*Proceedings of the Royal Society of Edinburgh*, **154**, 810–829, *2024*
DOI:10.1017/prm.2023.38

# Estimates for the nonlinear viscoelastic damped wave equation on compact Lie groups

**Arun Kumar Bhardwaj**
Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, Assam, India (arunkrbhardwaj@gmail.com)

**Vishvesh Kumar**
Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Krijgslaan 281, Building S8, B 9000 Ghent, Belgium (vishveshmishra@gmail.com)

**Shyam Swarup Mondal**
Department of Mathematics, Indian Institute of Technology Delhi, Delhi 110 016, India (mondalshyam055@gmail.com)

(Received 14 February 2023; accepted 22 March 2023)

Let $G$ be a compact Lie group. In this article, we investigate the Cauchy problem for a nonlinear wave equation with the viscoelastic damping on $G$. More precisely, we investigate some $L^2$-estimates for the solution to the homogeneous nonlinear viscoelastic damped wave equation on $G$ utilizing the group Fourier transform on $G$. We also prove that there is no improvement of any decay rate for the norm $\|u(t, \cdot)\|_{L^2(G)}$ by further assuming the $L^1(G)$-regularity of initial data. Finally, using the noncommutative Fourier analysis on compact Lie groups, we prove a local in time existence result in the energy space $\mathcal{C}^1([0, T], H^1_{\mathcal{L}}(G))$.

*Keywords:* nonlinear wave equation; viscoelastic damping; $L^2$-$L^2$-estimate; local well-posedness; compact Lie groups

2020 *Mathematics Subject Classification:* Primary: 35L15, 35L05
Secondary: 35L05

## 1. Introduction

Let $G$ be a compact Lie group and let $\mathcal{L}$ be the Laplace–Beltrami operator on $G$ (which also coincides with the Casimir element of the enveloping algebra of the Lie algebra of $G$). In this paper, we derive decay estimates for the solution to the Cauchy problem for a nonlinear wave equation with two types of damping terms, namely,

$$\begin{cases} \partial_t^2 u - \mathcal{L}u + \partial_t u - \mathcal{L}\partial_t u = f(u), & x \in G, t > 0, \\ u(0, x) = \varepsilon u_0(x), & x \in G, \\ \partial_t u(x, 0) = \varepsilon u_1(x), & x \in G, \end{cases} \tag{1.1}$$

where $\varepsilon$ is a positive constant describing the smallness of Cauchy data. Here, for the moment, we assume that $u_0$ and $u_1$ are taken from the energy space $H^1_{\mathcal{L}}(G)$ and concerning the nonlinearity of $f(u)$, we shall deal only with the typical case such as $f(u) := |u|^p$, $p > 1$ without loosing the essence of the problem. Equation (1.1) is known as the viscoelastic damped wave equation associated with the Laplace–Beltrami operators on compact Lie groups.

The linear viscoelastic damped wave equation in the setting of the Euclidean space has been well studied in the literature. Several prominent researchers have devoted considerable attention to the following Cauchy problem for linear damped wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

due to its application of this model in the theory of viscoelasticity and some fluid dynamics. In his seminal work, Matsumura [**18**] first established basic decay estimates for the solution to the linear equation (1.2) and after that, many researchers have concentrated on investigating a typical important nonlinear problem, namely, the following semilinear damped wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u = |u|^p, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.3}$$

In this case, there exists a real number $p_F \in (1, \infty)$ such that if $p > p_F$, then for some range of $p$ the corresponding Cauchy problem (1.3) has a small global in time solution $u(t, x)$ for the small initial data $u_0$ and $u_1$. On the other hand, when $p \in (1, p_F]$, under some condition on the initial data ($\int_{\mathbb{R}^n} u_i(x)\,\mathrm{d}x > 0$, $i = 0, 1$), the corresponding problem (1.3) does not have any nontrivial global solutions. In general, such a number $p_F$ is called the Fujita critical exponent. For a detailed study related to the Fujita exponent, we refer to [**1**, **9**, **12**, **20**, **33**] and references therein.

Further, the study of the semilinear damped wave equation (1.3) is further generalized by the following strongly damped wave equation

$$\begin{cases} \partial_t^2 u - \Delta u + \Delta \partial_t u = \mu f(u) & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.4}$$

by several researchers recently. When $\mu = 0$, in the case, for the dissipative structures of the Cauchy problem (1.4), Ponce [**24**] and Shibata [**31**] derived some $L^p(\mathbb{R}^n) - L^q(\mathbb{R}^n)$ decay estimates for the solution to (1.4) with $\mu = 0$. In the last decade, some $L^2(\mathbb{R}^n) - L^2(\mathbb{R}^n)$ estimates with additional $L^1(\mathbb{R}^n)$-regularity were also derived by several authors in [**2**, **4**, **8**, **11**]. In the same period, the authors of [**8**] proved global (in time) existence of small data solution to the corresponding semilinear Cauchy problem to (1.4) with power nonlinearity on the right-hand side. Recently, Ikehata *et al.* [**15**] and Ikehata [**10**] have caught an asymptotic profile of solutions to the problem (1.4), which is well-studied in the field of the Navier–Stokes equation case.

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

The study of the semilinear wave equation has also been extended in the non-Euclidean framework. Several papers have studied linear probability distribution equation (PDE) in non-Euclidean structures in the last decades. For example, the semilinear wave equation with or without damping has been investigated for the Heisenberg group [**19**, **26**]. In the case of graded groups, we refer to the recent works [**25**, **30**, **32**]. Concerning the damped wave equation on compact Lie groups, we refer to [**7**, **21**–**23**] (see also [**5**] for the fractional wave equation). Here, we would also like to highlight that estimates for the linear viscoelastic damped wave equation on the Heisenberg group were studied in [**16**].

Recently, Ikehata and Sawada [**13**] and Ikehata and Takeda [**14**] considered and studied the following Cauchy problem, which has two types of damping terms

$$\begin{cases} \partial_t^2 u - \Delta u + \partial_t u - \Delta \partial_t u = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

Such types of related problems with slight variants are extensively investigated by authors [**2**, **3**, **17**].

An interesting and viable problem is to consider such types of (i.e., Cauchy problem 1.7) viscoelastic damped wave equations in the setting of non-Euclidean spaces, in particular, for compact Lie groups. So far, to the best of our knowledge, in the framework of compact Lie groups, the viscoelastic damped wave equation has not been studied yet. Our main aim of this article is to study the Cauchy problem for the nonlinear wave equation with two types of damping terms on the compact Lie group $G$, namely,

$$\begin{cases} \partial_t^2 u - \mathcal{L} u + \partial_t u - \mathcal{L} \partial_t u = f(u), & x \in G, t > 0, \\ u(0,x) = \varepsilon u_0(x), & x \in G, \\ \partial_t u(x,0) = \varepsilon u_1(x), & x \in G. \end{cases}$$

### 1.1. Main results

Throughout the paper we denote $L^q(G)$, the space of $q$-integrable functions on $G$ with respect to the normalized Haar measure for $1 \leqslant q < \infty$ (respectively, essentially bounded for $q = \infty$) and for $s > 0$ and $q \in (1, \infty)$ the Sobolev space $H^{s,q}_{\mathcal{L}}(G)$ is defined as the space

$$H^{s,q}_{\mathcal{L}}(G) \doteq \left\{ f \in L^q(G) : (-\mathcal{L})^{s/2} f \in L^q(G) \right\} \tag{1.6}$$

endowed with the norm $\|f\|_{H^{s,q}_{\mathcal{L}}(G)} \doteq \|f\|_{L^q(G)} + \left\|(-\mathcal{L})^{s/2} f\right\|_{L^q(G)}$. We simply denote $H^s_{\mathcal{L}}(G)$ as the Hilbert space $H^{s,2}_{\mathcal{L}}(G)$.

