HIERARCHY OF PERIODIC SOLUTIONS FOR HAMILTONIAN SYSTEMS AND THEIR APPLICATIONS TO CELESTIAL MECHANICS

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ABSTRACT

A systematic investigation has been carried out for periodic solutions for standard-form Hamiltonian systems containing a small parameter/the principal problem of dynamics/. An efficient method of investigation of conditions for periodicity of solutions has been developed. Besides fitting the initial conditions of the action-angle variables, the idea of fitting the values of the parameters of the problem is used. Constructive conditions are obtained for the existence of periodic solutions in both principal and degenerate cases, as well as necessary conditions for their stab—ility; algorithms have been developed for constructing these solutions as series in integer powers of the small parameter. To study particular periodic solutions /by high order resonances/, canonical transformations of the initial equations to a special form are used.

1. Consider a one-degree nonautonomous Hamiltonian system

$$\frac{dI}{dt} = \frac{\partial H}{\partial \psi} \qquad \frac{d\psi}{dt} = -\frac{\partial H}{\partial I}$$

$$H = H_o(I,A,B) + \sum_{\sigma=1}^{\infty} \mu^{\sigma} H_{\sigma}(I,\psi,A,B,\Omega,t) \qquad (1)$$

Here H_o is the unperturbed Hamiltonian being a function of the position variable I and of the problem parameters A, B. The Hamiltonian H in (1) is a holomorphous function of a

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small nonnegative parameter μ in some vicinity of $\mu = 0$. The functions $H_\sigma(I,\psi,A,B,\Omega t)$ are 2π periodic in the angular variable ψ and Ωt is a given fixed frequency, $T = 2\pi/\Omega$ is the period.

For $\mu = 0$ we obtain from (1) a generating periodic solution [1]

$$I = a, \quad \psi = n^{(o)}t+w, \quad n^{(o)} = -\frac{\partial H_o}{\partial a}, \quad q_o n^{(o)}(a,\alpha_o,\beta_o) = q\Omega$$

(2)

Here w is an arbitrary constant, $n^{(o)}$ is the unperturbed frequency, q_o and q are integer numbers. The generating values of the variable I and of the parameters A,B are designated as a and α_o,β_o.

Let $a+\delta$ and $w+\gamma$ be the initial values of the variables, and $A = \alpha_o+\alpha$, $B = \beta_o+\beta$ are the parameter values for the sought periodic solutions. According to the proposed method, the quantities δ,γ, and in certain conditions the quantities α,β as well, are constructed as series in integer or fractional powers of the small parameter μ. The proof of existence of periodic solutions is basing on the analysis of the periodicity conditions and of their resolvability with respect to some two out of four small quantities $v = (\delta,\gamma,\alpha,\beta)$ [2].

Let us designate as $\overline{f}$ and $\tilde{f}$ the constant and the conditionally periodic parts, respectively, of the function $f(I,\psi,A,B,\Omega t)$ being 2π periodic in ψ and in Ωt and calculated for the generating values of the parameters (2), i.e.

$$\overline{f}(a,w,\alpha_o,\beta_o) = \frac{1}{T} \int_0^T f(a,n^{(o)}t+w,\alpha_o,\beta_o,\Omega t)dt, \quad \tilde{\overline{f}} = 0$$

The main forms of the existence conditions of periodic solutions of the system (1) are obtained as a result of an analysis of the main terms in the expansion of the periodicity conditions in powers of the small quantities v.

Besides the condition of commensurability of frequencies $n^{(o)},\Omega$, common here is the condition of H. Poincare [1]

$$\frac{\partial \overline{H}_1}{\partial w}(a,w,\alpha_o,\beta_o) = 0$$

(3)

to which one of the following four additional conditions must be added:

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1.1 $\quad \dfrac{\partial n^{(o)}}{\partial a} \neq 0$ $\qquad\qquad \dfrac{\partial^2 \overline{H}_1}{\partial w^2} \neq 0$

1.2 $\quad \dfrac{\partial n^{(o)}}{\partial \alpha_o} \neq 0$ $\qquad\qquad \dfrac{\partial^2 \overline{H}_1}{\partial w^2} \neq 0$

1.3 $\quad \dfrac{\partial n^{(o)}}{\partial \alpha_o} \dfrac{\partial^2 \overline{H}_1}{\partial a\, \partial w} - \dfrac{\partial n^{(o)}}{\partial a} \dfrac{\partial^2 \overline{H}_1}{\partial \alpha_o\, \partial w} \neq 0$

1.4 $\quad \dfrac{\partial n^{(o)}}{\partial \alpha_o} \dfrac{\partial^2 \overline{H}_1}{\partial \beta_o\, \partial w} - \dfrac{\partial n^{(o)}}{\partial \beta_o} \dfrac{\partial^2 \overline{H}_1}{\partial \alpha_o\, \partial w} \neq 0$

In the conditions 1.2 and 1.3 the parameter α_o can be replaced by β_o.

