Can. J. Math., Vol. XXVII, No. 1, 1975, pp. 176–185

NORM CONVERGENCE OF RIESZ-BOCHNER MEANS FOR RADIAL FUNCTIONS

G. V. WELLAND

1. Introduction. It is well known now that certain spherical methods in k ($\geqq 2$) dimensions are rather poor for reconstructing a function from its Fourier transform. Consider a function f in $L^1(\mathbf{R}^k)$, $k \geqq 2$,

$$(1.1)\quad \hat{f}(z) = \frac{1}{(2\pi)^{k/2}} \int f(x) e^{ix\cdot z} dx$$

and

$$(1.2)\quad S_R^{\alpha} f(x) = \frac{1}{(2\pi)^{k/2}} \int_{|z| \leqq R} \left(1 - \frac{|z|^2}{R^2}\right)^{\alpha} \hat{f}(z) e^{-ix\cdot z} dz$$

where both integrals are integrals in $\mathbf{R}^k$, the first over the whole space the second over the ball of radius R; $x \cdot y$ is the usual Euclidean inner product of x and y in $\mathbf{R}^k$ and $|z|^2 = z \cdot z$.

When $\alpha = 0$ we have the spherical method alluded to above. Fefferman [**3**] showed (using the extended definition of Fourier transform) that only for $p = 2$ is it true that $S_R^0 f$ converges in L^p norm to the function f, as $R \to \infty$. On the other hand it is known (see, e.g., [**11**, p. 172]) that $S_R^{\alpha} f$ converges to f in L^p norm ($1 \leqq p < \infty$) as long as α exceeds the "critical index" $(k-1)/2$.

For the more difficult range $0 \leqq \alpha \leqq (k-1)/2$ there are two types of results giving a range of values of p for which convergence holds. In [**5**] Herz pointed out that for radial functions $S_R f$ converges in L^p to the function f, provided

$$\frac{2k}{k+1} < p < \frac{2k}{k-1},$$

where f belongs to L^p and appropriate restrictions on f are made so that $\hat{f}$ exists. (Here, a radial function means an f such that $f(x) = f(|x|)$, where f also denotes a function defined on $(0, \infty)$.) For $p \leqq 2k/(k+1)$, the result fails. Stein [**8**, p. 487] proved a convergence result for general (non-radial) functions for p in the range

$$\frac{2(k-1)}{k-1+2\alpha} < p < \frac{2(k-1)}{k-1-2\alpha}.$$

Received August 10, 1973 and in revised form, July 29, 1974. This research was supported in part by NSF grant GP-34630.

https://doi.org/10.4153/CJM-1975-023-1 Published online by Cambridge University Press

Here we consider radial functions and get convergence for a wider range of values of p. Our result is as follows:

THEOREM 1. *Let $L^p(\mathbf{R}^k, r)$ denote the class of radial functions in $\mathbf{R}^k$ which are in L^p. The operator S_R^α is defined on a dense class in $L^p(\mathbf{R}^k, r)$ and we have for*

$$\frac{2k}{k+1+2\alpha} < p < \frac{2k}{k-1-2\alpha}, \quad 0 < \alpha \frac{k-1}{2},$$

$$(1.3) \quad ||S_R^\alpha f||_p \leqq A_{\alpha,p}||f||_p,$$

$A_{\alpha,p}$ independent of f and R, and the definition of S_R^α can be extended to all of $L^p(\mathbf{R}^k, r)$ by continuity.

To prove the theorem we use an interpolation theorem due to Stein [**10**]; he used this to interpolate between L^2 results for $\alpha = 0$ and L^1 results for $\alpha > (k-1)/2$. Because of the lack of L^p results for $\alpha = 0$, $p \neq 2$ [**3**], this technique cannot be extended further for general (non-radial) f. However, for radial functions L^p, results do exist for $\alpha = 0$ and $p < 2$ [**5**] and the interpolation technique gives essentially the best possible results.

