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FINITE TYPE PSEUDO-UMBILICAL SUBMANIFOLDS IN A HYPERSPHERE

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The notion of finite type submanifolds was introduced by B.Y. Chen. In this article we study 2- and 3-type pseudo-umbilical submanifolds in a hypersphere. Two theorems in this respect are obtained.

1. INTRODUCTION

A submanifold M of a Euclidean m-space E^m is said to be of finite type if each component of its position vector X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M, that is,

$$X = X_0 + \sum_{t=1}^{k} X_{i_t}, \tag{1.1}$$

where X_0 is a constant vector and $\Delta X_{i_t} = \lambda_{i_t} X_{i_t}$, $t = 1, 2, \ldots, k$. If in particular all eigenvalues $\{\lambda_{i_1}, \ldots, \lambda_{i_k}\}$ are mutually different, then M is said to be of k-type. A k-type submanifold is said to be null if one of the λ_{i_t}, $t = 1, 2, \ldots, k$, is null. It is easy to see that if M is compact, then X_0 in (1.1) is exactly the centre of mass in E^m. A submanifold M of a hypersphere S^{m-1} of E^m is called mass-symmetric in S^{m-1} if the centre of mass of M in E^m is the centre of the hypersphere S^{m-1} in E^m (see [1] for details).

In terms of finite type submanifolds, a well-known result of Takahashi [6] says that a submanifold M in E^m is of 1-type if and only if it is either a minimal submanifold of E^m or a minimal submanifold of a hypersphere of E^m. In the first case M is of null 1-type and in the last case M is mass-symmetric in S^{m-1}. In [3] it was proved that a compact 2-type hypersurface of a hypersphere is mass-symmetric if and only if it has constant mean curvature. In [5] it was proved that every 2-type pseudo-umbilical submanifold of E^m with constant mean curvature is either spherical or null 2-type. And in [4] some examples were given for spherical 2-type pseudo-umbilical surfaces, which is mass-symmetric.

In this article we prove the following.

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THEOREM 1. *Let M be a pseudo-umbilical submanifold of a hypersphere S^{m-1} in E^m. If M is of 2-type, then M is a non-null 2-type submanifold of S^{m-1} with constant mean curvature. In particular, if M is compact, then M is mass-symmetric.*

THEOREM 2. *Let M be a pseudo-umbilical submanifold of a hypersphere S^{m-1} in E^m with constant mean curvature. If M is of 3-type, then $\|DH\|$ is constant if and only if $\|DH\|^2$ is harmonic. In particular, if M is closed, then M is mass-symmetric.*

2. Some basic formulas

Let M be an n-dimensional (connected) submanifold of the unit hypersphere $S^{m-1}(1)$ of E^m centred at the origin. Then the position vector X of M in E^m is normal to M as well as to $S^{m-1}(1)$. Let A, h, D and H denote the Weingarten map, the second fundamental form, the normal connection and the mean curvature vector of M in E^m, respectively, and by A', h', D' and H' the corresponding invariants of M in $S^{m-1}(1)$. A submanifold M is said to be pseudo-umbilical if its Weingarten map with respect to the mean curvature vector is proportional to the identity map. It is clear that M is pseudo-umbilical in E^m if and only if so it is in S^{m-1}. We put $\alpha^2 = \langle H, H\rangle$ and $\beta^2 = \langle H', H'\rangle$, where $\langle\, ,\, \rangle$ denotes the inner product of E^m. We have

$$H = H' - X, \quad H' = \beta\xi, \quad \alpha^2 = \beta^2 + 1, \tag{2.1}$$

for some unit normal vector ξ. Let $\{e_1, \dots, e_n\}$ be an orthonormal tangent frame of M. For any vector η normal to the submanifold M we put

$$\widetilde{A}(\eta) = \sum_{i=1}^{n} h(e_i, A_\eta e_i), \tag{2.2}$$

$$\operatorname{tr}(\overline{\nabla} A_\eta) = \sum_{i=1}^{n}\{(\nabla_{e_i} A_\eta)e_i + \operatorname{tr} A_{D\eta}\}, \tag{2.3}$$

where $\operatorname{tr} A_{D\eta} = \sum_{i=1}^{n} A_{D_{e_i}\eta} e_i$. Then we have the following useful formulas obtained in [1, 2].

$$\Delta H = \Delta^D H + \widetilde{A}(H) + \operatorname{tr}(\overline{\nabla} A_H), \tag{2.4}$$

$$\operatorname{tr}(\overline{\nabla} A_H) = \frac{n}{2}\operatorname{grad}\alpha^2 + 2\operatorname{tr} A_{DH}, \tag{2.5}$$

where Δ^D denotes the Laplacian operator associated with the normal connection D.