By employing the tools from the Fourier analysis for compact Lie groups, our first result below is concerned with the existence of the global solution to the homogeneous Cauchy problem (1.1) (i.e., when $f = 0$) satisfying the suitable decay properties. More precisely, our goal is to derive $L^2(G)$-decay estimates for the Cauchy data, as it is stated in the following theorem.

THEOREM 1.1. *Let $u_0$, $u_1 \in H^1_{\mathcal{L}}(G)$ and let $u \in \mathcal{C}^1([0, \infty), H^1_{\mathcal{L}}(G))$ be the solution to the homogeneous Cauchy problem*

$$\begin{cases} \partial_t^2 u - \mathcal{L}u + \partial_t u - \mathcal{L}\partial_t u = 0, & x \in G, \ t > 0, \\ u(0, x) = u_0(x), & x \in G \\ \partial_t u(x, 0) = u_1(x), & x \in G. \end{cases} \tag{1.7}$$

*Then, $u$ satisfies the following $L^2$-estimates*

$$\|u(t, \cdot)\|_{L^2(G)} \leqslant C \left(\|u_0\|_{L^2(G)} + \|u_1\|_{L^2(G)}\right), \tag{1.8}$$

$$\|(-\mathcal{L})^{1/2} u(t, \cdot)\|_{L^2(G)} \leqslant C(1+t)^{-\frac{1}{2}} \left(\|u_0\|_{H^1_{\mathcal{L}}(G)} + \|u_1\|_{L^2(G)}\right), \tag{1.9}$$

$$\|\partial_t u(t, \cdot)\|_{L^2(G)} \leqslant C(1+t)^{-1} \left(\|u_0\|_{H^1_{\mathcal{L}}(G)} + \|u_1\|_{L^2(G)}\right), \tag{1.10}$$

$$\|\partial_t (-\mathcal{L})^{1/2} u(t, \cdot)\|_{L^2(G)} \leqslant C(1+t)^{-\frac{3}{2}} \left(\|u_0\|_{H^1_{\mathcal{L}}(G)} + \|u_1\|_{H^1_{\mathcal{L}}(G)}\right), \tag{1.11}$$

*for any $t \geqslant 0$, where $C$ is a positive multiplicative constant.*

REMARK 1.2. From the statement of Theorem 1.1 one can find that the regularity $u_1 \in H^1_{\mathcal{L}}(G)$ is necessary to remove the singularity of $\|\partial_t(-\mathcal{L})^{1/2} u(t, \cdot)\|_{L^2(G)}$ near $t = 0$.

REMARK 1.3. We also show that there is no improvement of any decay rate for the norm $\|u(t, \cdot)\|_{L^2(G)}$ in Theorem 1.1 even if we assume $L^1(G)$-regularity for $u_0$ and $u_1$.

Next, we prove the local well-posedness of the Cauchy problem (1.1) in the energy evolution space $\mathcal{C}^1([0, T], H^1_{\mathcal{L}}(G))$. In particular, a Gagliardo–Nirenberg type inequality (proved in [**29**]) will be used in order to estimate the power nonlinearity in $L^2(G)$. The following result is about the local existence for the solution of the Cauchy problem (1.1).

THEOREM 1.4. *Let $G$ be a compact, connected Lie group and let $n$ be the topological dimension of $G$. Assume that $n \geqslant 3$. Suppose that $u_0$, $u_1 \in H^1_{\mathcal{L}}(G)$ and $p > 1$ such that $p \leqslant \frac{n}{n-2}$. Then, there exists $T = T(\varepsilon) > 0$ such that the Cauchy problem (1.1) admits a uniquely determined mild solution $u$ in the space $\mathcal{C}^1([0, T], H^1_{\mathcal{L}}(G))$.*

As in [**21**], we note that in the statement of Theorem 1.4, the restriction on the upper bound for the exponent $p$, which is $p \leqslant \frac{n}{n-2}$ is necessary in order to apply Gagliardo–Nirenberg type inequality (5.4) in (5.6) in the proof of Theorem 1.4. The other restriction $n \geqslant 3$ is also technical and is made to fulfill the assumptions for the employment of such inequality. This could be avoided if one looks for a solution in a different space such as $\mathcal{C}^1([0, T], H^s_{\mathcal{L}}(G))$, $s \in (0, 1)$ than that of $\mathcal{C}^1([0, T], H^1_{\mathcal{L}}(G))$.

It is customary to study the corresponding nonlinear homogeneous problem, i.e., when $f = 0$ prior to investigate the nonhomogeneous problem (1.1). In this process, we first establish a $L^2$- energy estimate for the solution to the homogeneous viscoelastic damped wave equation on the compact Lie group $G$. Having these

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

estimates in our hand, we implement a Gagliardo–Nirenberg type inequality on compact Lie group [**21**–**23**, **29**] to prove the local well-posedness result for the solution to (1.1). We also show that, even if we assume $L^1(G)$-regularity for $u_0$ and $u_1$, there is no additional decay rate that can be gained for the $L^2$ norm of the solution of the corresponding homogeneous Cauchy problem.

Apart from the introduction, the paper is organized as follows. In § 2, we recall some essentials from the Fourier analysis on compact Lie groups which will be frequently used throughout the paper. In § 3, we prove Theorem 1.1 by deriving some $L^2$ decay estimates for the solution of the homogeneous nonlinear viscoelastic damped wave equation on the compact Lie group $G$. We also show that there is no additional gain in the decay rate of the $L^2$ norm of the solution to the corresponding homogeneous Cauchy problem even if we assume $L^1(G)$-regularity for $u_0$ and $u_1$ in § 4. Finally, in § 5, we briefly recall the notion of mild solutions in our framework and prove the local well-posedness of the Cauchy problem (1.1) in the energy evolution space $\mathcal{C}^1([0, T], H^\alpha_{\mathcal{L}}(G))$.

### 1.2. Notations

Throughout the article, we use the following notations:

- $f \lesssim g$ : There exists a positive constant $C$ (whose value may change from line to line in this manuscript) such that $f \leqslant Cg$.
- $G$ : Compact Lie group.
- $\mathrm{d}x$ : The normalized Haar measure on the compact group $G$.
- $\mathcal{L}$ : The Laplace–Beltrami operator on $G$.
- $\mathbb{C}^{d\times d}$ : The set of matrices with complex entries of order $d$.
- $\mathrm{Tr}(A) = \sum_{j=1}^{d} a_{jj}$ : The trace of the matrix $A = (a_{ij})_{1\leqslant i,j\leqslant d} \in \mathbb{C}^{d\times d}$.
- $I_d \in \mathbb{C}^{d\times d}$ : The identity matrix of order $d$.