The operator S_R^α is a convolution operator whose kernel is $\sigma_R^\alpha(x) = cR^k J_{k/2+\alpha}(R|x|)(R|x|)^{-(k/2+\alpha)}$, where c is a constant depending on the parameters k and α. For the following statements, fix α, $0 \leqq \alpha < (k-1)/2$. Results of non-radial nature hold for $\alpha \geqq (k-1)/2$ as can be seen in [**8**] and [**9**]. Because of the asymptotic expansion on page 199 of [**12**] one has

$$J_{k/2+\alpha}(R|x|) = c_1(\cos(R|x| + c_2))(R|x|)^{-1/2} + o(R|x|)^{-3/2}),$$

where c_1 and c_2 are constants. One can use this to show that if f is the characteristic function of the unit ball then $(S_1^\alpha f)$ is not in L^p for $p \leqq 2k/(k+1+2\alpha)$. This is the same idea used in [**5**] to get negative results. Hence, one sees that the best possible interval of values of $p \leqq 2$ for which a boundedness result is possible in $2k/(k+1+2\alpha) < p \leqq 2$.

From this point on we suppress the dependence of $A_{\alpha,p}$ on α and write A_p.

The theorem has as an immediate consequence that $S_R^\alpha f$ converges in norm to f. This is proved by means of the usual arguments using a dense class of smooth functions for which pointwise and norm convergence can be proved as $R \to \infty$. For example in [**1**, p. 119, Theorem 4.53], it is shown that convergence holds, for $\alpha \geqq 0$, for the case of Fourier series of a function with at least k continuous derivatives. Similar techniques work in the case of Fourier integrals.

Stein's interpolation lemma is stated in section 2. In sections 3 and 4 the family of operators to which interpolation is applied is introduced and shown to have the analyticity and boundedness properties required for the applicability of the lemma. Theorem 1 follows easily in the case $2k/(k+1+2\alpha) < p \leqq 2$. The case $2 \leqq p < 2k/(k-1-2\alpha)$ is handled by a duality argument (see, e.g., [**11**, Chapter 1, Theorem 3.20]).

In section 5 we point out an open problem in the theory of Fourier series.

2. The interpolation lemma. The technique essential to the proof of the theorem is the following interpolation lemma. Stein introduced it in [**10**] and used it effectively in [**8**] and [**9**] to prove results similar to the result here.

First one must introduce the notion of an *analytic family of operators* $\{T_z\}$. Let (M, dm) and (N, dn) be two measure spaces. A family of operators $\{T_z\}$ depending on a complex parameter z is called *analytic* if:

(i) For each z, T_z is a linear transformation of "simple" functions (i.e., those measurable functions which take on only a finite number of nonzero values and have support on a set of finite measure; in our case we consider only functions of bounded support) on M to measurable functions on N.

(ii) If ψ is a simple function on M, and ϕ is a simple function on N, then

$$\Phi(z) = \int_N T_z(\psi)\phi dn$$

is analytic in $0 < R(z) < 1$ and continuous in $0 \leqq R(z) \leqq 1$.

Since we deal with radial functions, we see that after a change to polar coordinates (in our case) M and N may be considered to be $(0, \infty)$ with the measures equal to a constant times $t^{k-1}dt$. However, we will freely switch to $\mathbf{R}^k$ with the usual Lebesgue measure when convenient. There should be no confusion.

An analytic family T_z is of *admissible growth* if $\Phi(z)$ is of admissible growth; that is, if

$$\sup_{|y|<r} \sup_{0\leqslant x\leqslant 1} \log|(\Phi(x+iy))| \leqq Ae^{ar},$$

where $a < \pi$ and A is a constant. Both A and a may depend on the functions ϕ and ψ.

The interpolation lemma is the following:

LEMMA. *Let $\{T_z\}$ be an analytic family of linear operators of admissible growth defined in the strip $0 \leqq R(z) \leqq 1$. Suppose that $1 \leqq p_1, p_2, q_1, q_2 \leqq \infty$, and that*

$$\frac{1}{p} = (1-t)\cdot\frac{1}{p_1} + \frac{t}{p_2}, \quad \frac{1}{q} = (1-t)\cdot\frac{1}{q_1} + \frac{t}{q_2},$$

where $0 \leqq t \leqq 1$. Finally suppose that

$$||T_{iy}f||_{q_1} \leqq A_0(y)||f||_{p_1} \text{ and } ||T_{1+iy}f||_{q_2} \leqq A_1(y)||f||_{p_2}$$

for every simple f. We also assume that:

$$\log|A_i(y)| \leqq A_i^{a|y|}, \qquad a < \pi;\ i = 0, 1,$$

where a, A_0 and A_1 do not depend on f.

