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GIRTH AND INDEPENDENCE RATIO

BY

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ABSTRACT. Lower bounds are given for the independence ratio in graphs satisfying certain girth and maximum degree requirements. In particular, the independence ratio of a graph with maximum degree Δ and girth at least six is at least $(2\Delta-1)/(\Delta^2+2\Delta-1)$. Sharper bounds are given for cubic graphs.

Professor Erdös [3] has shown that there exist graphs with very large girth and yet very small independence ratio. Such graphs must of course have very large maximum degree, for if the maximum degree is bounded above by p, then the independence ratio is bounded below by $1/(p+1)$. As a consequence of Brooks' Theorem [2], the independence ratio is bounded below by $1/p$ if one assumes maximum degree p, no complete subgraphs on $p+1$ vertices, and $p \geq 3$. More recently, Albertson, Bollobas and Tucker [1], assuming no complete subgraphs on p vertices have improved the inequality yielded by Brooks' Theorem to a strict inequality, except in two cases, which are demonstrated. Specializing to the case $p=3$, Fajtlowicz [4] and Staton [6, 7] have determined constants larger than $\frac{1}{3}$ which serve as lower bounds for the independence ratio in cubic triangle-free graphs. Staton's constant, $\frac{5}{14}$, is shown to be best possible by an example of Fajtlowicz in which this ratio is achieved.

In [5], Fajtlowicz showed that graphs with maximum degree p containing no complete subgraphs with q vertices have independence ratio at least $2/(p+q)$. When $q=3$, this says that graphs with girth at least four have independence ratio at least $2/(p+3)$. Taking this result as our point of departure, we investigate independence ratio in graphs with fixed lower bounds on girth and fixed upper bounds on the maximum degree. We concentrate primarily on cubic graphs.

We will employ the ideas and notation introduced by Fajtlowicz in [4] and extended by him in [5]. In particular, if G is a finite graph, then α_0 will be the size of a maximum independent subset of G, n will be the number of vertices of G, and the ratio α_0/n will be called the independence ratio of G. If I is a maximum independent vertex set in G, and if p is the maximum degree of G, then for $1 \leq i \leq p$, $G_i(I)=G_i$ will be the set of all vertices of G which are

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adjacent to exactly i vertices of I. The cardinality of G_i will be denoted $\alpha_i=\alpha_i(I)$. Note that the G_i's are disjoint, so that $\sum_{i=1}^p \alpha_i=n-\alpha_0$. For $1\le i\le p$, Γ_i will be the set of all vertices in I adjacent to at least one vertex of G_i. Note that the Γ_i's need not be disjoint. If X and Y are disjoint sets of vertices, then $[X, Y]$ will denote the collection of edges with one end in X and one end in Y.

If G is cubic, then we have $\alpha_1+\alpha_2+\alpha_3=n-\alpha_0$, and counting the number of edges in $[I, G-I]$, we get $\alpha_1+2\alpha_2+3\alpha_3=3\alpha_0$. Eliminating α_2 from these two equations, and solving for the independence ratio, we get

$$\frac{\alpha_0}{n}=\frac{2}{5}-\frac{\alpha_1-\alpha_3}{5n},$$

an equation to which we will refer frequently. In [4], Fajtlowicz shows that if G contains no triangle, and if I is a maximum independent set, then each vertex in Γ_1 has exactly one neighbor in G_1, so that $|\Gamma_1|=\alpha_1$. This observation is easily proven, and we feel free to use it in what follows.