## 2. Preliminaries: Fourier analysis on compact Lie groups

In this section, we recall some basics of Fourier analysis on compact (Lie) groups to make the manuscript self-contained. A complete account of representation theory of the compact Lie groups can be found in [**7**, **27**, **28**]. However, we mainly adopt the notation and terminology given in [**27**].

Let us first recall the definition of a representation of a compact group $G$. A unitary representation of $G$ is a pair $(\xi, \mathcal{H})$ such that the map $\xi : G \to U(\mathcal{H})$, where $U(\mathcal{H})$ denotes the set of unitary operators on complex Hilbert space $\mathcal{H}$, such that it satisfies the following properties:

- The map $\xi$ is a group homomorphism, that is, $\xi(xy) = \xi(x)\xi(y)$.

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

- The mapping $\xi : G \to U(\mathcal{H})$ is continuous with respect to strong operator topology (SOT) on $U(\mathcal{H})$, that is, the map $g \mapsto \xi(g)v$ is continuous for every $v \in \mathcal{H}$.

The Hilbert space $\mathcal{H}$ is called the representation space. If there is no confusion, we just write $\xi$ for a representation $(\xi, \mathcal{H})$ of $G$. Two unitary representations $\xi$, $\eta$ of $G$ are called equivalent if there exists a unitary operator, called intertwiner, $T$ such that $T\xi(x) = \eta(x)T$ for any $x \in G$. The intertwiner is an irreplaceable tool in the theory of representation of compact groups and helpful in the classification of representation. A (linear) subspace $V \subset \mathcal{H}$ is said to be invariant under the unitary representation $\xi$ of $G$ if $\xi(x)V \subset V$ for any $x \in G$. An irreducible unitary representation $\xi$ of $G$ is a representation such that the only closed and $\xi$-invariant subspaces of $\mathcal{H}$ are trivial once, that is, $\{0\}$ and the full space $\mathcal{H}$.

The set of all equivalence classes $[\xi]$ of continuous irreducible unitary representations of $G$ is denoted by $\widehat{G}$ and called the unitary dual of $G$. Since $G$ is compact, $\widehat{G}$ is a discrete set. It is known that an irreducible unitary representation $\xi$ of $G$ is finite dimensional, that is, the Hilbert space $\mathcal{H}$ is finite dimensional, say, $d_\xi$. Therefore, if we choose a basis $\mathfrak{B} := \{e_1, e_2, \dots, e_{d_\xi}\}$ for the representation space $\mathcal{H}$ of $\xi$, we can identify $\mathcal{H}$ as $\mathbb{C}^{d_\xi}$ and consequently, we can view $\xi$ as a matrix-valued function $\xi : G \to U(\mathbb{C}^{d_\xi \times d_\xi})$, where $U(\mathbb{C}^{d_\xi \times d_\xi})$ denotes the space of all unitary matrices. The matrix coefficients $\xi_{ij}$ of the representation $\xi$ with respect to $\mathfrak{B}$ are given by $\xi_{ij}(x) := \langle \xi(x)e_j, e_i \rangle$ for all $i, j \in \{1, 2, \dots, d_\xi\}$. It follows from the Peter-Weyl theorem that the set

$$\left\{ \sqrt{d_\xi}\xi_{ij} : 1 \leqslant i, j \leqslant d_\xi, [\xi] \in \widehat{G} \right\}$$

forms an orthonormal basis of $L^2(G)$.

The group Fourier transform of $f \in L^1(G)$ at $\xi \in \widehat{G}$, denoted by $\widehat{f}(\xi)$, is defined by

$$\widehat{f}(\xi) := \int_G f(x)\xi(x)^* \,\mathrm{d}x,$$

where $\mathrm{d}x$ is the normalised Haar measure on $G$. It is apparent from the definition that $\widehat{f}(\xi)$ is matrix valued and therefore, this definition can be interpreted as weak sense, that is, for $u, v \in \mathcal{H}$, we have

$$\langle \widehat{f}(\xi)u, v \rangle := \int_G f(x)\langle \xi(x)^* u, v \rangle \,\mathrm{d}x.$$

It follows from the Peter–Weyl theorem that, for every $f \in L^2(G)$, we have the following Fourier series representation:

$$f(x) = \sum_{[\xi] \in \widehat{G}} d_\xi \operatorname{Tr}(\xi(x)\widehat{f}(\xi)).$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

The Plancherel identity for the group Fourier transform on $G$ takes the following form:

$$\|f\|_{L^2(G)} = \left( \sum_{[\xi] \in \widehat{G}} d_\xi \|\widehat{f}(\xi)\|_{\mathrm{HS}}^2 \right)^{1/2} := \|\widehat{f}\|_{\ell^2(\widehat{G})}, \tag{2.1}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of a matrix $A := (a_{ij}) \in \mathbb{C}^{d_x i \times d_\xi}$ defined as

$$\|A\|_{\mathrm{HS}}^2 = \mathrm{Tr}\,(AA^*) = \sum_{i,j=1}^{d_\xi} |a_{ij}|^2.$$

We would like to emphasize here that the Plancherel identity is one of the crucial tools to establish $L^2$-estimates of the solution to PDEs.

Let $\mathcal{L}$ be the Laplace–Beltrami operator on $G$. It is important to understand the action of the group Fourier transform on the Laplace–Beltrami operator $\mathcal{L}$ for developing the machinery for the proofs. For $[\xi] \in \widehat{G}$, the matrix elements $\xi_{ij}$ are the eigenfunctions of $\mathcal{L}$ with the same eigenvalue $-\lambda_\xi^2$. In other words, we have, for any $x \in G$,

$$-\mathcal{L}\xi_{ij}(x) = \lambda_\xi^2 \xi_{ij}(x), \qquad \text{for all } i, j \in \{1, \ldots, d_\xi\}.$$

The symbol $\sigma_{\mathcal{L}}$ of the Laplace–Beltrami operator $\mathcal{L}$ on $G$ is given by

$$\sigma_{\mathcal{L}}(\xi) = -\lambda_\xi^2 I_{d_\xi}, \tag{2.2}$$

for any $[\xi] \in \widehat{G}$ and therefore, the following holds:

$$\widehat{\mathcal{L}f}(\xi) = \sigma_{\mathcal{L}}(\xi)\widehat{f}(\xi) = -\lambda_\xi^2 \widehat{f}(\xi)$$

for any $[\xi] \in \widehat{G}$.