Note also that for the periodic solutions determined by the conditions (2),(3) and 1.2 - 1.4 the parameters A, B are constructed as series in integer powers of the small parameter μ.

Let the Hamiltonian system (1) be such that the function does not contain resonant terms, i.e. $\overline{H}_1 = \overline{H}_1(w)$. This situation takes place in many problems of Celestial Mechanics, e.g., in high-order resonances. In this case, as a result of deeper analysis of periodicity conditions, new conditions were obtained for the existence of periodic solutions, generated from the solutions (2). For these groups the common condition is

$$\frac{\partial \overline{K}}{\partial w}(a,w,\alpha_o,\beta_o) = 0,$$

which must be supplemented by one of the following four conditions:

2.1 $\quad \dfrac{\partial n^{(o)}}{\partial a} \neq 0$ $\qquad\qquad \dfrac{\partial^2 \overline{K}}{\partial w^2} \neq 0,$

2.2 $\quad \dfrac{\partial n^{(o)}}{\partial \alpha_o} \neq 0$ $\qquad\qquad \dfrac{\partial^2 \overline{K}}{\partial w^2} \neq 0,$

2.3 $\quad \dfrac{\partial n^{(o)}}{\partial \alpha_o} \dfrac{\partial^2 \overline{K}}{\partial a\, \partial w} - \dfrac{\partial n^{(o)}}{\partial a} \dfrac{\partial^2 \overline{K}}{\partial \alpha_o\, \partial w} \neq 0$

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2.4
$$\frac{\partial n^{(o)}}{\partial \alpha_o} \frac{\partial^2 \overline{K}}{\partial \beta_o \partial w} - \frac{\partial n^{(o)}}{\partial \beta_o} \frac{\partial^2 \overline{K}}{\partial \alpha_o \partial w} \neq 0,$$

where
$$\overline{K} = \frac{\overline{\partial^2 \tilde{H}_1}}{\partial a \partial w} \tilde{I}_1 + \frac{\overline{\partial^2 \tilde{H}_1}}{\partial w^2} \tilde{\psi}_1 + \frac{\partial \overline{H}_2}{\partial w} ,$$

and $\mu\tilde{I}_1$, $\mu\tilde{\psi}_1$ are purely periodic components of the first-order perturbations in the sought solution.

Of interest is a more general case of degeneracy. When resonant terms appear only in construction of perturbation of a certain arbitrary order p. Using the method of canonical transformations /e.g., the method of Deprit-Hori [3,4,5]/, the initial differential equations (1) in a sufficiently general case can be reduced to a canonical system with a Hamiltonian

$$H = H_o(I,A,B) + \sum_{\sigma=x}^{P-1} \mu^\sigma H_\sigma(I,A,B) + \sum_{\sigma=p}^{\infty} \mu^\sigma H_\sigma(I,\psi,A,B,\Omega t)$$

(4)

Periodic solutions of the transformed equations with the Hamiltonian (4) are generated from the solutions (2) at small values of μ if, besides the conditions

$$q_o n^{(o)}(a,\alpha_o,\beta_o) = q\Omega , \quad \frac{\partial \overline{H}_p}{\partial w}(a,w,\alpha_o,\beta_o) = 0$$

(5)

anyone of the following four conditions is fulfilled:

3.1
$$\frac{\partial n^{(o)}}{\partial a} \neq 0, \qquad \frac{\partial^2 \overline{H}_p}{\partial w^2} \neq 0,$$

3.2
$$\frac{\partial n^{(o)}}{\partial \alpha_o} \neq 0, \qquad \frac{\partial^2 \overline{H}_p}{\partial w^2} \neq 0,$$