LEMMA 1. *If G has maximum degree p, then*

$$\frac{\alpha_0}{n}\ge\frac{2}{p+2}-\frac{\alpha_1}{n(p+2)}.$$

Proof. Counting vertices, we get $\alpha_1+\alpha_2+\cdots+\alpha_p=n-\alpha_0$, and counting those edges with an end in I, we get $\alpha_1+2\alpha_2+\cdots+p\alpha_p\le p\alpha_0$. Doubling the equation and subtracting from the inequality yields

$$(p+2)\alpha_0-2n\ge-\alpha_1+\alpha_3+2\alpha_4+\cdots+(p-2)\alpha_p.$$

It follows that

$$\frac{\alpha_0}{n}\ge\frac{2}{p+2}-\frac{\alpha_1}{n(p+2)}+\frac{\sum_{k=2}^p (k-2)\alpha_k}{n(p+2)},$$

the last summand of which is nonnegative. ∎

LEMMA 2. *If G has maximum degree p, no 3-cycle, and no 5-cycle, then, for $k\ge 2$, each vertex in G_k has at least one neighbor in $I-\Gamma_1$.*

Proof. Suppose that v is a vertex in G_k and that $x_1, x_2,\ldots,x_k$ are the neighbors of v which lie in I. If each x_i is in Γ_1, then there are vertices $y_1, y_2,\ldots,y_k$ in G_1 such that x_i is adjacent to y_i for $1\le i\le k$. Since G has no 3-cycle, no y_i is adjacent to v, and, since G has no 5-cycle, no two of the y_i's are adjacent. Since the only vertices in I adjacent to the y_i's are the x_i's, the set

$$(I-\{x_1, x_2,\ldots,x_k\})\cup\{v, y_1, y_2,\ldots,y_k\}$$

is an independent set larger than I. This contradiction shows that at least one of the neighbors of v in I must fail to be in Γ_1. ∎

THEOREM 3. *If G has maximum degree p, no 3-cycle, and no 5-cycle, then*

$$\frac{\alpha_0}{n} \geq \frac{2p-1}{p^2+2p-1}.$$

Proof. By Lemma 2, each vertex in $G_2 \cup G_3 \cup \cdots \cup G_p$ has at least one neighbor in $I-\Gamma_1$. Since each vertex in $I-\Gamma_1$ has degree at most p, it follows that

$$|I-\Gamma_1| \geq \frac{\alpha_2+\alpha_3+\cdots+\alpha_p}{p}.$$

But $|I-\Gamma_1| = \alpha_0-\alpha_1$, and $\alpha_2+\alpha_3+\cdots+\alpha_p = n-\alpha_0-\alpha_1$. Thus

$$\alpha_0-\alpha_1 \geq \frac{n-\alpha_0-\alpha_1}{p}, \quad \text{or} \quad \alpha_1 \leq \frac{p+1}{p-1}\alpha_0 - \frac{n}{p-1}.$$

Invoking Lemma 1, we get

$$\frac{\alpha_0}{n} \geq \frac{2}{p+2} - \frac{\frac{p+1}{p-1}\alpha_0 - \frac{n}{p-1}}{n(p+2)},$$

which implies

$$\frac{\alpha_0}{n} \geq \frac{2p-1}{p^2+2p-1}. \quad \blacksquare$$

We note that the ratio $(2p-1)/(p^2+2p-1)$ is larger than the $2/(p+3)$ obtained by Fajtlowicz for triangle free graphs. For $p=3$, Theorem 3 gives an independence ratio of at least $\frac{5}{14}$. We note that in [7], it was shown that the $\frac{5}{14}$ ratio may be gotten even under the milder assumption of no triangles. However, specializing to cubic graphs, we now obtain a bound larger than $\frac{5}{14}$.

LEMMA 4. *If G is cubic and contains no 3-cycle, and if I is a maximum independent set with the property that $\alpha_3(I)$ is maximum, then $|\Gamma_1-\Gamma_3| \leq |\Gamma_1 \cap \Gamma_3|$.*