By using the same method of the proofs of (2.4) and (2.5) given in [1, 2] we may obtain the following.

Lemma 2.1. *Let M be an n-dimensional submanifold of E^m. Then for any vector field η normal to M we have*

$$\Delta\eta = \Delta^D\eta + \widetilde{A}(\eta) + \operatorname{tr}(\overline{\nabla} A_\eta), \tag{2.6}$$

$$\operatorname{tr}(\overline{\nabla} A_\eta) = n\sum_{i=1}^{n}\langle\eta, D_{e_i}H\rangle e_i + 2\operatorname{tr} A_{D\eta}. \tag{2.7}$$

From (2.4), (2.5) and Lemma 2.1, we may obtain the following.

Lemma 2.2. *([5]). If M is a pseudo-umbilical submanifold of E^m, then we have*

$$\Delta H = \Delta^D H + n\alpha^2 H + \frac{4-n}{2}\operatorname{grad}\alpha^2. \tag{2.8}$$

3. Proof of Theorem 1

Let M be a 2-type submanifold of $S^{m-1}(1)$ of E^m centred at the origin. Then (1.1) becomes

$$X = X_0 + X_{i_1} + X_{i_2}, \quad \Delta X_{i_t} = \lambda_{i_t} X_{i_t}, \quad t = 1, 2. \tag{3.1}$$

Since $\Delta X = -nH$, we have

$$\Delta H = bH + c(X - X_0), \tag{3.2}$$

where $b = \lambda_{i_1} + \lambda_{i_2}$, $c = \lambda_{i_1}\lambda_{i_2}/n$.

If M is pseudo-umbilical in E^m, then from Lemma 2.2 and (3.2) we have

$$\Delta^D H + n\alpha^2 H + \frac{4-n}{2}\operatorname{grad}\alpha^2 = bH + c(X - X_0), \tag{3.3}$$

which implies

$$\frac{4-n}{2}\operatorname{grad}\alpha^2 = -c(X_0)^T, \tag{3.4}$$

$$\langle\Delta^D H, X\rangle - \left(n\alpha^2 - b + c\right) + c\langle X_0, X\rangle = 0, \tag{3.5}$$

where $(X_\alpha)^T$ denotes the tangent component of X_0.

Since $\langle\Delta^D H, X\rangle = \Delta\langle H, X\rangle = 0$, (3.5) becomes

$$n\alpha^2 - b + c = c\langle X_0, X\rangle. \tag{3.6}$$

Taking the gradient of both sides of (3.6), we have

$$n\operatorname{grad}\alpha^2 = c(X_0)^T. \tag{3.7}$$

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Combining (3.4) and (3.7), we obtain

$$\frac{4+n}{2}\operatorname{grad}\alpha^2 = 0, \tag{3.8}$$

which implies that α is constant on M and, moreover, by (3.7)

$$c(X_0)^T = 0. \tag{3.9}$$

In (3.9), if $c = 0$, that is, if M is of null 2-type, then by (3.6) and (3.3), we have $\Delta^D H = 0$. Consequently, we have

$$0 = \langle \Delta^D H, H\rangle = \frac{1}{2}\Delta\alpha^2 + \|DH\|^2 = \|DH\|^2. \tag{3.10}$$

Since we have shown above that α is constant on M, (3.10) and (2.1) imply that $D\xi = 0$ and $\beta =$ constant. Put $\eta = \xi + \beta X$ and take the covariant derivative of η in E^m with respect to any vector Y tangent to M, we have

$$\tilde{\nabla}_Y\eta = \tilde{\nabla}_Y\xi + \beta\tilde{\nabla}_Y X = -A_\xi Y - \beta A_X Y = 0. \tag{3.11}$$

Thus the normal vector field η on M is constant. It implies that M is in the intersection of S^{m-1} and a hyperplane of E^m, that is, in a S^{m-2} of E^{m-1}. Now we may consider M as a pseudo-umbilical submanifold of S^{m-2} which is still of null 2-type. Then by induction M would be an open portion of S^n in E^{n+1}, but which is not of 2-type. This is a contradiction.

Now we may assume that M is of non-null 2-type, then from (3.9) we have $(X_0)^T = 0$. Consequently, we have

$$\operatorname{grad}\langle X - X_0, X_0\rangle = (X_0)^T = 0, \tag{3.12}$$

which implies that either $X_0 = 0$, or $X_0 \neq 0$ and M is in the intersection of S^{m-1} and a hyperplane of E^m. But the latter case could not occur since by an argument similar to the one mentioned above it would lead to a contradiction. Consequently, if M is compact, since in this case X_0 is the centre of mass of M in E^m, then M is mass-symmetric in S^{m-1}.