3.3
$$\frac{\partial n^{(o)}}{\partial \alpha_o} \frac{\partial^2 \overline{H}_p}{\partial a \partial w} - \frac{\partial n^{(o)}}{\partial a} \frac{\partial^2 \overline{H}_p}{\partial \alpha_o \partial w} \neq 0,$$

3.4
$$\frac{\partial n^{(o)}}{\partial \alpha_o} \frac{\partial^2 \overline{H}_p}{\partial \beta_o \partial w} \frac{\partial n^{(o)}}{\partial \beta_o} \frac{\partial^2 \overline{H}_p}{\partial \alpha_o \partial w} \neq 0$$

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2.   Periodic solutions determined by the conditions 1.1-1.4, 2.1 - 2.4 and 3.1 - 3.4 are represented by series in integer powers of the small parameter μ converging by its small value.

As a result of a deeper analysis of the periodicity conditions, periodic solutions of the equations (1) were studied; these solutions can be represented as series in fractional powers of the small parameter μ.

The conditions of existence of periodic solutions which were obtained as a result of an analysis of two and three multiple roots of the corresponding periodicity conditions, can be written down as an ensemble of conditions $\theta_i^{(z)}$ and $\phi_j^{(z,u)}$:

$$(\ \theta_i^{(z)} \ , \ \phi_j^{(z,u)}) \tag{6}$$

$(z = (a, \alpha_o, \beta_o), \ u = (a, w, \alpha_o, \beta_o), \ z \neq u; \ i, j = 1, 2, \ldots, 15)$

The conditions $\theta_i^{(z)}$, in their turn, are given in Table 1, and the conditions $\phi_j^{(z,u)}$ are obtained from the conditions by a formal substitution of ψ_i to $F_i^{(z,j)}$, $i = 0,1,2,3,4$, of arguments z to u, and of the quantities, in particular, $z_1^{(i)}$ to $u_1^{(i)}$. Here $z_1^{(i)} = (\delta_1^{(i)}, \alpha_1^{(i)}, \beta_1^{(i)})$ and $u_1^{(i)} = (\delta_1^{(i)}, \gamma_1^{(i)}, \alpha_1^{(i)}, \beta_1^{(i)})$. The integer exponents $k^{(z)}$, characterizing the multiplicity of the corresponding roots of the periodicity conditions, are replaced by $k^{(u)}$. Some formulae, allowing to describe in detail the structure of the existence conditions (6) are given in the Appendix.

If the conditions (6) are fulfilled, then the differential equations (1) admit periodic solutions which can be represented by series in, generally speaking, fractional powers of the parameter μ:

$$I^{(z,u;i,j)} = a + \sum_{\zeta=1}^{\infty} \mu^{\frac{\zeta}{k_i^{(z)} k_j^{(u)}}} I_\zeta^{(z,u;i,j)}(t)$$

$$\psi^{(z,u;i,j)} = n^{(o)}(a)t + w + \sum_{\zeta=1}^{\infty} \mu^{\frac{\zeta}{k_i^{(z)} k_j^{(u)}}} \psi_\zeta^{(z,u;i,j)},$$

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Table 1: The conditions $\Theta_j^{(z)}$ (j = 1, 2, ..., 15)

$\Theta_1^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} \neq 0$	$k_1^{(z)} = 1$
$\Theta_2^{(z)}, \Theta_3^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} \neq 0$, $\psi_1 \neq 0$	$k_2^{(z)} = k_3^{(z)} = 2$
$\Theta_4^{(z)}, \Theta_5^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} \neq 0$, $\psi_1 = 0$	$k_4^{(z)} = k_5^{(z)} = 1$
$\Theta_6^{(z)}, \Theta_7^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} \neq 0$, $\psi_1 = 0$	$k_6^{(z)} = k_7^{(z)} = 2$
$\Theta_8^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} = 0$, $\dfrac{\partial^3 \psi_0}{\partial z^3} \neq 0$, $\psi \neq 0$	$k_8^{(z)} = 3$
$\Theta_9^{(z)}, \Theta_{10}^{(z)}, \Theta_{11}^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} = 0$, $\dfrac{\partial^3 \psi_0}{\partial z^3} \neq 0$,	$k_9^{(z)} = k_{10}^{(z)} = 2$
	$\psi_1 = 0$, $\dfrac{\partial \psi_1}{\partial z} \neq 0$,	$k_{11}^{(z)} = 1$
$\Theta_{12}^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} = 0$, $\dfrac{\partial^3 \psi_0}{\partial z^3} \neq 0$,	
	$\psi_1 = 0$, $\dfrac{\partial \psi_1}{\partial z} = 0$, $\psi_2 \neq 0$,	$k_{12}^{(z)} = 3$
$\Theta_{13}^{(z)}, \Theta_{14}^{(z)}, \Theta_{15}^{(z)}$	$\dfrac{\partial \psi_0}{\partial z} = 0$, $\dfrac{\partial^2 \psi_0}{\partial z^2} = 0$, $\dfrac{\partial^3 \psi_0}{\partial z^3} \neq 0$,	
	$\psi_1 = 0$, $\dfrac{\partial \psi_1}{\partial \alpha} = 0$, $\psi_2 = 0$,	$k_{13}^{(z)} = k_{14}^{(z)} =$
		$k_{15}^{(z)} = 1$