For $s > 0$, the Sobolev space $H_{\mathcal{L}}^s(G)$ of order $s$ is defined as follows:

$$H_{\mathcal{L}}^s(G) := \left\{ u \in L^2(G) : \|u\|_{H_{\mathcal{L}}^s(G)} < +\infty \right\},$$

where $\|u\|_{H_{\mathcal{L}}^s(G)} = \|u\|_{L^2(G)} + \left\|(-\mathcal{L})^{s/2} u\right\|_{L^2(G)}$ and $(-\mathcal{L})^{s/2}$ is defined in terms of the group Fourier transform by the following formula:

$$(-\mathcal{L})^{\alpha/2} f := \mathcal{F}^{-1}\left(\lambda_\xi^{2\alpha}(\mathcal{F}u)\right), \quad \text{for all } [\xi] \in \widehat{G}.$$

Further, using Plancherel identity, for any $s > 0$, we have that

$$\left\|(-\mathcal{L})^{s/2} f\right\|_{L^2(G)}^2 = \sum_{[\xi] \in \widehat{G}} d_\xi \lambda_\xi^{2s} \|\widehat{f}(\xi)\|_{\mathrm{HS}}^2.$$

We also recall the definition of the space $\ell^\infty(\widehat{G})$. We denote $\mathcal{S}'(\widehat{G})$ as the space of slowly increasing distributions on the unitary dual $\widehat{G}$ of $G$. Then the space $\ell^\infty(\widehat{G})$

is defined as

$$\ell^\infty(\widehat{G}) = \{H = \{H([\xi])\}_{[\xi]\in\widehat{G}} : \|H\|_{\ell^\infty(\widehat{G})} < \infty\},$$

where $H([\xi]) \in \mathbb{C}^{d_\xi \times d_\xi}$ for any $[\xi] \in \widehat{G}$ and

$$\|H\|_{\ell^\infty(\widehat{G})} := \sup_{[\xi]\in\widehat{G}} d_\xi^{-\frac{1}{2}} \|H(\xi)\|_{HS} < \infty. \tag{2.3}$$

Then $\ell^\infty(\widehat{G})$ is a subspace of $\mathcal{S}'(\widehat{G})$. Moreover, for any $f \in L^1(G)$, from the group Fourier transform it is true that

$$\|\widehat{f}\|_{\ell^\infty(\widehat{G})} \leqslant \|f\|_{L^1(G)}. \tag{2.4}$$

We must mention that implementation of (2.4) is very important in order to use the $L^1(G)$-regularity for the Cauchy data. A detailed study on the construction of the space $\ell^\infty(\widehat{G})$ can be found in section 10.3.2 of [**27**] (see also section 2.1.3 of [**6**]).

## 3. $L^2$-estimates for the solution to the homogeneous problem

In this section, we derive $L^2(G) - L^2(G)$ estimates for the solutions to (1.7) when $f = 0$, namely, the homogeneous problem on $G$ :

$$\begin{cases} \partial_t^2 u - \mathcal{L}u + \partial_t u - \mathcal{L}\partial_t u = 0, & x \in G, t > 0, \\ u(0,x) = u_0(x), & x \in G, \\ \partial_t u(x,0) = u_1(x), & x \in G. \end{cases} \tag{3.1}$$

We employ the group Fourier transform on the compact Lie group $G$ with respect to the space variable $x$ together with the Plancherel identity in order to estimate $L^2$-norms of $u(t,\ )$, $(-\mathcal{L})^{\frac{1}{2}} u(t,\cdot)$, $\partial_t u(t,\ )$ and $\partial_t(-\mathcal{L})^{1/2} u(t,\cdot)$.

Let $u$ be a solution to (3.1). Let $\widehat{u}(t,\xi) = (\widehat{u}(t,\xi)_{kl})_{1\leqslant k,l\leqslant d_\xi} \in \mathbb{C}^{d_\xi\times d_\xi}$, $[\xi] \in \widehat{G}$ denote the Fourier transform of $u$ with respect to the $x$ variable. Invoking the group Fourier transform with respect to $x$ on (3.1), we deduce that $\widehat{u}(t,\xi)$ is a solution to the following Cauchy problem for the system of ordinary differential equations (with size of the system that depends on the representation $\xi$)

$$\begin{cases} \partial_t^2 \widehat{u}(t,\xi) - \sigma_{\mathcal{L}}(\xi)\widehat{u}(t,\xi) + \partial_t\widehat{u}(t,\xi) - \sigma_{\mathcal{L}}(\xi)\partial_t\widehat{u} = 0, & [\xi] \in \widehat{G}, t > 0, \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi), & [\xi] \in \widehat{G}, \\ \partial_t\widehat{u}(0,\xi) = \widehat{u}_1(\xi), & [\xi] \in \widehat{G}, \end{cases} \tag{3.2}$$

where $\sigma_{\mathcal{L}}$ is the symbol of the Laplace–Beltrami operator $\mathcal{L}$ defined in (2.2). Using identity (2.2), system (3.2) is decoupled in $d_\xi^2$ independent ODEs, namely,

$$\begin{cases} \partial_t^2 \widehat{u}(t,\xi)_{kl} + (1+\lambda_\xi^2)\partial_t\widehat{u}(t,\xi)_{kl} + \lambda_\xi^2\widehat{u}(t,\xi)_{kl} = 0, & [\xi] \in \widehat{G}, t > 0, \\ \widehat{u}(0,\xi)_{kl} = \widehat{u}_0(\xi)_{kl}, & [\xi] \in \widehat{G}, \\ \partial_t\widehat{u}(0,\xi)_{kl} = \widehat{u}_1(\xi)_{kl}, & [\xi] \in \widehat{G}, \end{cases} \tag{3.3}$$

for all $k, l \in \{1, 2, \ldots, d_\xi\}$.

Then, the characteristic equation of (3.3) is given by

$$\lambda^2 + (1+\lambda_\xi^2)\lambda + \lambda_\xi^2 = 0,$$

and consequently, the characteristic roots of (3.3) are

$$\lambda = \frac{-(1+\lambda_\xi^2) \pm |1-\lambda_\xi^2|}{2}.$$

We note that if $\lambda_\xi^2 \neq 1$, then there are two distinct roots, say, $\lambda^+ = -1$ and $\lambda^- = -\lambda_\xi^2$, and if $\lambda_\xi^2 = 1$ then both the roots are same and equal to $\lambda = -1$. We analyse the following two cases for the solution to system (3.3).

**Case I.** Let $\lambda_\xi^2 \neq 1$. The solution of (3.3) is given by

$$\widehat{u}(t,\xi)_{kl} = \mathcal{K}_0(t,\xi)\widehat{u}_0(\xi)_{kl} + \mathcal{K}_1(t,\xi)\widehat{u}_1(\xi)_{kl}, \tag{3.4}$$

where

$$\begin{cases} \mathcal{K}_0(t,\xi) = \frac{e^{-\lambda_\xi^2 t} - \lambda_\xi^2 e^{-t}}{1-\lambda_\xi^2}, \\ \mathcal{K}_1(t,\xi) = \frac{e^{-\lambda_\xi^2 t} - e^{-t}}{1-\lambda_\xi^2}. \end{cases} \tag{3.5}$$

**Case II.** Let $\lambda_\xi^2 = 1$. The solution of (3.3) is given by

$$\widehat{u}(t,\xi)_{kl} = \mathcal{K}_0(t,\xi)\widehat{u}_0(\xi)_{kl} + \mathcal{K}_1(t,\xi)\widehat{u}_1(\xi)_{kl}, \tag{3.6}$$

where

$$\begin{cases} \mathcal{K}_0(t,\xi) = (1+t)e^{-t}, \\ \mathcal{K}_1(t,\xi) = te^{-t}. \end{cases} \tag{3.7}$$

