lambda$. Then there are also only a finite number of orthogonal primitive idempotents in Λ. This implies (2) $\Longrightarrow$ (3).

The implications (1) $\Longleftrightarrow$ (3) in the following theorem were proved in [9], Theorem 4 in case C is Noetherian.

THEOREM 1. 3. *Let Λ be a separable R-algebra and C the center of Λ. Then the following conditions are equivalent:*

(1) *C is directly indecomposable*

(2) *Any finitely generated projective non-zero left (right) Λ-module is Λ-completely faithful.*

(3) *Λ is a simple algebra.*

Proof. It suffices to show that Λ satisfies the assumptions in (1. 2). However, as Λ is a separable R-algebra, Λ is a finitely generated C-module by [2], (1. 2) and $\Lambda/\mathfrak{m}\Lambda$ is a central separable $C/\mathfrak{m}C$-algebra for any maximal ideal $\mathfrak{m}$ of C by [2], (1. 4). This completes our proof.

LEMMA 1. 4. *Let R be a complete local ring with a maximal ideal $\mathfrak{m}$ and Λ be an R-algebra which is a finitely generated R-module. Then Λ is directly indecomposable if and only if $\Lambda/\mathfrak{m}\Lambda$ is so.*

Proof. We have only to prove the only if part. Let $\bar{e}$ be a central idempotent of $\Lambda/\mathfrak{m}\Lambda$. As R is complete, there exists an idempotent e of Λ whose image in $\Lambda/\mathfrak{m}\Lambda$ coincides with $\bar{e}$. Now it suffices to show that e is central. Since e is central in $\Lambda/\mathfrak{m}\Lambda$, we have $\lambda e-e\lambda \in \mathfrak{m}\Lambda$ for any λ of Λ. Then we have $e\lambda e-e\lambda, \lambda e-e\lambda e \in \mathfrak{m}\Lambda$. If we put $e\lambda e-e\lambda=\sum_{i=1}^{t} m_i\lambda_i$, $m_i \in \mathfrak{m}$, $\lambda_i \in \Lambda$, then we have $e\lambda e-e\lambda=e(e\lambda e-e\lambda)-(e\lambda e-e\lambda)e=\sum_{i=1}^{t} m_i(e\lambda_i-\lambda_i e) \in \mathfrak{m}^2\Lambda$. By repeating the same procedure, we see $e\lambda e-e\lambda \in \mathfrak{m}^l\Lambda$, for any $l>0$. As $\bigcap_{l=1}^{\infty} \mathfrak{m}^l\Lambda=0$, we obtain $e\lambda e=e\lambda$. Similarly we can show $e\lambda e=\lambda e$. So $\lambda e=e\lambda$ for any λ of Λ, which completes our proof.

LEMMA 1. 5. *Let Λ be a central R-algebra which is a finitely generated R-module, and S be a commutative R-algebra which is a flat R-module. Then $S\otimes_R\Lambda$ is a central S-algebra.*

Proof. See [6], Chap. V, 6, Lemma 3.

Now we give, as our main result, the following

THEOREM 1. 6. *Let R be a Noetherian ring and Λ a semi-simple R-algebra which is a finitely generated R-module. Let C be the center of Λ.*
Then the following conditions are equivalent:

(1) *C is directly indecomposable.*

(2) *Any finitely generated projective non-zero left (right) Λ-module is Λ-completely faithful.*

(3) *Λ is a simple algebra.*

So, a semi-simple algebra over a Noetherian ring R, which is a finitely generated R-module, is expressible as the direct sum of a finite number of simple R-algebras.

Proof. It suffices to show that Λ satisfies the assumptions in (1. 2). As Λ is a central semi-simple C-algebra, $\Lambda_{\mathfrak{m}}$ is also a central semi-simple $C_{\mathfrak{m}}$-algebra for any maximal ideal $\mathfrak{m}$ of C by (1. 5). Let $\hat{C}_{\mathfrak{m}}$ be the completion of $C_{\mathfrak{m}}$. and put $\hat{\Lambda}_{\mathfrak{m}}=\hat{C}_{\mathfrak{m}}\underset{C_{\mathfrak{m}}}{\otimes}\Lambda$. Then, again by (1.5), $\hat{\Lambda}_{\mathfrak{m}}$ has no non-trivial central idempotent, and then, by (1. 4), $\hat{\Lambda}_{\mathfrak{m}}/\mathfrak{m}\hat{\Lambda}_{\mathfrak{m}}$ also has no non-trivial central idempotent.