where $I_\zeta^{(\ldots)}$, $\psi_\zeta^{(\ldots)}$ are periodic functions of time, determined sequentially by means of simple quadratures.

A list of particular cases of existence of periodic solutions of the Hamiltonian systems (1) can be extended as much as one may like. Derivation of new conditions of existence must base on a detailed analysis of the periodicity conditions and of the conditions of their resolvability. Thereby, as we did earlier, the idea of selection of the problem parameters in form of corresponding series in integer or fractional powers of μ can be efficiently used.

3. The above-mentioned method of investigation of periodic solutions allows to study the existence of periodic solutions in vicinities of fixed points. Here it is advisable to use normal forms of Hamiltonians [5]:

$$H = H_o(I,A,B) + \mu^2 H_2(I,A,B) +\ldots+ \mu^{2n-2} H_{2n}(I,A,B) +$$

$$+ \mu^{2n-1} H_{2n+1}(I,\psi,A,B,\Omega t) + O(\mu^{2n}) \tag{7}$$

for the resonances $(2n+1)n^{(o)} = m\Omega$,

$$H = H_o(I,A,B) + \mu^2 H_2(I,A,B) +\ldots+ \mu^{2n-2} H_{2n}(I,A,B) +$$

$$+ \mu^{2n} H_{2n+2}(I,\psi,A,B,\Omega t) + O(\mu^{2n+1}) \tag{8}$$

for the resonances $2(n+1)n^{(o)} = m\Omega$ (n,m are integer numbers).

For the function $\overline{H}$ we have the following expressions, depending on the type of resonance

$$\overline{H}_{2n+1} = h_{2n+1}(\alpha_o,\beta_o)a^{n+\frac{1}{2}} \cos(2n+1)w,$$

$$\text{if} \quad (2n+1)n^{(o)} = m\Omega,$$

$$\overline{H}_{2(n+1)} = \alpha^{n+1}[g_{2(n+1)}(\alpha_o,\beta_o) + \tau_{2(n+1)}(\alpha_o,\beta_o)\cos 2(n+1)w],$$

$$\text{if} \quad 2(n+1)n^{(o)} = m\Omega,$$

where $h_{2n+1}, g_{2(n+1)}, \tau_{2(n+1)}$ are constant coefficients of the normal forms. The parameter μ is introduced into (7), (8) artificially, by a simple substitution of variables.

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The structure of the Hamiltonians (7),(8) corresponds to (4); therefore, for an investigation of periodic solutions here the conditions (5), 3.1 - 3.4 are valid. By means of these conditions we obtain the following four groups of conditions of existence of periodic solutions of canonical equations with the Hamiltonians (7),(8):

4.1 $\dfrac{\partial n^{(o)}}{\partial \alpha_o} \dfrac{\partial h_{2n+1}}{\partial \beta_o} - \dfrac{\partial n^{(o)}}{\partial \beta_o} \dfrac{\partial h_{2n+1}}{\partial \alpha_o} \neq 0$

   $\alpha \neq 0$, $(2n+1)w \neq \pi k$, $(k = 0,1,2,\ldots)$

   $h_{2n+1}(\alpha_o, \beta_o) = 0$, $(2n+1)n^{(o)}(\alpha_o, \beta_o) = m$

4.2 $\dfrac{\partial n^{(o)}}{\partial \alpha_o} \dfrac{\partial \tau_{2(n+1)}}{\partial \beta_o} - \dfrac{\partial n^{(o)}}{\partial \beta_o} \dfrac{\partial \tau_{2(n+1)}}{\partial \alpha_o} \neq 0$