Proof. Let v be a vertex in $\Gamma_1-\Gamma_3$, and let x be its neighbor in G_1. We claim that x has no neighbor in G_2. For if y is a neighbor of x and $y \in G_2$, then $J=(I-\{v\}) \cup \{x\}$ is a maximum independent set. But $G_3(J) = G_3(I) \cup \{y\}$, which contradicts the maximality of $\alpha_3(I)$. Thus x has no neighbor in G_2, so it has two neighbors in G_1, say y and z. Now y and z have neighbors y_1 and z_1 respectively in Γ_1. If both y_1 and z_1 were in $\Gamma_1-\Gamma_3$, then the set $K=(I-\{y_1, z_1\}) \cup \{y, z\}$ would be a maximum independent set with $G_3(K) = G_3(I) \cup \{x\}$, which would contradict the maximality of $\alpha_3(I)$. Thus, at least one of y_1 and z_1 is in $\Gamma_1 \cap \Gamma_3$. In this manner we may associate with the vertex v of $\Gamma_1-\Gamma_3$ a vertex of $\Gamma_1 \cap \Gamma_3$. Any such association will be one-to-one as may be verified readily. It follows that $|\Gamma_1-\Gamma_3| \leq |\Gamma_1 \cap \Gamma_3|$. $\blacksquare$

COROLLARY 5. *If G is cubic and contains no* 3*-cycle, and if I is a maximum independent set with* $\alpha_3(I)$ *maximum, then*

i) $\alpha_1 \le 6\alpha_3$; Fajtlowicz [4]

ii) *if G has no* 5*-cycle,* $\alpha_1 \le 4\alpha_3$.

Proof. i) Clearly $|\Gamma_1 \cap \Gamma_3| \le 3\alpha_3$, and $\alpha_1 = |\Gamma_1 - \Gamma_3| + |\Gamma_1 \cap \Gamma_3|$.

ii) By Lemma 2, each vertex in G_3 has at least one neighbor in $I - \Gamma_1$, and thus at most two neighbors in $\Gamma_1 \cap \Gamma_3$. Hence $|\Gamma_1 \cap \Gamma_3| \le 2\alpha_3$. ■

LEMMA 6. *If G is cubic and contains no* 3*-cycle and no* 5*-cycle, then* $\alpha_1 \le \frac{1}{2}\alpha_2 + \alpha_3$.

Proof. Note that Γ_1 may be split into three disjoint classes: A consisting of vertices with a G_1 neighbor and two G_2 neighbors; B consisting of vertices with a G_1 neighbor, a G_2 neighbor, and a G_3 neighbor; and C consisting of vertices with a G_1 neighbor and two G_3 neighbors. Thus $|A| + |B| + |C| = \alpha_1$. Consider the edge set $[\Gamma_1, G_2]$. The number of edges in $[\Gamma_1, G_2]$ is $2|A| + |B|$, and, by Lemma 2, this number is not bigger than α_2. Thus $2|A| + |B| \le \alpha_2$. The number of edges in $[\Gamma_1, G_3]$ is $|B| + 2|C|$, and, again by Lemma 2, this number is not bigger than $2\alpha_3$. So $|B| + 2|C| \le 2\alpha_3$. Adding the two inequalities yields $2|A| + 2|B| + 2|C| \le \alpha_2 + 2\alpha_3$, or $\alpha_1 \le \frac{1}{2}\alpha_2 + \alpha_3$. ■

THEOREM 7. *If G is cubic and contains no* 3*-cycle and no* 5*-cycle, then* $\alpha_0/n \ge \frac{19}{52}$.

Proof. Recall that $\alpha_0/n = \frac{2}{5} - (\alpha_1 - \alpha_3)/5n$. If I is a maximum independent set with $\alpha_3(I)$ maximum, we have $\alpha_1 \le 4\alpha_3$, and so $\alpha_0/n \ge \frac{2}{5} - 3\alpha_3/5n$. And, since $\alpha_1 \le \frac{1}{2}\alpha_2 + \alpha_3 = \frac{1}{2}(\alpha_2 + 2\alpha_3) = \frac{1}{2}(4\alpha_0 - n)$, we have

$$\frac{\alpha_0}{n} \ge \frac{2}{5} - \frac{\frac{1}{2}(4\alpha_0 - n) - \alpha_3}{5n}, \quad \text{or} \quad \frac{\alpha_0}{n} \ge \frac{5}{14} + \frac{\alpha_3}{7n}.$$

Consider two cases:

(i) if $\alpha_3/n \ge \frac{3}{52}$, then $\alpha_0/n \ge \frac{5}{14} + \frac{1}{7}(\frac{3}{52}) = \frac{19}{52}$; and

(ii) if $\alpha_3/n \le \frac{3}{52}$, then $\alpha_0/n \ge \frac{2}{5} - \frac{3}{5}(\frac{3}{52}) = \frac{19}{52}$.