Since $\Lambda/\mathfrak{m}\Lambda$ is semi-simple and we have $\Lambda/\mathfrak{m}\Lambda \cong \hat{\Lambda}_{\mathfrak{m}}/\hat{\Lambda}_{\mathfrak{m}} \cong \Lambda_{\mathfrak{m}}/\mathfrak{m}\Lambda_{\mathfrak{m}}$, $\Lambda/\mathfrak{m}\Lambda$ must be simple. This completes our proof.

2. Separability of a central semi-simple algebra.

We first give

PROPOSITION 2. 1. *Let R be an Artinian ring and Λ be a central semi-simple algebra over R which is a finitely generated R-module. Then the following conditions are equivalent:*

(1) *Λ is a separable algebra.*

(2) *Λ is a projective R-module.*

(3) *Λ is a completely faithful R-module.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) of our theorem follow from [2], (1. 2) and [1], (A. 3). Hence we have only to prove (3) $\Longrightarrow$ (1). Without loss of generality we may assume, according to (1. 6), that R is a local ring with a maximal ideal $\mathfrak{m}$ and that Λ is a central simple R-algebra with a unique maximal two-sided ideal $\mathfrak{m}\Lambda$. Suppose that Λ is a completely faithful R-module. Then we have $\Lambda=R\oplus M$ for an R-module M. Let $\bar{\alpha}$ be an element of the center of $\Lambda/\mathfrak{m}\Lambda$ and α a representative of $\bar{\alpha}$ in Λ. Then we have $\alpha\lambda-\lambda\alpha \in \mathfrak{m}\Lambda$ for any λ of Λ. Let l be a non-negative integer such that $\mathfrak{m}^l \neq 0$

but $\mathfrak{m}^{l+1}=0$.

Then we have $\mathfrak{m}^l(\alpha\lambda-\lambda\alpha)=0$ for any λ of Λ. As R is the center of Λ, we have $\mathfrak{m}^l\alpha\subseteq R$. Now put $\alpha=r+u$, $r\in R$, $u\in M$. Then, as $\Lambda=R\oplus M$ as R-modules, we have $\mathfrak{m}^l u=0$. Since $\mathfrak{m}\Lambda$ is a unique maximal two-sided ideal of Λ, we have $u=\alpha-r\in\mathfrak{m}\Lambda$, and so we have $\bar{\alpha}=\bar{r}\in R/\mathfrak{m}\subseteq\Lambda/\mathfrak{m}\Lambda$. This proves that $\Lambda/\mathfrak{m}\Lambda$ is a central simple $R/\mathfrak{m}$-algebra. By virtue of the classical result, $\Lambda/\mathfrak{m}\Lambda$ is a separable $R/\mathfrak{m}$-algebra. According to [2], (4. 7), Λ is also a separable R-algebra. This proves the implication (3) $\Longrightarrow$ (1).

COROLLARY 2. 2. *Let R be a (quasi-) Frobenius ring and Λ a central semi-simple R-algebra which is a finitely generated R-module. Then Λ is a separable R-algebra and Λ is also a Frobenius ring.*

Proof. As R is Frobenius, Λ is R-completely faithful, as is well known. By (2. 1) Λ is a separable R-algebra, and is a projective R-module. Then, according to [5], (3. 6), Λ is a Frobenius R-algebra, and so Λ is a Frobenius ring.

This corollary is an affirmative answer, in case R is Frobenius, to the following

PROBREM H. *Is any central semi-simple R-algebra (which is a finitely generated R-module) a separable R-algebra?*

THEOREM 2. 3. *If the answer to Problem H is affirmative for any Artinian ring R, then it is also affirmative for any Noetherian ring R.*

Proof. Let R be a Noetherian ring and Λ a central semi-simple R-algebra. Now it suffices, by [2], (4.7), to show that, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda/\mathfrak{m}\Lambda$ is a central simple $R/\mathfrak{m}$-algebra. Hence, by (1. 6) we may suppose that R is a complete local ring with a maximal ideal $\mathfrak{m}$ and that Λ is a central simple R-algebra with a unique maximal two-sided ideal $\mathfrak{m}\Lambda$. Therefore we have only to prove that $\Lambda/\mathfrak{m}\Lambda$ is a central $R/\mathfrak{m}$-algebra.