Then we have

$$||T_t(f)||_q \leqq A_t||f||_p$$

https://doi.org/10.4153/CJM-1975-023-1 Published online by Cambridge University Press

where

$$(2.1)\quad \log A_t = \int_{-\infty}^{\infty} w(1-t, y) \log A_0(y) dy + \int_{-\infty}^{\infty} w(t, y) \log A_1(y) dy.$$

The function $w(1-t, y)$ is the Poisson kernel for the region $0 < \operatorname{Re}(z) < 1$ in the complex plane, and is given by

$$w(1-t, y) = \frac{1}{2} \frac{\tan(\pi t/2)}{[\tan^2(\pi t/2) + \tanh^2(\pi y/2)] \cosh^2(\pi y/2)}.$$

3. An analytic family of operators. In this section we describe the analytic family $\{T_z\}$ which we use. But first, we make a simple observation. Using (1.1) and (1.2) it is easy to show by means of changes of variables that $S_R^\alpha f(x) = S_1^\alpha f_R(Rx)$ where $f_R(x) = f(x/R)$ and the function f is such that all integrals involved exist. If one then supposes that

$$||S_1^\alpha f(\cdot)||_p \leqq A_p ||f||_p$$

it follows that

$$||S_R^\alpha f(\cdot)||_p^p = \int |S_1^\alpha f_R(Rx)|^p dx = R^{-k} \int |S_1^\alpha f_R(x)|^p dx$$
$$\leqq A_p^p R^{-k} \int |f_R(x)|^p dx = A_p^p ||f||_p^p.$$

Hence, it suffices to prove the theorem in the case $R = 1$. For the remainder of the proof we restrict our attention to operators S_1^α, we will presently extend the definition of α to include complex values.

Let f be a radial function on $\mathbf{R}^k$ and let f also denote the associated function on $(0, \infty)$ such that $f(x) = f(|x|)$. For simple functions one has (see for example [**7**, Formula 6, p. 52]) that

$$(3.1)\quad S_1^\alpha f(x) = c \int_{\mathbf{R}^k} f(t)(|x-t|)^{-(k/2+\alpha)} J_{k/2+\alpha}(|x-t|) dt,$$

where $c = 2\alpha\Gamma(\alpha+1)(2\pi)^{-k/2}$. Let $2\alpha_0 = k-1$ and let $\alpha(z) = \alpha_0(1-z) + \epsilon$, $\epsilon > 0$ and $0 < \operatorname{Re}(z) < 1$. With this notation we consider the family of operators $\{S_1^{\alpha(z)}\}$ on the family of simple functions of bounded support which is dense in $L^p(\mathbf{R}^k, r)$.

We need the following facts about Bessel functions:

$$(3.2)\quad J_\zeta(t) = \frac{(t/2)^\zeta}{\Gamma(1/2)\Gamma(\zeta+1/2)} \int_0^1 (1-u^2)^{\zeta-1/2} \cos ut\, du, \quad \operatorname{Re}(\zeta) > -\frac{1}{2}$$

$$(3.3)\quad |J_{\xi+i\eta}(t)| \leqq A_\xi e^{\pi|\eta|} \cdot t^{-1/2}, \quad t \geqq 1, \xi \geqq 0$$

$$(3.4)\quad |J_{\xi+i\eta}(t)| \leqq A_\xi e^{\pi|\eta|} t^\xi, \quad t > 0, \xi \geqq 0.$$

Formula (3.2) follows from that on page 38 of [**12**]. Inequality (3.3) is obtained from the asymptotic expansion on page 199 of [**12**]. Inequality (3.4) can be obtained from (3.2) by using Hankel's formula for $1/\Gamma(z)$ which can be found in [**2**, p. 227]. Here we will use $\zeta = \alpha(z) + k/2$ with $0 < \operatorname{Re}(z) < 1$.