Thus, we have

$$\widehat{u}(t,\xi)_{kl} = \begin{cases} \frac{e^{-\lambda_\xi^2 t} - \lambda_\xi^2 e^{-t}}{1-\lambda_\xi^2}\widehat{u}_0(\xi)_{kl} + \frac{e^{-\lambda_\xi^2 t} - e^{-t}}{1-\lambda_\xi^2}\widehat{u}_1(\xi)_{kl}, & \lambda_\xi^2 \neq 1, \\ (1+t)e^{-t}\widehat{u}_0(\xi)_{kl} + te^{-t}\widehat{u}_1(\xi)_{kl}, & \lambda_\xi^2 = 1. \end{cases} \tag{3.8}$$

Also, we note that

$$\partial_t^\ell \mathcal{K}_0(t,\xi) = \frac{(-\lambda_\xi^2)^\ell e^{-\lambda_\xi^2 t} + (-1)^{\ell+1}\lambda_\xi^2 e^{-t}}{1-\lambda_\xi^2},$$

and

$$\partial_t^\ell \mathcal{K}_1(t,\xi) = \frac{(-\lambda_\xi^2)^\ell e^{-\lambda_\xi^2 t} + (-1)^{\ell+1} e^{-t}}{1-\lambda_\xi^2}.$$

First, we determine an explicit expression for the $L^2(G)$ norms of $u(t,\cdot)$, $(-\mathcal{L})^{1/2}$ $u(t,\cdot)$, $\partial_t u(t,\cdot)$ and $\partial_t(-\mathcal{L})^{1/2}u(t,\cdot)$. We apply the group Fourier transform with respect to the spatial variable $x$ together with the Plancherel identity in order to determine the $L^2(G)$ norms.

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

To simplify the presentation, we introduce the following partition of the unitary dual $\widehat{G}$ as:

$$\mathcal{R}_1 = \{[\xi] \in \widehat{G} : \lambda_\xi^2 = 0\},$$
$$\mathcal{R}_2 = \{[\xi] \in \widehat{G} : 0 < \lambda_\xi^2 < 1\},$$
$$\mathcal{R}_3 = \{[\xi] \in \widehat{G} : \lambda_\xi^2 = 1\} \text{ and}$$
$$\mathcal{R}_4 = \{[\xi] \in \widehat{G} : \lambda_\xi^2 > 1\}.$$

Here, we note that some of the above sets may be empty.

### 3.1. Estimate for $\|u(t,\cdot)\|_{L^2(G)}$

By the Plancherel formula, we have

$$\|u(t,\cdot)\|_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{k,l=1}^{d_\xi} |\widehat{u}(t,\xi)_{kl}|^2. \tag{3.9}$$

**Estimate on $\mathcal{R}_1$.** Using $\lambda_\xi^2 = 0$ in (3.5) we get

$$|\mathcal{K}_0(t,\xi)|,\ |\mathcal{K}_1(t,\xi)| \lesssim 1. \tag{3.10}$$

Hence, (3.4) implies that

$$|\widehat{u}(t,\xi)_{kl}| \lesssim |\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|. \tag{3.11}$$

**Estimate on $\mathcal{R}_2$.** Since the set $\{\lambda_\xi^2\}_{[\xi]\in\widehat{G}}$ is a discrete set, there exist $\delta_1$ and $\delta_2$ such that

$$0 < \delta_1 \leqslant \lambda_\xi^2 \leqslant \delta_2 < 1, \quad [\xi] \in \mathcal{R}_2, \tag{3.12}$$

consequently $\frac{1}{1-\lambda_\xi^2}$ is bounded on $\mathcal{R}_2$ and by (3.5) we have

$$|\mathcal{K}_0(t,\xi)|,\ |\mathcal{K}_1(t,\xi)| \lesssim e^{-\delta_1 t}. \tag{3.13}$$

Hence, by (3.4) we get

$$|\widehat{u}(t,\xi)_{kl}| \lesssim e^{-\delta_1 t}\left[|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right]. \tag{3.14}$$

**Estimate on $\mathcal{R}_3$.** By (3.7) we have

$$|\mathcal{K}_0(t,\xi)|,\ |\mathcal{K}_1(t,\xi)| \lesssim (1+t)e^{-t}. \tag{3.15}$$

Hence, by (3.6) we get

$$|\widehat{u}(t,\xi)_{kl}| \lesssim (1+t)e^{-t}\left[|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right]. \tag{3.16}$$

**Estimate on $\mathcal{R}_4$.** Again discreteness of the set $\{\lambda_\xi^2\}_{[\xi]\in\widehat{G}}$ implies that there exists $\delta_3$ such that

$$1 < \delta_3 \leqslant \lambda_\xi^2, \qquad [\xi] \in \mathcal{R}_4. \tag{3.17}$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

Hence, $\frac{1}{\lambda_\xi^2-1}$ and $\frac{\lambda_\xi^2}{\lambda_\xi^2-1}$ are bounded on $\mathcal{R}_4$, consequently (3.5) yields

$$|\mathcal{K}_0(t,\xi)|, \quad |\mathcal{K}_1(t,\xi)| \lesssim e^{-t}. \tag{3.18}$$

Using (3.4) we obtain

$$|\widehat{u}(t,\xi)_{kl}| \lesssim e^{-t} \left[|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right]. \tag{3.19}$$

Combining (3.11), (3.14), (3.16) and (3.19) we get

$$|\widehat{u}(t,\xi)_{kl}| \lesssim |\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|, \quad [\xi] \in \widehat{G}. \tag{3.20}$$

Substituting (3.20) into (3.9) we obtain

$$\|u(t,\cdot)\|_{L^2(G)} \leqslant C\left(\|u_0\|_{L^2(G)} + \|u_1\|_{L^2(G)}\right). \tag{3.21}$$

Remark 3.1. Note that we do not get any decay on the right-hand side of (3.21) due to the fact that the set $\mathcal{R}_1$ is always nonempty (in fact singleton).

### 3.2. Estimate for $\|(-\mathcal{L})^{1/2}u(t,\cdot)\|_{L^2(G)}$

By the Plancherel formula, we get

$$\|(-\mathcal{L})^{1/2}u(t,\cdot)||_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} d_\xi \|\sigma_{(-\mathcal{L})^{1/2}}(\xi)\widehat{u}(t,\xi)\|_{\mathrm{HS}}^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2.$$

**Estimate on $\mathcal{R}_1$.** We have

$$\lambda_\xi^2|\widehat{u}(t,\xi)_{kl}|^2 = 0.$$

**Estimate on $\mathcal{R}_2$.** By (3.13) we obtain

$$\lambda_\xi^2|\widehat{u}(t,\xi)_{kl}|^2 \lesssim e^{-2\delta_1 t}[|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2]. \tag{3.22}$$

**Estimate on $\mathcal{R}_3$.** By (3.15) we have

$$\lambda_\xi^2|\widehat{u}(t,\xi)_{kl}|^2 \lesssim (1+t)^2 e^{-2t}\left[|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right]. \tag{3.23}$$

**Estimate on $\mathcal{R}_4$.** Again using the fact that $\frac{1}{\lambda_\xi^2-1}$ and $\frac{\lambda_\xi^2}{\lambda_\xi^2-1}$ are bounded on $\mathcal{R}_4$ we obtain

$$\lambda_\xi^2|\widehat{u}(t,\xi)_{kl}|^2 \lesssim e^{-2t}\left[\lambda_\xi^2|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right]. \tag{3.24}$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