Thus, in any case, $\alpha_0/n \ge \frac{19}{52}$.

THEOREM 8. *If G is cubic and contains no* 3*-cycle, no* 5*-cycle, and no* 7*-cycle, then* $\alpha_0/n \ge \frac{20}{53}$.

Proof. Let I be a maximum independent set with $\alpha_3(I)$ maximal. We partition I into seven subsets as follows:

A: vertices with two neighbors in G_2, and one in G_1;

B: vertices with one neighbor in G_1, one in G_2, one in G_3;

C: vertices with one neighbor in G_1, two in G_3;

D: vertices with three neighbors in G_2;

E: vertices with two neighbors in G_2, one in G_3;
F: vertices with two neighbors in G_3, one in G_2;
H: vertices with three neighbors in G_3.

We let a, b, c, d, e, f, and h be the respective cardinalities of these sets. Thus,

(1) $a+b+c+d+e+f+h=\alpha_0$
(2) $a+b+c=\alpha_1$
(3) $2a+b+3d+2e+f=2\alpha_2$
(4) $b+2c+e+2f+3h=3\alpha_3$.

Now, let v be a vertex in D. All three of the neighbors of v are in G_2; call them x, y, and z. We claim that not more than one of these may have a neighbor in Γ_1. For, suppose that x has a neighbor x_1 in Γ_1, and that y has a neighbor y_1 in Γ_1. Let x_2 and y_2 be the respective G_1 neighbors of x_1 and y_1. See Fig. 1.

The set $J=(I-\{v, x_1, y_1\})\cup\{x, x_2, y, y_2\}$ is an independent set larger than I, which is a contradiction. Therefore, each vertex in D has at least two G_2 neighbors with no Γ_1 neighbors. A similar argument shows that each vertex in E has at least one G_2 neighbor with no Γ_1 neighbor. Consider the edge set $[\Gamma_1, G_2]$, which has exactly $2a+b$ edges. From Lemma 2, we know that each vertex in G_2 is incident with at most one edge in $[\Gamma_1, G_2]$. But some vertices in G_2 contribute no edges at all to $[\Gamma_1, G_2]$. In particular, there are at least $\frac{1}{2}(2d+e)$ vertices in G_2 which do not contribute to $[\Gamma_1, G_2]$. Thus $2a+b\le\alpha_2-\frac{1}{2}(2d+e)$, or $4a+2b+2d+e\le 2\alpha_2$, and, invoking equation (3),

$$4a+2b+2d+e\le 2a+b+3d+2e+f$$

or $2a+b\le d+e+f$. But note that $\alpha_0-\alpha_1=d+e+f+h$. Hence $2a+b\le\alpha_0-\alpha_1$. From the proof of Lemma 6, we have $b+2c\le 2\alpha_3$. Hence, adding, we obtain

$$2a+2b+2c\le\alpha_0-\alpha_1+2\alpha_3,$$

or

$$2\alpha_1\le\alpha_0-\alpha_1+2\alpha_3,$$

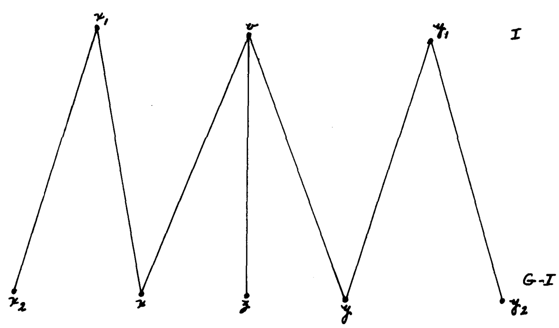

Figure 1

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or
$$\alpha_1 \le \tfrac{1}{3}\alpha_0 + \tfrac{2}{3}\alpha_3.$$