   $\alpha \neq 0$, $2(n+1)w \neq \pi k$

   $\tau_{2(n+1)}(\alpha_o, \beta_o) = 0$, $2(n+1)n^{(o)}(\alpha_o, \beta_o) = m$

4.3 $\dfrac{\partial n^{(o)}}{\partial \alpha_o} \neq 0$, $\neq 0$, $(2n+1)w = k\pi$

   $h_{2n+1}(\alpha_o, \beta_o) \neq 0$, $(2n+1)n^{(o)}(\alpha_o, \beta_o) = m$

4.4 $\dfrac{\partial n^{(o)}}{\partial \alpha_o} \neq 0$, $\neq 0$, $(2n+1)w = k\pi$

   $\tau_{2(n+1)}(\alpha_o, \beta_o) \neq 0$, $2(n+1)n^{(o)}(\alpha_o, \beta_o) = m$

4.   The conditions 4.1 and 4.2 have allowed to establish rather readily the existence of plane periodic motions of a nonspherical satellite in an elliptic orbit/in the vicinity of a circular synchronous motion/in a central gravity field. Periodic solutions were found in the cases of commensurability of the main frequence $n^{(o)}$ and of the mean orbital motion Ω of the form $n^{(o)} = \dfrac{\Omega}{3}$, $n^{(o)} = \dfrac{2}{3}\Omega$ and $n^{(o)} = \dfrac{1}{4}\Omega$. To these solutions correspond the following generating values of the eccentricity of the satellite's orbit $e = e_o$ and of the dynamic compression $\alpha = \alpha_o$:

$$e_o = 0, \; \alpha_o = \frac{1}{9} \; ; \; e_o = 0, \; \alpha_o = \frac{4}{9} \; ; \; e_o = 0, \; \alpha_o = \frac{1}{16}$$

To these values of the parameters, correspond nonsymmetric periodic solutions.

The existence of symmetric periodic solutions, determined by the conditions 4.3, 4.4 in dependence of the type of resonance was also established from the coefficients of the normal form of the Hamiltonian of the problem; these Hamiltonians were constructed in [6].

5.   The results obtained in §§1-3 are valid for an investigation of periodic solutions of Schwarzschild type of two-degree autonomous Hamiltonian systems. For this purpose, it is sufficient to use their reduction to a system of canonical equations with one and a half degrees of freedom, i.e. to the equations of the form (1).

The same approach allowed to obtain constructive conditions of existence of periodic solutions in the vicinities of stationary solutions of two-degree autonomous Hamiltonian systems, in particular, for the resonances 0:1, 1:1, 1:3, 1:2, etc. The conditions of existence of these periodic solutions were written down by means of expressions for frequencies and coefficients of normal forms of the Hamiltonians.

The results obtained allowed to study periodic solutions in the problem of motion of a point satellite in the equatorial plane of a rotating nonspherical planet. To these solutions correspond satellite's periodic motions in the vicinity of the planet's libration points. Planet's dynamic compressions α, β were used as active/varied/ parameters of the problem. As a result, symmetric period solutions were found in the vicinity of resonant curves f $m/n(\alpha_o, \beta_o) = 0$, corresponding to the commensurabilitys of the frequencies of the problem M:n = 1:2, 1:3, as well as nonsymmetric periodic solutions in the vicinities of some particular points of these resonant curves.

By means of the above-mentioned conditions, was proved also the existence of periodic solutions in the restricted circular problem of three point bodies. To this solution correspond periodic motions of a point with a negligible mass in the vicinity of the triangular libration point by the commensurability of frequencies. Earlier, these motions were studied by means of numerical and analytical methods in the works of Henrard, Schmidt, et al. [7].

6.   The restricted space of this report does not permit to consider in detail similar problems of investigation of periodic solutions of multidimensional Hamiltonian systems. Among the works of this research branch, devoted to a study of

existence, construction, and stability of periodic solutions
of Hamiltonian systems in particular and degenerate cases
/including the studies with an active use of the problem par-
ameters/ are the works [2],[9],[10] etc.