Therefore,

$$\begin{aligned}\|(-\mathcal{L})^{1/2}u(t,\cdot)\|_{L^2(G)}^2 &= \sum_{[\xi]\in\mathcal{R}_1} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 + \sum_{[\xi]\in\mathcal{R}_2} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 \\ &\quad + \sum_{[\xi]\in\mathcal{R}_3} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 + \sum_{[\xi]\in\mathcal{R}_4} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 \\ &\lesssim e^{-2\delta_1 t} \sum_{[\xi]\in\mathcal{R}_2} d_\xi \sum_{k,l=1}^{d_\xi} \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\ &\quad + (1+t)^2 e^{-2t} \sum_{[\xi]\in\mathcal{R}_3} d_\xi \sum_{k,l=1}^{d_\xi} \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\ &\quad + e^{-2t} \sum_{[\xi]\in\mathcal{R}_4} d_\xi \sum_{k,l=1}^{d_\xi} \left(\lambda_\xi^2|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\ &\lesssim (1+t)^2 e^{-2\delta_1 t} \left(\|u_0\|_{H^1_{\mathcal{L}}(G)}^2 + \|u_1\|_{L^2(G)}^2\right) \\ &\lesssim (1+t)^{-1} \left(\|u_0\|_{H^1_{\mathcal{L}}(G)}^2 + \|u_1\|_{L^2(G)}^2\right). \end{aligned} \tag{3.25}$$

### 3.3. Estimate for $\|\partial_t u(t,\cdot)\|_{L^2(G)}$

By Plancherel theorem, we have

$$\|\partial_t u(t,\cdot)\|_{L^2(G)}^2 = \sum_{[\xi]\in\widehat{G}} d_\xi \sum_{k,l=1}^{d_\xi} |\partial_t \widehat{u}(t,\xi)_{kl}|^2.$$

We note that

$$\partial_t \widehat{u}(t,\xi)_{kl} = \begin{cases} \frac{\lambda_\xi^2}{1-\lambda_\xi^2}(e^{-t} - e^{-\lambda_\xi^2 t})\widehat{u}_0(\xi)_{kl} + \frac{e^{-t}-\lambda_\xi^2 e^{-\lambda_\xi^2 t}}{1-\lambda_\xi^2}\widehat{u}_1(\xi)_{kl}, & \lambda_\xi^2 \neq 1, \\ -te^{-t}\widehat{u}_0(\xi)_{kl} + (1-t)e^{-t}\widehat{u}_1(\xi)_{kl}, & \lambda_\xi^2 = 1. \end{cases} \tag{3.26}$$

**Estimate on $\mathcal{R}_1$.** From (3.26), we have

$$|\partial_t \widehat{u}(t,\xi)_{kl}| = e^{-t}|\widehat{u}_1(\xi)_{kl}|.$$

**Estimate on $\mathcal{R}_2$.** By (3.26) and (3.12) we obtain

$$|\partial_t \widehat{u}(t,\xi)_{kl}| \lesssim e^{-\delta_1 t}\left(|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right).$$

**Estimate on $\mathcal{R}_3$.** By (3.26) we get

$$|\partial_t \widehat{u}(t,\xi)_{kl}| \lesssim e^{-t}\left(t|\widehat{u}_0(\xi)_{kl}| + |1-t||\widehat{u}_1(\xi)_{kl}|\right) \lesssim (1+t)e^{-t}\left(|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right).$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

**Estimate on $\mathcal{R}_4$.** Using (3.26) and (3.17) and the fact that $\frac{1}{\lambda_\xi^2-1}$ and $\frac{\lambda_\xi^2}{\lambda_\xi^2-1}$ are bounded on $\mathcal{R}_4$, we obtain

$$|\partial_t\widehat{u}(t,\xi)_{kl}| \lesssim e^{-t}\left(|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right).$$

Combining these, we get

$$|\partial_t\widehat{u}(t,\xi)_{kl}| \lesssim (1+t)e^{-\delta_1 t}\left(|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right),\ \forall\ [\xi]\in\widehat{G}.$$

Thus,

$$\begin{aligned}\|\partial_t u(t,\cdot)\|^2_{L^2(G)} &\lesssim (1+t)^2e^{-2\delta_1 t}\left(\|u_0\|^2_{L^2(G)} + \|u_1\|^2_{L^2(G)}\right)\\ &\lesssim (1+t)^{-2}\left(\|u_0\|^2_{L^2(G)} + \|u_1\|^2_{L^2(G)}\right).\end{aligned} \tag{3.27}$$

### 3.4. Estimate for $\|\partial_t(-\mathcal{L})^{1/2}u(t,\cdot)\|_{L^2(G)}$

By Plancherel theorem we have

$$\|\partial_t(-\mathcal{L})^{1/2}u(t,\cdot)\|^2_{L^2(G)} = \sum_{[\xi]\in\widehat{G}} d_\xi\sum_{k,l=1}^{d_\xi}\lambda_\xi^2|\partial_t\widehat{u}(t,\xi)_{kl}|^2.$$

**Estimate on $\mathcal{R}_1$.** We have

$$\lambda_\xi^2|\partial_t\widehat{u}(t,\xi)_{kl}|^2 = 0.$$

**Estimate on $\mathcal{R}_2$.** By (3.13) and (3.26) we obtain

$$\lambda_\xi^2|\partial_t\widehat{u}(t,\xi)_{kl}|^2 \lesssim e^{-2\delta_1 t}[|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2]. \tag{3.28}$$

**Estimate on $\mathcal{R}_3$.** By (3.26) we have

$$\lambda_\xi^2|\partial_t\widehat{u}(t,\xi)_{kl}|^2 \lesssim (1+t)^2e^{-2t}\left[|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right]. \tag{3.29}$$

**Estimate on $\mathcal{R}_4$.** Using (3.26), (3.17) and the fact that $\frac{1}{\lambda_\xi^2-1}$ and $\frac{\lambda_\xi^2}{\lambda_\xi^2-1}$ are bounded on $\mathcal{R}_4$, we obtain

$$\lambda_\xi^2|\partial_t\widehat{u}(t,\xi)_{kl}| \lesssim e^{-t}\lambda_\xi^2\left(|\widehat{u}_0(\xi)_{kl}| + |\widehat{u}_1(\xi)_{kl}|\right).$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

Therefore,

$$
\begin{aligned}
\|\partial_t(-\mathcal{L})^{1/2}u(t,\cdot)\|^2_{L^2(G)} &= \sum_{[\xi]\in\mathcal{R}_1} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 + \sum_{[\xi]\in\mathcal{R}_2} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 \\
&\quad + \sum_{[\xi]\in\mathcal{R}_3} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 + \sum_{[\xi]\in\mathcal{R}_4} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 |\widehat{u}(t,\xi)_{kl}|^2 \\
&\lesssim e^{-2\delta_1 t} \sum_{[\xi]\in\mathcal{R}_2} d_\xi \sum_{k,l=1}^{d_\xi} \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\
&\quad + (1+t)^2 e^{-2t} \sum_{[\xi]\in\mathcal{R}_3} d_\xi \sum_{k,l=1}^{d_\xi} \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\
&\quad + e^{-2t} \sum_{[\xi]\in\mathcal{R}_4} d_\xi \sum_{k,l=1}^{d_\xi} \lambda_\xi^2 \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\
&\lesssim (1+t)^2 e^{-2\delta_1 t} \left(\|u_0\|^2_{H^1_{\mathcal{L}}(G)} + \|u_1\|^2_{H^1_{\mathcal{L}}(G)}\right) \\
&\lesssim (1+t)^{-3} \left(\|u_0\|^2_{H^1_{\mathcal{L}}(G)} + \|u_1\|^2_{H^1_{\mathcal{L}}(G)}\right). \qquad (3.30)
\end{aligned}
$$

Now we are in a position to prove Theorem 1.1.