Recalling that $\alpha_0/n = \frac{2}{5} - (\alpha_1 - \alpha_3)/5n$, we get

$$\frac{\alpha_0}{n} \ge \frac{2}{5} - \frac{\frac{1}{3}\alpha_0 - \frac{1}{3}\alpha_3}{5n},$$

or

$$\frac{16}{15}\frac{\alpha_0}{n} \ge \frac{2}{5} + \frac{\alpha_3}{15n},$$

or

(5) $$\frac{\alpha_0}{n} \ge \frac{3}{8} + \frac{\alpha_3}{16n}.$$

Recall that in the proof of Theorem 7, we obtained

(6) $$\frac{\alpha_0}{n} \ge \frac{2}{5} - \frac{3\alpha_3}{5n}.$$

Consider two cases:

(i) if $\alpha_3/n \ge \frac{2}{53}$, we use (5) to get $\alpha_0/n \ge \frac{3}{8} + \frac{1}{16}(\frac{2}{53}) = \frac{160}{424} = \frac{20}{53}$;

(ii) if $\alpha_3/n \le \frac{2}{53}$, we use (6) to get $\alpha_0/n \ge \frac{2}{5} - \frac{3}{5}(\frac{2}{53}) = \frac{100}{265} = \frac{20}{53}$.

Thus, in any case, $\alpha_0/n \ge \frac{20}{53}$. ■

We turn our attention at this point to cubic graphs with large girth. If u_j is the greatest lower bound for independence ratios of cubic graphs with girth at least j, then one may ask for $\lim_{j\to\infty} u_j$. We will show this limit is at least $\frac{7}{18}$.

LEMMA 9. *If G is cubic and has girth bigger than $4k+1$, then*

$$\alpha_1 \le \frac{\alpha_2}{k} + f + c - a.$$

Proof. Let l be an edge in $[\Gamma_1, G_2]$. Let v be the end vertex of l in Γ_1, and let w_1 be the end vertex of l in G_2. Let u be the neighbor of v in G_1. Now let $y_1 \ne v$ be the other neighbor of w_1 in I. By Lemma 2, y_1 is not in Γ_1, so y_1 is in $D \cup E \cup F$. So either y_1 has another neighbor $w_2 \ne w_1$ in G_2, or y_1 is in F. If y_1 is in F, we stop. If y_1 is not in F, we consider w_2 and its neighbor $y_2 \ne y_1$ in I. Imitating the proof of Lemma 2, one may show that y_2 is not in Γ_1. So again, either y_2 is in F, or y_2 has a neighbor $w_3 \ne w_2$ in G_2. Continuing in this manner, we obtain, after at most k steps, either a vertex z in F or a sequence w_1, $w_2, \ldots, w_k$ of vertices in G_2. Hence, to each of the $2a+b$ edges in $[\Gamma_1, G_2]$, we may assign either a vertex z in F or a sequence $w_1, \ldots, w_k$ of distinct vertices in G_2.

Suppose that, beginning with l in $[\Gamma_1, G_2]$ we obtain $v, w_1, w_2, \ldots, w_i, z$, with $z \in F$, each w in G_2. And, beginning with another edge l' in $[\Gamma_1, G_2]$