These same works in fact indicate pathways and approac-
hes to a more complete and thorough investigation of the pe-
riodicity conditions and of the periodic solutions themselves
in a large variety of degenerate cases, in particular, to a
study of periodic solutions of autonomous and nonautonomous
Hamiltonian systems of the second, third and arbitrary orders
in the vicinities of stationary and periodic reference solu-
tions. It seems thereby possible to use efficiently the res-
ults of construction of the Hamiltonian's normal forms both
for general-form sy stems and for specific problems of Cele-
stial Mechanics, stated in [5],[11], etc.

7.   By means of the H. Poincaré's classical conditions and of
the above-mentioned constructive elaboration, a systematic
study was done of periodic solutions in the plane and spatial
problems of the motion of two mutually gravitating bodies,
which either possess quasiconcentric density distributions
or are represented by homogeneous ellipsoids with small com-
pressions. Here new families of symmetric and nonsymmetric
periodic solutions were found both in general and in degene-
rate cases, with an application of the procedure of the prob-
lem parameters' fitting, in the nonrestricted as well as in
the restricted statements of the problem.

For these solutions, the motions of the centres of mass-
es are close to the Eulerian ones of the bodies with spheri-
cal ellipsoids of inertia. In a few orbital revolutions, each
body commits an integer number of revolutions around its own
centre of masses.[10].

The existence was studied of periodic solutions in the
planetary versions of the problems of three and n rigid bodies
with small dimensions and with the ellipsoids of inertia close
to spheres. These solutions generalize the H. Poincaré's clas-
sical periodic solution in the problem of three point bodies
to a planetary sys tem of nonspherical rigid bodies [10][12].

An application is presented of the found periodic solu-
tions to a study of resonant phenomena in the motions of the
Moon, Mercury, and Venus. It is shown, in particular, that
the observed in the Venus' motion resonance is well described
by one of the found first-kind periodic solutions in a nonre-
stricted three-body problem /Venus being a nonspherical rig-
id body, and the Sun and the Earth - point masses/, with the
necessary stability conditions fulfilled for this solution
[10].

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APPENDIX

Expansion for Z and formulae for $z_1^{(j)}$, $z_2^{(j)}$.

$$z = z_1^{(j)} \; \mu^{\frac{1}{k_j(z)}} + z_2^{(j)} \; \mu^{\frac{2}{k_j(z)}} + \ldots,$$

$j = 1; \; k_1(z) = 1$

$$z_1^{(j)} = - \; \psi_1 \left(\frac{\partial \psi_o}{\partial z} \right)^{-1} , \quad z_2^{(j)} = -\left(\frac{1}{2} \frac{\partial^2 \psi_o}{\partial z^2} z_1^{(j)2} + \frac{\partial \psi_1}{\partial z} z_1^{(j)} + \psi_2 \right)$$
$$\times \left(\frac{\partial \psi_o}{\partial z} \right)^{-1}$$

$j = 2,3; \; k_2^{(z)} = k_3^{(z)} = 1$

$$z_1^{(j)} = \pm \sqrt{-2\psi_1 \left(\frac{\partial^2 \psi_o}{\partial z^2} \right)^{-1}} , \quad z_2^{(j)} = -\left(\frac{\partial \psi_1}{\partial z} + \frac{1}{6} \frac{\partial^3 \psi_o}{\partial z^3} \right)$$
$$\times \left(\frac{\partial^2 \psi_o}{\partial z^2} \right)^{-1}$$

$j = 4,5; \; k_4^{(z)} = k_5^{(z)} = 1$

$$P_2(z_1^{(j)}) = \frac{1}{2} \frac{\partial^2 \psi_o}{\partial z^2} z_1^{(j)2} + \frac{\partial \psi_1}{\partial z} z_1^{(j)} + \psi_2 = 0,$$

$$z_2^{(j)} = -P_3 \left(\frac{\partial P_2}{\partial z_1^{(j)}} \right)^{-1} , \quad j = 6,7; \; k_6^{(z)} = k_7^{(\dot{z})} = 2$$

$$P_2(z_1^{(j)}) = 0, \quad \frac{\partial P_2}{\partial z_1^{(j)}} = \frac{\partial \psi_1}{\partial z} + \frac{\partial^2 \psi_o}{\partial z^2} z_1^{(j)} = 0, \quad z_2^{(j)} = 0,$$

$$z_3^{(j)} = - \; P_3 \left(\frac{1}{2} \frac{\partial^2 \psi_o}{\partial z^2} \right)^{-1}$$