Let $\bar{C}_l$ be the center of $\Lambda/\mathfrak{m}^l\Lambda$ for any positive integer l. Since $\Lambda/\mathfrak{m}^l\Lambda$ is a simple $R/\mathfrak{m}^l$-algebra, $\Lambda/\mathfrak{m}^l\Lambda$ is a central simple $\bar{C}_l$-algebra. As $R/\mathfrak{m}^l$ is Artinian, $\bar{C}_l$ is also an Artinian local ring. If the answer to Problem H is affirmative for Artinian rings, then $\Lambda/\mathfrak{m}^l\Lambda$ is a central separable $\bar{C}_l$-algebra. Then $\mathfrak{m}\bar{C}_l$ is a maximal ideal of $\bar{C}_l$ and we have $\mathfrak{m}^l\bar{C}_{l+1}=\mathfrak{m}^l(\Lambda/\mathfrak{m}^{l+1}\Lambda)\cap\bar{C}_{l+1}$. By [2], (1. 4), we have $\bar{C}_l=\bar{C}_{l+1}/\mathfrak{m}^l\bar{C}_{l+1}$. If we put $C=\varprojlim\bar{C}_l$, we have $C\subseteq\Lambda$ as R is complete. Let α be an element of C. Then, for any positive integer l we have $\alpha\lambda-\lambda\alpha\in$

$\mathfrak{m}^l C$ for any λ of Λ, and so we have $\alpha\lambda-\lambda\alpha=0$ as $\bigcap_{l=1}^{\infty}\mathfrak{m}^l\Lambda=0$. Hence we have $\alpha\in R$, which proves $C\subseteq R$. Since for any l, $C\to\bar{C}_l$ is an epimorphism induced by $\Lambda\to\Lambda/\mathfrak{m}^l\Lambda$, we have $\bar{C}_l=C/\mathfrak{m}^l\Lambda\cap C$, and, especially we have $\bar{C}_1=C/\mathfrak{m}\Lambda\cap C \subseteq R/\mathfrak{m}\Lambda\cap R=R/\mathfrak{m}$. Thus we see $\bar{C}_1=R/\mathfrak{m}$. This shows that $\Lambda/\mathfrak{m}\Lambda$ has $R/\mathfrak{m}$ as its center, which completes our proof.

3. p-trivial simple algebras.

We see that a simple algebra Λ over a Noetherian ring R is p-trivial ([4]) if and only if there is only one division algebra to which Λ belongs. In [11] it was shown that any simple algebra over a complete local ring is p-trivial (In [11], p-trivial algebras are called 'strongly simple' algebras). However, this can not be generalized to simple algebras over a non-complete local ring without further assumptions. In this section we shall give another class of p-trivial simple algebras.

PROPOSITION 3. 1. *Let R be a Noetherian integrally closed integral domain and Λ a simple R-algebra which is a finitely generated projective R-module. Then the center of Λ is also a Noetherian integrally closed integral domain.*

Proof. Let C be the center of Λ. Then C is an indecomposable Noetherian ring. Let K be the quotient field of R and put $\Sigma=K\otimes_R\Lambda$. Then Σ is a semi-simple K-algebra. Let e be a central idempotent of Σ. As Λ is a simple R-algebra, $\Lambda_{\mathfrak{p}}$ is a semi-simple $R_{\mathfrak{p}}$-algebra for any prime ideal $\mathfrak{p}$ of height 1 in R. Since $R_{\mathfrak{p}}$ is a discrete valuation ring, $\Lambda_{\mathfrak{p}}$ is a maximal $R_{\mathfrak{p}}$-order in Σ and $C_{\mathfrak{p}}$ is a hereditary ring (cf. [7], [9]). Therefore we have $e\in C_{\mathfrak{p}}$. As Λ is a finitely generated projective R-module, we have $\Lambda=\bigcap_{ht\mathfrak{p}=1}\Lambda_{\mathfrak{p}}$, and, from this we obtain $C=\bigcap_{ht\mathfrak{p}=1}C_{\mathfrak{p}}$. Hence we must have $e\in C$. Since C is directly indecomposable, this shows that Σ is a simple K-algebra. Consequently C must be an integral domain. Since each $C_{\mathfrak{p}}$ is hereditary, C is also integrally closed.