By (3.2) we have J_ζ is analytic in the strip $0 < \operatorname{Re}(z) < 1$. In fact, for z in a compact subset of $0 < \operatorname{Re}(z) < 1$

$$B(z, t) = \int_0^1 (1 - u^2)^{\zeta - 1/2} \cos utdu$$

is a bounded function for all t with bound independent of t. By holding z in an open ball which has compact closure in the strip $0 < \operatorname{Re}(z) < 1$ the integral

$$\int_{\mathbf{R}^k} f(t)(|x - t|)^{-\zeta} t^\zeta B(z, t)dt$$

can be considered as the limit of a uniformly convergent sequence of analytic functions in this open ball when f is a simple function. Again using Hankel's formula for $1/\Gamma(z)$ we see that $S_1^{\alpha(z)} f(x)$ is a product of analytic functions in this ball and by analytic continuation throughout all of the strip $0 < \operatorname{Re}(z) \leqq 1$. A similar argument proves continuity throughout $0 \leqq \operatorname{Re}(z) \leqq 1$.

To see that the admissible growth condition of the lemma is satisfied it suffices to estimate $S_1^{\alpha(z)} f(x)$ in the L^∞ norm. For this, we break the integral in (3.1) into two parts corresponding to $|x - t| \geqq 1$ and $|x - t| \leqq 1$. Using (3.3) for the first of these we have

$$\left| c \int_{|x-t| \geqslant 1} f(t)|x - t|^{-\zeta} J_\zeta(|x - t|)dt \right| \leqq cA_\xi e^{\pi|\eta|} \times \int_{|x-t| \geqslant 1} |f(t)||x - t|^{-\xi - 1/2} dt$$

where $\zeta = \xi + i\eta = \alpha(z) + k/2$. Using (3.4) for the second of these gives

$$\left| c \int_{|x-t| \leqslant 1} f(t)|x - t|^{-\zeta} J_\zeta(|x - t|)dt \right| \leqq cA_\xi e^{\pi|\eta|} \int_{|x-t| \leqslant 1} |f(t)|dt.$$

Since f is a bounded simple function, both of the above integrals converge and hence $|S_1^\alpha f(x)| \leqq cA_\xi e^{\pi|\eta|}$, where c is a constant which depends on the simple function f. From this it follows that $\{S_1^{\alpha(z)}\}$ satisfies the admissible growth condition.

4. Completion of the proof of the theorem. In this section, we prove the bounds on the boundary of the strip $0 < \operatorname{Re}(z) < 1$ which are necessary for the application of the lemma. We will restrict our attention to $1 \leqq p \leqq 2$. The results valid for conjugate values of p are obtained by the use of a duality argument, a discussion of which can be found in [**11**, Chapter 1, Theorem 3.20]. We take $p_1 = q_1 = 1$ and $p_2 = q_2 > 2k/(k + 1)$.

The norm inequality which corresponds to $\mathrm{Re}(z) = 0$ is

(4.1) $\|S_1^{\alpha(iy)}f\|_1 \leqq A_\xi e^{\pi|\eta|}\|f\|_1.$

We note that A_ξ may vary in meaning from time to time, those A_ξ which appear on two occasions will be related by a constant multiple depending only on the dimension k. For $\mathrm{Re}(z) = 0$, $\mathrm{Re}(\alpha(z)) > (k-1)/2$, in which case (3.1) shows that $S_1^{\alpha(z)}$ is a kernel operator with kernel

$$K_z(x) = c|x|^{-(k/2+\alpha(z))} \cdot J_{\alpha(z)+k/2}(|x|).$$

Conditions (3.3) and (3.4) together are enough to show that $K_z(x)$ is an L^1 function with $\|K_z\|_1 \leqq A_\xi e^{\pi|\eta|}$ where $\xi + i\eta = \alpha(z) + k/2$. This implies (4.1).