*Proof of theorem* 1.1. The proof follows from the estimates (3.21), (3.25), (3.27) and (3.30) for $\|u(t,\cdot)\|_{L^2(G)}$, $\left\|(-\mathcal{L})^{1/2}u(t,\cdot)\right\|_{L^2(G)}$, $\|\partial_t u(t,\cdot)\|_{L^2(G)}$ and $\|\partial_t(-\mathcal{L})^{1/2}u(t,\cdot)\|_{L^2(G)}$, respectively. □

## 4. $L^1(G) - L^2(G)$ estimates for the solution to the homogeneous problem

In this section, we show that there is no improvement of any decay rate for the norm $\|u(t,\cdot)\|_{L^2(G)}$ when further we assume $L^1(G)$-regularity for $u_0$ and $u_1$. Note that in Theorem 1.1, we employed data on $L^2(G)$ basis. Since $G$ is a compact group, the Haar measure of $G$ is finite. This implies that $L^2(G)$ is continuously embedded in $L^1(G)$, and therefore, one might be curious to know which changes will occur if we further implement $L^1(G)$-regularity for $u_0$ and $u_1$.

From (3.14), (3.16) and (3.19), it immediately follows that

$$
\begin{aligned}
\sum_{[\xi]\in\widehat{G}\setminus\mathcal{R}_1} d_\xi \sum_{k,l=1}^{d_\xi} |\widehat{u}(t,\xi)_{kl}|^2 &\lesssim (1+t)^2 e^{-2\delta_1 t} \sum_{[\xi]\in\widehat{G}\setminus\mathcal{R}_1} d_\xi \sum_{k,l=1}^{d_\xi} \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\
&\lesssim (1+t)^2 e^{-2\delta_1 t} \left(\|\widehat{u}_0(\xi)_{kl}\|^2_{L^2(G)} + \|\widehat{u}_1(\xi)_{kl}\|^2_{L^2(G)}\right)
\end{aligned}
$$

for some suitable constant $\delta_1$. Therefore, the contribution to the sum in (3.9) corresponding to $\mathcal{R}_1$ refrains us to get a decay rate for $\|u(t,\cdot)\|^2_{L^2(G)}$. Thus, if we want

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

to employ $L^1(G)$-regularity rather than $L^2(G)$-regularity, then we must apply it to obtain the estimation of the terms with $[\xi] \in \mathcal{R}_1$. Here, we must note that the set $\mathcal{R}_1$ is a singleton.

Note that for the multiplier in (3.5), the best estimate that one can obtain on the set $\mathcal{R}_1$ is

$$|\mathcal{K}_0(t,\xi)\,|, \quad |\,\mathcal{K}_1(t,\xi)\,| \lesssim 1.$$

Since the set $\mathcal{R}_1$ is singleton, using the definition defined in (2.3), we obtain

$$\begin{aligned}\sum_{[\xi]\in\mathcal{R}_1} d_\xi \sum_{k,l=1}^{d_\xi} |\widehat{u}(t,\xi)_{kl}|^2 &\lesssim d_\xi \sum_{k,l=1}^{d_\xi} \left(|\widehat{u}_0(\xi)_{kl}|^2 + |\widehat{u}_1(\xi)_{kl}|^2\right) \\ &\lesssim d_\xi \left(\|\widehat{u}_0(\xi)\|_{\mathrm{HS}}^2 + \|\widehat{u}_1(\xi)\|_{\mathrm{HS}}^2\right) \\ &\lesssim \left(\sup_{[\xi]\in\widehat{G}} d_\xi^{-\frac{1}{2}} \left(\|\widehat{u}_0(\xi)\|_{\mathrm{HS}} + \|\widehat{u}_1(\xi)\|_{\mathrm{HS}}\right)\right)^2 \\ &\lesssim \left(\|\widehat{u}_0(\xi)_{kl}\|_{\ell^\infty(\widehat{G})} + \|\widehat{u}_1(\xi)\|_{\ell^\infty(\widehat{G})}\right)^2 \\ &\lesssim \left(\|\widehat{u}_0(\xi)_{kl}\|_{L^1(G)}^2 + \|\widehat{u}_1(\xi)_{kl}\|_{L^1(G)}^2\right).\end{aligned}$$

This shows that even if we use $L^1(G)$-regularity, we are not able to get any decay rate for the norm $\|u(t,\,\cdot)\|_{L^2(G)}$.

The main reason behind this behaviour is that we cannot neglect the eigenvalue 0 as the Plancherel measure on a compact Lie group turns out to be a weighted counting measure.

Remark 4.1. In the noncompact setting such as the Euclidean space and the Heisenberg group, one can get a global existence result for a nonempty range for $p$ by asking an additional $L^1$-regularity for the initial data. Consequently, we get an improved decay rate for the estimates of the $L^2$-norm of the solution to the corresponding linear homogeneous problem. One can see [**16**, **21**] for the illustration and discussion on this matter.

## 5. Local existence

This section is devoted to prove Theorem 1.4, i.e., the local well-posedness of the Cauchy problem (1.1) in the energy evolution space $\mathcal{C}^1([0,\,T],\,H^1_{\mathcal{L}}(G))$. To present the proof of Theorem 1.4, first, we recall the notion of mild solutions in our setting.

Consider the space

$$X(T) := \mathcal{C}^1\left([0,T], H^1_{\mathcal{L}}(G)\right),$$

equipped with the norm

$$\begin{aligned}\|u\|_{X(T)} := \sup_{t\in[0,T]} \big(&\|u(t,\cdot)\|_{L^2(G)} + \|(-\mathcal{L})^{1/2}u(t,\cdot)\|_{L^2(G)} + \|\partial_t u(t,\cdot)\|_{L^2(G)} \\ &+ \|\partial_t(-\mathcal{L})^{1/2}u(t,\cdot)\|_{L^2(G)}\big).\end{aligned} \tag{5.1}$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

The solution to the nonlinear inhomogeneous problem

$$\begin{cases} \partial_t^2 u - \mathcal{L}u + \partial_t u - \mathcal{L}\partial_t u = F(t,x), & x \in G, t > 0, \\ u(0,x) = u_0(x), & x \in G, \\ \partial_t u(0,x) = u_1(x), & x \in G, \end{cases} \tag{5.2}$$

can be expressed, by using Duhamel's principle, as

$$u(t,x) := u_0(x) *_{(x)} E_0(t,x) + u_1(x) *_{(x)} E_1(t,x) + \int_0^t F(s,x) *_{(x)} E_1(t-s,x)\, \mathrm{d}s,$$

where $*_{(x)}$ denotes the convolution with respect to the $x$ variable, $E_0(t,\, x)$ and $E_1(t,\, x)$ are the fundamental solutions to the homogeneous problem (5.2), i.e., when $F = 0$ with initial data $(u_0,\, u_1) = (\delta_0,\, 0)$ and $(u_0,\, u_1) = (0,\, \delta_0)$, respectively. For any left-invariant differential operator $L$ on the compact Lie group $G$, we applied the property that $L(v *_{(x)} E_1(t,\, \cdot)) = v *_{(x)} L(E_1(t,\, \cdot))$ and the invariance by time translations for the viscoelastic wave operator $\partial_t^2 - \mathcal{L} + \partial_t - \mathcal{L}\partial_t$ in order to get the previous representation formula.