LEMMA 3. 2. *Let R be a Noetherian local integral domain and K the quotient field of R. Let Λ be a simple R-algebra which is a finitely generated projective R-module such that $K\otimes_R\Lambda$ is a division K-algebra. Then any finitely generated projective Λ-module is free.*

Proof. This follows immediately from [3], Theorem 2, since $\Lambda/\mathfrak{m}\Lambda$ is a semi-simple $R/\mathfrak{m}$-algebra for a maximal ideal $\mathfrak{m}$ of R.

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THEOREM 3. 3. *Let R be a semi-local regular domain with Krull dimension $\leqq 2$. Then any simple R-algebra Λ, which is a finitely generated projective R-module, is p-trivial.*

Proof. Let K be the quotient field of R. Then, by (3. 1), $K \otimes_R \Lambda$ is a simple K-algebra. Let e be a primitive idempotent of $K \otimes \Lambda$ and put $\mathfrak{l} = (K \otimes_R \Lambda)e \cap \Lambda$. Then $\Lambda/\mathfrak{l}$ is a torsion-free R-module and we have $K \otimes_R \mathfrak{l} = (K \otimes_R \Lambda)e$. Since gl. dim $\Lambda \leqq$ gl. dim $R \leqq 2$, we have $\mathrm{dh}_\Lambda \Lambda/\mathfrak{l} \leqq 1$, and so $\mathfrak{l}$ is projective. $\mathfrak{l}$ is obviously Λ-faithful, so it is also Λ-completely faithful. Now put $\Omega = \mathrm{End}_\Lambda(\mathfrak{l})$. Then $K \otimes_R \Omega$ is a division K-algebra and Ω is a simple (division) R-algebra which is Morita-equivalent to Λ. According to (3.2), any finitely generated projective $\Omega_\mathfrak{m}$-module is free for any maximal ideal $\mathfrak{m}$ of R. As R is semi-local, any finitely generated projective Ω-module is free (cf. [3], Theorem 1), and therefore Ω is p-trivial. Since Ω is Morita equivalent to Λ, Λ is also p-trivial ([4], [11]).

Remark. There exists a simple algebra over a non-semi-local principal ideal domain which is not p-trivial. Accordingly, Theorem 3.3 can not be generalized to a non-semi-local principal ideal domain.

4. Tensor products and Commutor theory.

This section is concerned with the commutor theory of a central separable algebra and its simple subalgebras.

First we give

PROPOSITION 4. 1. *Let Γ be an R-algebra which is a finitely generated projective R-module and Λ be a semi-simple R-subalgebra of Γ. Then Λ is a Λ-direct summand of Γ.*

Proof. By the semi-simplicity of Λ and R-projectivity of Γ, Γ is a Λ-projective and (clearly) Λ-faithful Λ-module, so a completely faithful Λ-module by [5], (6. 1). Then Λ is a Λ-direct summand of Γ.

A semi-simple subalgebras of a central separable algebra splits by the above proposition. So, the coherent condition of the Hattori's commutor theory ([7]) is always satisfied. Then we can give, as a supplement to a Hattori's theorem, the following

THEOREM 4.2 ([7]). *Let Γ be a central separable algebra over a commutative ring R, and Λ be a semi-simple R-subalgebra. Then:*

(1) $V_\Gamma(\Lambda)$ *is semi-simple.*(*)

(2) $V_\Gamma(V_\Gamma(\Lambda))=\Lambda$.

(3) $V_\Gamma(\Lambda)$ *is Morita-equivalent to* $\Gamma \underset{R}{\otimes} \Lambda^0$ *where* Λ^0 *denotes the opposite algebra of* Λ.

LEMMA 4. 3. *Let* Γ *be a central* R*-algebra having* R *as an* R*-direct summand, and* Λ *be an* R*-projective algebra. Then the center of* $\Lambda \underset{R}{\otimes} \Gamma$ *coincides with the center of* Λ.