For the boundary condition corresponding to $\mathrm{Re}(z) = 1$ we have to consider $p_2 = q_2 = p_0 > 2k/(k+1)$ and $p \leqq 2$. The operator $S_1^{\alpha(z)}$ becomes $S_1^{\epsilon+i\eta}$ where $\epsilon > 0$ and $\eta = -\alpha_0 y$. We have the following modified Hankel transform representation of $S_1^{\epsilon+i\eta}f(x)$:

$$(4.2)\quad S_1^{\epsilon+i\eta}f(x) = c|x|^{-(k-2)/2}\int_0^\infty f(t)t^{k/2}\int_0^1 (1-r^2)^{\epsilon+i\eta}J_{(k-2)/2} \times (|x|r)J_{(k-2)/2}(tr)rdrdt.$$

To see this, use the formula

$$J_n(z) = \frac{(z/2)^n}{\Gamma(n+1/2)\Gamma(1/2)}\int_0^\pi e^{\pm iz\cos\phi}\sin^{2n}\phi d\phi$$

which is valid for $\mathrm{Re}(n + 1/2) > 0$. This formula is found in [**13**, p. 366]. We first write

$$S_1^{\epsilon+i\eta}f(x) = \frac{1}{(2\pi)^k}\int_{|y|<1}(1-|y|^2)^{\epsilon+i\eta}\left[\int f(w)e^{iw\cdot y}dw\right]e^{-iy\cdot x}dy.$$

Because f is a radial function, its Fourier transform is also radial. Using this fact and allowing interchange of order of integrations whenever necessary, a procedure which is valid for "good" functions, we proceed using the following notation: $|w| = t$, $|y| = r$ and w_{k-1} is the $(k-1)$-dimensional volume of the $(k-1)$-ball. After a change to polar coordinates the inner integral is

$$\int_0^\infty (k-1)w_{k-1}f(t)\left[\int_0^\pi e^{irt\cos\theta}(\sin\theta)^{k-2}d\theta\right]t^{k-1}dt = F(r).$$

Similarly the whole integral is

$$(k-1)\frac{w_{k-1}}{(2\pi)^k}\int_0^1 (1-r^2)^{\epsilon+i\eta}r^{k-1}F(r)\left[\int_0^\pi e^{-i|x|r\cos\phi}(\sin\phi)^{k-2}d\phi\right]dr.$$

Using the above representation of the Bessel function in the two integrals involving θ and ϕ, and interchanging the order of integration one obtains (4.2).

We use the formula with $\nu = (k - 2)/2$ (see [**13**, p. 381])

$$rJ_\nu(|x|r)J_\nu(tr) = \frac{1}{|x|^2 - t^2}\frac{d}{dr}[rtJ_\nu(|x|r)J_\nu'(tr) - r|x|J_\nu'(|x|r)J_\nu(tr)]$$

and the derivative formula for J_ν (see [**13**, p. 361]) to see that the inner integral in (4.2) is $1/(|x|^2 - t^2)$ multiplied by the sum of

$$(4.3)\qquad \int_0^1 (1 - r^2)^{\epsilon+i\eta}\frac{d}{dr}[rtJ_\nu(|x|r)J_{\nu-1}(tr)]dr$$

and

$$(4.4)\qquad \int_0^1 (1 - r^2)^{\epsilon+i\eta}\frac{d}{dr}[r|x|J_{\nu-1}(|x|r)J_\nu(tr)]dr.$$

In (4.4), we integrate by parts to obtain

$$(4.5)\qquad \int_0^1 r|x|J_{\nu-1}(|x|r)J_\nu(tr)\cdot K(r,\eta)dr$$

where

$$K(r,\eta) = 2r(\epsilon + i\eta)(1 - r^2)^{\epsilon-1+i\eta}.$$

Because of the asymptotic behavior of J_ν and $J_{\nu-1}$ we find that (4.5) is dominated by

$$\begin{aligned} 2(|\eta| + \epsilon)&\int_0^1 r^2|x|B_1(|x|r)(|x|r)^{-1/2}B_2(tr)(tr)^{-1/2}(1 - r^2)^{\epsilon-1}dr \\ &\leqq 2(|\eta| + \epsilon)c\left(\frac{|x|}{t}\right)^{1/2}\int_0^1 (1 - r^2)^{\epsilon-1}rdr \\ &= \epsilon^{-1}(|\eta| + \epsilon)c\left(\frac{x}{t}\right)^{1/2}. \end{aligned}$$

The functions B_1 and B_2 are bounded functions and c represents the product of their supremums. The integral (4.3), in a similar manner, gives a term bounded by $\epsilon^{-1}c(|\eta| + \epsilon)(t/|x|)^{1/2}$.