$j = 8, \; k_8^{(z)} = 3$

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$$V_3(z_1^{(j)}) = \frac{1}{6} \frac{\partial^3 \psi_o}{\partial z^3} z_1^{(j)3} + \psi_1 = 0, \quad z_2^{(j)} = -V_4 \left(\frac{\partial V_3}{\partial z_1^{(j)}} \right)^{-1}$$

$$j = 9,10; \quad k_9^{(z)} = k_{10}^{(z)} = 2$$

$$u_3(z_1^{(j)}) = z_1^{(j)} \left(\frac{1}{6} \frac{\partial^3 \psi_o}{\partial z^3} z_1^{(j)2} + \frac{\partial \psi_1}{\partial z} \right) = 0,$$

$$z_2^{(j)} = -u_4 \left(\frac{\partial u_3}{\partial z_1^{(j)}} \right)^{-1}$$

$$j = 11; \quad k_{11}^{(z)} = 1$$

$$z_1^{(j)} = -\psi_2 \left(\frac{\partial \psi_1}{\partial z} \right)^{-1} , \quad z_2^{(j)} = -P_3 \left(\frac{\partial \psi_1}{\partial z} \right)^{-1}$$

$$j = 12; \quad k_{12}^{(z)} = 3$$

$$z_1^{(j)} = 0, \quad z_2^{(j)} = \left[-\psi_2 \left(\frac{1}{6} \frac{\partial^3 \psi_o}{\partial z^3} \right)^{-1} \right]^{1/3}$$

$$j = 13,14,15; \quad k_{13}^{(z)} = k_{14}^{(z)} = k_{15}^{(z)} = 1$$

$$P_3(z_1^{(j)}) = \frac{1}{6} \frac{\partial^3 \psi_o}{\partial z^3} z_1^{(j)3} + \frac{1}{2} \frac{\partial^2 \psi_1}{\partial z^2} z_1^{(j)2} + \frac{\partial \psi_2}{\partial z} z_1^{(j)} + \psi_3 = 0,$$

$$z_2^{(j)} = -P_4 \left(\frac{\partial P_3}{\partial z_1^{(j)}} \right)^{-1}$$

Polynomials $P_4(z_1^{(j)})$, $V_4(z_1^{(j)})$, $u_4(z_1^{(j)})$

$$P_4(z_1^{(j)}) = \frac{1}{4!} \frac{\partial^4 \psi_o}{\partial z^2} z_1^{(j)4} + \frac{1}{3!} \frac{\partial^3 \psi_1}{\partial z^3} z_1^{(j)3} + \frac{1}{2!} \frac{\partial^2 \psi_2}{\partial z^2} z_1^{(j)2}$$

$$+ \frac{\partial \psi_3}{\partial z} z_1^{(j)} + \psi_4$$

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$$V_4(z_1^{(j)}) = \frac{1}{4!}\frac{\partial^4 \psi_o}{\partial z^4} z_1^{(j)4} + \frac{\partial \psi_1}{\partial z} z_1^{(j)} + \psi_2$$

$$u_4(z_1^{(j)}) = \frac{1}{4!}\frac{\partial^4 \psi_o}{\partial z^4} z_1^{(j)4} + \frac{1}{2!}\frac{\partial^2 \psi_1}{\partial z^2} z_1^{(j)2} + \psi_2$$

2. Expressions for the functions $F_i^{(z,j)}$, $i = 0,1,2$.

For $j = 1,4,5,11,13,14,15$:

$$F_o^{(z,j)} = I_1, \qquad F_1^{(z,j)} = \frac{\partial I_1}{\partial z} z_1^{(j)} + I_2 ,$$

$$F_2^{(z,j)} = \frac{\partial I}{\partial z} z_2^{(j)} + \frac{1}{2}\frac{\partial^2 I_1}{\partial z^2} z_1^{(j)2} + \frac{\partial I_2}{\partial z} z_1^{(j)} + I_3 .$$

For $j = 2,3,6,7,9,10$:

$$F_o^{(z,j)} = I_1, \qquad F_1^{(z,j)} = \frac{\partial I_1}{\partial z} z_1^{(j)} ,$$

$$F_2^{(z,j)} = \frac{\partial I_1}{\partial z} z_2^{(j)} + \frac{1}{2}\frac{\partial^2 I_1}{\partial z^2} z_1^{(j)2} + I_2 .$$