Proof. At first, we assume that R is a local ring. Since Λ is free, any element of $\Lambda \underset{R}{\otimes} \Gamma$ is uniquely expressed by the form $\sum u_i \otimes \gamma_i$, where $\{u_i\}$ is a basis of Λ and $\gamma_i \in \Gamma$. If $z=\sum u_i \otimes \gamma_i$ lies in the center of $\Lambda \underset{R}{\otimes} \Gamma$, z commutes with all $1 \otimes \gamma$, so $\sum u_i \otimes (\gamma_i\gamma - \gamma\gamma_i)=0$. Then we get $\gamma_i \in$ the center of $\Gamma=R$(i.e. $z=\sum \gamma_i u_i \otimes 1$). Therefore we get $\sum((\gamma_i u_i)\lambda - \lambda(\gamma_i u_i)) \otimes 1 = 0$ for any $\lambda \in \Lambda$, so $\sum \gamma_i u_i \lambda = \lambda \sum \gamma_i u_i$ by the assumption of Lemma. Hence z is in the center of Λ. The converse inclusion is trivial.

In the case that R is global, for any maximal ideal $\mathfrak{m}$ of R, $\Lambda_{\mathfrak{m}}$ is $R_{\mathfrak{m}}$-projective. $\Gamma_{\mathfrak{m}}$ is a central $R_{\mathfrak{m}}$-algebra and $R_{\mathfrak{m}}$ is an $R_{\mathfrak{m}}$-direct summand of $\Gamma_{\mathfrak{m}}$.

Let z be in the center of $\Lambda \underset{R}{\otimes} \Gamma$. Then Z is in the center of $\Lambda_{\mathfrak{m}}$. So there exists $s \notin \mathfrak{m}$ (depending on $\lambda, \mathfrak{m}$) such that sz is contained in the center of Λ. Put $\mathfrak{c}=\{r \in R \mid rz$ is contained in the center of $\Lambda\}$. Then $\mathfrak{c}$ is an ideal of R which is not contained in any maximal ideal. So $1 \in \mathfrak{c}$. Hence z is in the center of Λ.

PROPOSITION 4. 4. *Let* R *be a commutative Noetherian ring. If* Γ *is a central separable* R*-algebra and* Λ *is a simple* R*-algebra which is a finitely generated* R*-projective module, then* $\Lambda \underset{R}{\otimes} \Gamma$ *is a simple* R*-algebra.*

Proof. $\Lambda \underset{R}{\otimes} \Gamma$ is semi-simple by [7], (2. 4). Since R is Noetherian, by virtue of (1. 6), we have only to show the indecomposability of the center of $\Lambda \underset{R}{\otimes} \Gamma$. But it is an immediate consequence of (4.3).

Let Λ be a simple subalgebra of a central separable algebra over a commutative Noetherian ring R. Since Λ is a Λ-direct summand of Γ, $V_\Gamma(\Lambda)$ is Morita-equivalent to $\Lambda^0 \underset{R}{\otimes} \Gamma$. An algebra which is Morita-equivalent to a simple algebra is also simple ([11]), so we get the following

THEOREM 4. 5. *Let* Γ *be a central separable algebra over a commutative Noetherian ring* R*, and* Λ *be a simple subalgebra of* Γ.

(*) $V_\Gamma(\Lambda)=\{\gamma \in \Gamma / \gamma\lambda = \lambda\gamma$, for any $\lambda \in \Lambda\}$.

Then,

(1) $V_\Gamma(\Lambda)$ *is a simple algebra.*

(2) $V_\Gamma(V_\Gamma(\Lambda))=\Lambda$.

(3) $\Lambda^0 \otimes_R \Gamma$ *is simple and is Morita-equivalent to* $V_\Gamma(\Lambda)$.

Remark. In Theorem 4. 5, the simplicity of $V_\Gamma(\Lambda)$ can be proved by a more simple argument. That is: the center of $V_\Gamma(\Lambda)$ contains the center of Λ, and the center of $V_\Gamma(V_\Gamma(\Lambda))(=\Lambda)$ contains the center of $V_\Gamma(\Lambda)$, then the center of $V_\Gamma(\Lambda)$ coincides with the center of Λ. Hence $V_\Gamma(\Lambda)$ is simple whenever Λ is so.

By virtue of this remark, Proposition 4.4 in the case that Λ is a simple subalgebra of Γ is proved without help of Lemma 4. 3.

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Added in proof. Lemma 1.4 holds for a Henselian local ring R. So we can prove the first part of Theorem 1.6 without the assumption that R is Noetherian, by using the Henselization instead of the completion, Then Theorem 1.3 is a special case of Theorem 1.6. Also we can omit this assumption from Proposition 4.4 and Theorem 4.5. (For Henselian rings see M. Nagata, 'Local rings', Interscience, 1962).

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