We now place these estimates for (4.3) and (4.4) in (4.2) and obtain

$$(4.6)\qquad |S_1^{\epsilon+i\eta}(f,x)| \leqq c\epsilon^{-1}(|\eta| + \epsilon)|x|^{-(k-2)/2}\int_0^\infty \frac{f(t)t^{k/2}}{|x|^2 - t^2} \times\left[\left(\frac{|x|}{t}\right)^{1/2} + \left(\frac{t}{|x|}\right)^{1/2}\right]dt.$$

We split the right-hand-side (dropping the constant for the moment) into

https://doi.org/10.4153/CJM-1975-023-1 Published online by Cambridge University Press

two terms:

$$\tilde{f}_1(|x|) = |x|^{-(k-3)/2} \int_0^\infty \frac{f(t)t^{(k-1)/2}}{|x|^2 - t^2} dt$$

and

$$\tilde{f}_2(|x|) = |x|^{-(k-1)/2} \int_0^\infty \frac{f(t)t^{(k+1)/2}}{|x|^2 - t^2} dt.$$

We are interested in proving $||\tilde{f}_i||_p \leqq A_p||f||_p$, $i = 1, 2$; for this purpose we note that f is in $L^p(t^{k-1}dt)$ when considered as a function on $(0, \infty)$, and we want to prove $\tilde{f}_i$ is in $L^p(t^{k-1}dt)$ when considered as a function on $(0, \infty)$. Making the changes of variable $|x|^2 = \sigma$ and $t^2 = \tau$ and letting

$$\psi_i(\sigma) = \sigma^{(1/p)(k/2-1)}\tilde{f}_i(\sigma^{1/2}), \quad \phi(\tau) = \frac{1}{2}\tau^{(1/p)(k/2-1)}f(\tau^{1/2}) \text{ and}$$

$$\gamma = \frac{k}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$$

our problem is to show that there exists A_p such that $||\psi_i||_p \leqq A_p||\phi||_p$, where

$$\psi_i(\sigma) = \int_0^\infty \frac{\phi(\tau)}{\sigma - \tau}\left(\frac{\tau}{\sigma}\right)^{\alpha_i} d\tau$$

and $\alpha_1 = -\gamma + 1/p - 3/4$ and $\alpha_2 = -\gamma + 1/p - 1/4$. We see that ψ_i can be expressed as the difference of

$$(4.7) \quad \int_0^\infty \frac{\phi(\tau)}{\sigma - \tau} d\tau$$

and

$$(4.8) \quad \int_0^\infty \frac{\phi(\tau)}{\sigma - \tau}\left(1 - \left(\frac{\tau}{\sigma}\right)^{\alpha_i}\right) d\tau, \quad i = 1, 2.$$

It follows from [**4**, Theorem 319] using the method indicated in [**5**, p. 998] that the integral (4.8) is in L^p and $||\psi_i||_p \leqq A_p||\phi||_p$ provided $\alpha_i < 1/p$, i.e. provided that $p < 2k/(k + 6)$ and $p > 2k/(k + 1)$ for $i = 1$ and 2 respectively. The integral (4.7) satisfies a similar L^p norm inequality (see [**4**]) for $1 < p < \infty$ since it is the Hilbert transform of ϕ.

Thus we have shown that $||S_1^{\epsilon+i\eta}f||_p \leqq A_p c(|\eta|\epsilon^{-1} + 1)||f||_p$, for $2k/(k + 1) < p \leqq 2$, so we see that $S_1^{\epsilon+i\eta}$ satisfies the second boundary condition of the interpolation lemma.