For $j = 8, 12$:

$$F_o^{(z,j)} = I_1 , \qquad F_1^{(z,j)} = \frac{\partial I_1}{\partial z} z_1^{(j)} ,$$

$$F_2^{(z,j)} = \frac{\partial I_1}{\partial z} z_2^{(j)} + \frac{1}{2}\frac{\partial^2 I_1}{\partial z^2} z_1^{(j)2} .$$

3. Expressions for ψ_i, I_i $(i = 0,1,2)$

$$\psi_o = -T\left(\frac{\partial H_o}{\partial a} + \frac{q}{q_o}\Omega\right), \qquad I_1 = T\frac{\partial \overline{H}_1}{\partial w} ,$$

$$\psi_1 = -T\frac{\partial^2 H_o}{\partial a^2}\left(\frac{\partial \overline{H}_1}{\partial w} - \int \frac{\partial \overline{H}_1}{\partial w} dt\Big|_o\right) - \frac{\partial \overline{H}_1}{\partial a} T ,$$

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$$\psi_2 = - T\left(\frac{\partial^2 H_o}{\partial \alpha^2} \overline{I_2(t)} + \frac{1}{2} \frac{\partial^3 H_o}{\partial \alpha^3} \overline{(I_1(t))^2} \right.$$

$$\left. + \frac{\partial^2 H_1}{\partial \alpha^2} I_1 + \frac{\overline{\partial^2 H_1}}{\partial \alpha \partial w} \psi_1 + \frac{\partial \overline{H}_2}{\partial \alpha} \right),$$

$$I_2 = T\left(\frac{\overline{\partial^2 H_1}}{\partial \alpha \partial w} I_1 + \frac{\overline{\partial^2 H_1}}{\partial w^2} \psi_1 + \frac{\partial \overline{H}_2}{\partial w} \right)$$

REFERENCES

1   Poincare, H. New Methods in Celestial Mechanics. Selec-
     ted Works. Nauka, Moscow, 1971, Vol. 1, p.7-326.
2   Barkin, Yu.V., Pankratov, A.A., Periodic solutions of
     Hamiltonian s ystems and their application to satelli-
     tes ' dynamics. Kosmicheskie issledovaniya. - 1986,
     Vol.24, No.3, p.345-357.
3   Deprit, A, Canonical transformations depending on a
     small parameter - Celestial Mechanics, 1969, Vol.1,No.1.
4   Hori, G.I., Theory of general perturbations with unspe-
     cified canonical variables, Publ. Astron. Soc.Japan,
     1966, Vol. 18, No. 4, p.287-296.
5   Markeev, A.P., Libration points in Celestial Mechanics
     and co smodynamics, Nauka, Moscow, 1978, pp.312.
6   Zlatoustov, V.A., Markeev, A.P., Stability of planar
     oscillations of a satellite in an elliptic orbit,
     Celestial Mechanics, 1973, Vol.7, p.31-45.
7   Schmidt, D.S., Periodic solutions near a resonant equil-
     brium of a Hamiltonian system, Celestial Mechanics,1974,
     Vol. 9, p.81- 103.
8   Barkin, Yu. V., Constructive methods of investigation
     of periodic solutions of multifrequency oscillatory
     systems and their applications to mechanics, All Union
     Conference on Nonlinear Oscillations of Mechanical Sys-
     tems, Abstracts of papers, Gorky, 1987.
9   Barkin, Yu. V., Pankratov, A.A., Periodic solutions of
     Hamiltonian systems in some cases of degeneracy,
     Prikladnaya matematika i mekhanika, 1987, Vol. 51,No.2,
     p. 235-243.
10   Barkin, Yu, V., Dynamics of a system of nonspherical
     Celestial bodies and the theory of the Moon's rotation.
     Doctoral thesis. Moscow State University, Sternberg
     Astronomical Institute, 1989, 412 pp.
11   Markeev, A.P., Medvedev, S.V., Sokol'skij, A.G., Methods
     and algorithms for normalization of differential equa-
     tions, Moscow Aviation Institute, 19   , 74 pp.
12   Barkin, Yu, V., Resonant motions of rigid bodies in the
     solar system, Dynamics of the Solar System/Edited by
     M.Sidlichovsky, Proceedings of the Tenth European Regi-
     nal Astronomy Meeting of the IAU/Prague, 24-29 August,
     1987/, Prague, 1987, Vol. 3, p.63-69.