4. Proof of Theorem 2

Let M be a 3-type submanifold of $S^{m-1}(1)$ of E^m centred at origin. Then, as in Section 3, we have

$$X = X_0 + X_{i_1} + X_{i_2} + X_{i_3}, \quad \Delta X_{i_t} = \lambda_{i_t} X_{i_t}, \quad t = 1, 2, 3. \tag{4.1}$$

$$\Delta^2 H = a\Delta H + bH + c(X - X_0), \tag{4.2}$$

where $a = \sum_t \lambda_{i_t}$, $b = -\sum_{t<s} \lambda_{i_t}\lambda_{i_s}$ and $c = \lambda_{i_1}\lambda_{i_2}\lambda_{i_3}/n$.

If M is a pseudo-umbilical submanifold in E^m with constant mean curvature, then by Lemmas 2.2 and 2.1, we have

$$\Delta H = \Delta^D H + n\alpha^2 H, \tag{4.3}$$

$$\begin{aligned} \Delta^2 H &= \Delta(\Delta^D H) + n\alpha^2 \Delta H \\ &= \Delta^D\Delta^D H + \widetilde{A}(\Delta^D H) + \operatorname{tr}(\overline{\nabla} A_{\Delta^D H}) + n\alpha^2\Delta^D H + n^2\alpha^4 H. \end{aligned} \tag{4.4}$$

From (4.2)—(4.4), we have

$$\begin{aligned} &\langle \Delta^D\Delta^D H, X\rangle + \langle \widetilde{A}(\Delta^D H), X\rangle + n\alpha^2\langle \Delta^D H, X\rangle + n^2\alpha^4\langle H, X\rangle \\ &\quad = a\langle \Delta^D H, X\rangle + (n\alpha^2 a + b)\langle H, X\rangle + c - c\langle X_0, X\rangle. \end{aligned} \tag{4.5}$$

In (4.5), we have

$$\langle \Delta^D H, X\rangle = \Delta\langle H, X\rangle = 0, \quad \langle \Delta^D\Delta^D H, X\rangle = \Delta\langle \Delta^D H, X\rangle = 0, \tag{4.6}$$

$$\begin{aligned} \langle \widetilde{A}(\Delta^D H), X\rangle &= \langle \Delta^D H, \widetilde{A}(X)\rangle = \langle \Delta^D H, \sum_i h(e_i, A_X e_i)\rangle \\ &= -n\langle \Delta^D H, H\rangle = -n\,\|DH\|^2. \end{aligned} \tag{4.7}$$

Thus (4.5) implies

$$\|DH\|^2 = -(n^2\alpha^4 - n\alpha^2 a - b + c)/n + c\langle X, X_0\rangle/n. \tag{4.8}$$

Taking the Laplacian of both sides of (4.8), we obtain

$$\Delta\,\|DH\|^2 = c\Delta\langle X, X_0\rangle/n = -c\langle H, X_0\rangle. \tag{4.9}$$

Consequently, $\|DH\|^2$ is harmonic if and only if $c\langle H, X_0\rangle = 0$, that is, (i) $c = 0$, or (ii) $c \neq 0$ and $\langle H, X_0\rangle = 0$.

In Case (i), M is of null 3-type and by (4.8) $\|DH\|^2 = -(n^2\alpha^4 - n\alpha^2 a - b)/n$ which is a constant.

In Case (ii), from (4.2) we have

$$\langle \Delta^2 H, X_0\rangle = a\langle \Delta H, X_0\rangle + b\langle H, X_0\rangle + c\langle X - X_0, X_0\rangle. \tag{4.10}$$

Since we have

$$\langle \Delta H, X_0\rangle = \Delta\langle H, X_0\rangle, \quad \langle \Delta^2 H, X_0\rangle = \Delta\langle \Delta H, X_0\rangle = \Delta^2\langle H, X_0\rangle, \tag{4.11}$$

(4.10) implies $\langle X - X_0, X_0\rangle = 0$. With an argument similar to the one mentioned in the proof of Theorem 1, we may conclude that X_0 must be the origin of E^m. Then (4.8) gives $\|DH\|^2 = -(n^2\alpha^4 - n\alpha^2 a - b + c)/n$ which is a constant. In particular, if M is closed (that is, M is compact and without boundary), then because λ_{i_1}, λ_{i_2} and λ_{i_3} are positive, M cannot be null. Moreover, in this case, because X_0 is the centre of mass of M in E^m, M is mass-symmetric in S^{m-1}.

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