We are now ready to apply the interpolation lemma. Let $2\alpha_0 = k - 1$, $p_0 = 2k/k + 1 + \epsilon$, and $\alpha(z) = \alpha_0(1 - z) + \epsilon$ where $\epsilon > 0$, and $0 < R(z) < 1$. With $1/p = (1 - t) + t/p_0$, we apply the lemma to the analytic family $S_1^{\alpha(z)}$ and find

$$(4.9) \quad ||S_1^{\alpha(t)}f||_p \leqq A_p||f||_p$$

https://doi.org/10.4153/CJM-1975-023-1 Published online by Cambridge University Press

where $\log A_p = \log A_\nu$ is given by (2.1). Easy estimates using the growth of $A_0(y)$ and $A_1(y)$ show that A_p is finite.

For each $\epsilon > 0$, the interpolation theorem gives (4.9) for those values of p which satisfy

$$1/p = (1 - t) + t/p_0, \qquad 0 \leqq t \leqq 1.$$

But t is restricted by the conditions which were imposed on α. By using $\alpha = \alpha_0(1 - t) + \epsilon$ we have $t = 1 - (\alpha - \epsilon)/\alpha_0$. Hence the result holds for those values of p which satisfy

$$\frac{1}{p} = \left[\frac{\alpha - \epsilon}{\alpha_0}\right]\left(1 - \frac{1}{p_0}\right) + \frac{1}{p_0}, \quad \epsilon \leqq \alpha \leqq \epsilon + \alpha_0.$$

Considering that ϵ may be arbitrarily small we see that if $0 < \alpha < \alpha_0$, then (4.9) holds for those values of p which satisfy $1/p < (2\alpha + k + 1)/2k$. The technique fails when $\epsilon = 0$, that is when $p = 2k/(k + 1 + 2\alpha)$. As was earlier remarked, this is sufficient to complete the proof of the theorem.

5. An open problem. In [**11**, p. 261–263], a technique for constructing multipliers for periodic functions in $L^p(\mathbf{T}^k)$ is given where $\mathbf{T}^k$ is the torus in k dimensions. The method modifies and extends a periodic function to Euclidean space. After applying a multiplier on $L^p(\mathbf{R}^k)$ and a limiting process a multiplier theorem on $L^p(\mathbf{T}^k)$ results. In our case, we only have a multiplier theorem for functions in $L^p(\mathbf{R}^k, r)$.

A class of "radial" periodic functions could be defined to be those periodic functions f in $L^1(\mathbf{T}^k)$ for which $\hat{f}(n)$ depends only on $|n|$. This leads to the question, whether a multiplier theorem for "radial" periodic functions can be obtained from such a theorem on $L^p(\mathbf{R}^k, r)$. In particular, does Theorem 1 imply a similar theorem for radial periodic functions? The technique of [**11**] fails because the modified and extended function is not *radial.*

References

1. K. Chandresekharan and S. Minakshisundaran, *Typical means* (Oxford University Press, Oxford, 1952).
2. E. T. Copson, *Theory of functions of a complex variable* (Oxford University Press, Oxford 1935).
3. C. L. Fefferman, *The multiplier problem for the ball,* Ann. of Math. *94* (1971), 330–336.
4. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities* (Cambridge Univ. Press, Cambridge, 1934).
5. C. S. Herz, *On the mean inversion of Fourier and Hankel transforms,* Proc. Nat. Acad. Sci. U.S.A. *40* (1954), 996–999.
6. R. A. Hunt, *On the convergence of Fourier series,* Proc. Conf. Orthogonal Expansions and their Continuous Analogues (Edwardsville, Ill., 1967), Southern Illinois University Press, Carbondale, Ill., 1968, pp. 235–255.
7. V. L. Shapiro, *Fourier series in several variables,* Bull. Amer. Math. Soc. *70* (1964), 48–93.
8. E. M. Stein, *Localization and summability of multiple Fourier series,* Acta Math. *100* (1958), 93–147.

9. ——— *On certain exponential sums arising in multiple Fourier series*, Ann. of Math. *73* (1961), 87–109.

10. ——— *Interpolation of linear operators*, Trans. Amer. Math. Soc. *83* (1956), 482–496.

11. E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Series (1971).

12. G. N. Watson, *A treatise on the theory of Bessel functions* (Cambridge University Press, Cambridge, 1922).

13. E. T. Whittaker and G. N. Watson, *A course of modern analysis* (Cambridge University Press, Cambridge, 1902).

University of Missouri-St. Louis,
St. Louis, Missouri