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suppose that we obtain v', w_1', w_2', $\ldots$, w_j', z', with z' in F, and each w' in G_2. If $z=z'$, then $w_i=w_j'$, since z and z' have only one edge into G_2. Hence $y_{i-1}=y_{j-1}'$. Let t be the smallest integer such that $y_t=y_m'$ for some m. The vertices $u, w_1, \ldots, w_t, w_m', w_{m-1}', \ldots, w_1', u'$ are independent, since the girth is at least $4k+2$. Then let $J=(I-\{v, y_1, \ldots, y_t=y_m', y_{m-1}', \ldots, y_1', v'\})\cup\{u, w_1, \ldots, w_t, w_m', w_{m-1}', \ldots, w_1', u'\}$ is an independent set larger than I. This contradiction shows that $z\neq z'$. In a similar manner, one may show that if beginning with l we obtain $w_1, w_2, \ldots, w_k$ in G_2, and if beginning with l' we obtain $w_1', w_2', \ldots, w_k'$ in G_2, then the sets $\{w_1, w_2, \ldots, w_k\}$ and $\{w_1', w_2', \ldots, w_k'\}$ are disjoint. It follows that $[\Gamma_1, G_2]$ has no more than $(1/k)\alpha_2+f$ edges. Hence $2a+b\leq(1/k)\alpha_2+f$. Adding c to both sides of this inequality yields $2a+b+c=\alpha_1+a\leq(1/k)\alpha_2+f+c$. ■

THEOREM 10. *There exists a sequence $\{\varepsilon_k\}$ of positive numbers converging to zero such that if G is a cubic graph with girth bigger than $4k+1$, then the independence ratio $\alpha_0/n\geq\frac{7}{18}-\varepsilon_k$.*

Proof. By Lemma 9, $\alpha_1+a\leq(1/k)\alpha_2+f+c$. Adding $b+c$ to both sides yields

(7) $$2\alpha_1\leq\frac{1}{k}\alpha_2+b+2c+f.$$

Equation (4) from the proof of Theorem 8 says $3\alpha_3=b+2c+e+2f+3h$, so $3\alpha_3-f\geq b+2c+f$. Substituting this inequality into (7) yields

(8) $$2\alpha_1\leq\frac{1}{k}\alpha_2+3\alpha_3-f.$$

Now, from the proof of Lemma 6, we take the inequality $b+2c\leq 2\alpha_3$, and substitute it into (7), yielding

(9) $$2\alpha_1\leq\frac{1}{k}\alpha_2+2\alpha_3+f.$$

Adding (8) and (9), and dividing by four gives

(10) $$\alpha_1\leq\frac{1}{2k}\alpha_2+\frac{5}{4}\alpha_3.$$

Recall that $\alpha_0/n=\frac{2}{5}-(\alpha_1-\alpha_3)/5n$. Substituting (10) into this equation we get

$$\frac{\alpha_0}{n}\geq\frac{2}{5}-\frac{1}{10nk}\alpha_2-\frac{1}{20n}\alpha_3.$$

Let $\varepsilon_k=1/18k$. Since $\alpha_2/n\leq 1$, we have

(11) $$\frac{\alpha_0}{n}\geq\frac{2}{5}-\frac{1}{10k}-\frac{1}{20}\frac{\alpha_3}{n}.$$

Recall from the proof of Theorem 8 that we have

$$\frac{\alpha_0}{n} \geq \frac{3}{8} + \frac{1}{16}\frac{\alpha_3}{n}.$$

Consider two cases. If $\alpha_3/n \geq \frac{2}{9} - 8/9k$, then (5) yields

$$\frac{\alpha_0}{n} \geq \frac{3}{8} + \frac{1}{16}\left(\frac{2}{9} - \frac{8}{9k}\right) = \frac{3}{8} + \frac{1}{72} - \frac{1}{18k}$$
$$= \frac{7}{18} - \frac{1}{18k} = \frac{7}{18} - \varepsilon_k.$$

If $\alpha_3/n \leq \frac{2}{9} - 8/9k$, then (11) yields

$$\frac{\alpha_0}{n} \geq \frac{2}{5} - \frac{1}{10k} - \frac{1}{20}\left(\frac{2}{9} - \frac{8}{9k}\right)$$
$$= \frac{2}{5} - \frac{1}{90} - \frac{1}{10k} + \frac{8}{180k} = \frac{7}{18} - \varepsilon_k. \quad \blacksquare$$

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