Definition 5.1. *The function $u$ is said to be a mild solution to* (5.2) *on* $[0,\, T]$ *if $u$ is a fixed point for the integral operator $N : u \in X(T) \to Nu(t,\, x)$ defined as*

$$\begin{aligned} Nu(t,x) &= \varepsilon u_0(x) *_{(x)} E_0(t,x) + \varepsilon u_1(x) *_{(x)} E_1(t,x) \\ &\quad + \int_0^t |u(s,x)|^p *_{(x)} E_1(t-s,x)\, \mathrm{d}s \end{aligned} \tag{5.3}$$

*in the evolution space $\mathcal{C}^1([0,\, T],\, H^1_{\mathcal{L}}(G))$, equipped with the norm defined in* (5.1).

As usual, the proof of the fact that the map $N$ admits a uniquely determined fixed point for sufficiently small $T = T(\varepsilon)$ is based on Banach's fixed point theorem with respect to the norm on $X(T)$ as defined above. More importantly, for $\|(u_0,\, u_1)\|_{H^1_{\mathcal{L}}(G)\times H^1_{\mathcal{L}}(G)}$ small enough, if we can show the validity of the following two inequalities:

$$\|Nu\|_{X(T)} \leqslant C\,\|(u_0,u_1)\|_{H^1_{\mathcal{L}}(G)\times H^1_{\mathcal{L}}(G)} + C\|u\|^p_{X(T)},$$

$$\|Nu - Nv\|_{X(T)} \leqslant C\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)} + \|v\|^{p-1}_{X(T)}\right),$$

for any $u,\, v \in X(T)$ and for some suitable constant $C > 0$ independent of $T$. Then by Banach's fixed point theorem, we can ensure that the operator $N$ admits a unique fixed point $u$. This function $u$ will be the mild solution to (5.2) on $[0,\, T]$.

In order to prove the local existence result, an important tool is the following Gagliardo–Nirenberg type inequality which can be derived from the general version of this inequality given in [**29**]. We also refer [**29**] for the detailed proof of this inequality for more general connected unimodular Lie groups.

LEMMA 5.2. *Let $G$ be a (connected) compact Lie group with topological dimension $n \geqslant 3$. Assume that $q \geqslant 2$ such that $q \leqslant \frac{2n}{n-2}$. Then the following Gagliardo–Nirenberg type inequality holds*

$$\|f\|_{L^q(G)} \lesssim \|f\|_{H^1_{\mathcal{L}}(G)}^{\theta(n,q)} \|f\|_{L^2(G)}^{1-\theta(n,q)} \tag{5.4}$$

*for all $f \in H^1_{\mathcal{L}}(G)$, where $\theta(n, q) = n(\frac{1}{2} - \frac{1}{q})$.*

One can also consult [**21**, **29**] for several immediate important remarks.

*Proof of theorem* 1.1. Expression (5.3) can be written as $Nu = u^\sharp + I[u]$, where

$$u^\sharp(t, x) = \varepsilon u_0(x) *_{(x)} E_0(t, x) + \varepsilon u_1(x) *_{(x)} E_1(t, x)$$

and

$$I[u](t, x) := \int_0^t |u(s, x)|^p *_x E_1(t - s, x)\, \mathrm{d}s.$$

Now for the part $u^\sharp$, from Theorem 1.1, immediately it follows that

$$\|u^\sharp\|_{X(T)} \lesssim \varepsilon \|(u_0, u_1)\|_{H^1_{\mathcal{L}}(G) \times H^1_{\mathcal{L}}(G)}. \tag{5.5}$$

On the other hand, for the part $I[u]$, using Minkowski's integral inequality, Young's convolution inequality, Gagliardo–Nirenberg type inequality (5.4), Theorem 1.1 and by time translation invariance property of the Cauchy problem (1.1), we get

$$\begin{aligned}
\|\partial_t^j (-\mathcal{L})^{i/2} I[u]\|_{L^2(G)} &= \left( \int_G \left| \partial_t^j (-\mathcal{L})^{i/2} \int_0^t |u(s, x)|^p *_x E_1(t - s, x)\, \mathrm{d}s \right|^2 dg \right)^{\frac{1}{2}} \\
&= \left( \int_G \left| \int_0^t |u(s, x)|^p *_x \partial_t^j (-\mathcal{L})^{i/2} E_1(t - s, x)\, \mathrm{d}s \right|^2 dg \right)^{\frac{1}{2}} \\
&\lesssim \int_0^t \| |u(s, \cdot)|^p *_x \partial_t^j (-\mathcal{L})^{i/2} E_1(t - s, \cdot)\|_{L^2(G)}\, \mathrm{d}s \\
&\lesssim \int_0^t \|u(s, \cdot)^p\|_{L^2(G)} \|\partial_t^j (-\mathcal{L})^{i/2} E_1(t - s, \cdot)\|_{L^2(G)}\, \mathrm{d}s \\
&\lesssim \int_0^t (1 + t - s)^{-j - \frac{i}{2}} \|u(s, \cdot)\|_{L^{2p}(G)}^p\, \mathrm{d}s \\
&\lesssim \int_0^t \|u(s, \cdot)\|_{H^1_{\mathcal{L}}(G)}^{p\theta(n,2p,)} \|u(s, \cdot)\|_{L^2(G)}^{p(1-\theta(n,2p))}\, \mathrm{d}s \\
&\lesssim t \|u\|_{X(t)}^p,
\end{aligned} \tag{5.6}$$

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press

for all $(i, j) \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. Again for $(i, j) \in \{(0, 0), (1, 0), (0, 1), (1, 1)\}$, similar calculations as in (5.6) together with Holder's inequality, we get

$$\begin{aligned}
&\|\partial_t^j (-\mathcal{L})^{i/2} \left(I[u] - I[v]\right) \|_{L^2(G)} \\
&\quad \lesssim \int_0^t (1+t-s)^{-j-\frac{i}{2}} \| |u(s,\cdot)|^p - |v(s,\cdot)|^p \|_{L^2(G)} \, \mathrm{d}s \\
&\quad \lesssim \int_0^t \|u(s,\cdot) - v(s,\cdot)\|_{L^{2p}(G)} \left( \|u(s,\cdot)\|^{p-1}_{L^{2p}(G)} + \|v(s,\cdot)\|^{p-1}_{L^{2p}(G)} \right) \mathrm{d}s \\
&\quad \lesssim t\|u - v\|_{X(t)} \left( \|u\|^{p-1}_{X(t)} - \|v\|^{p-1}_{X(t)} \right). 
\end{aligned} \tag{5.7}$$

Thus, combining (5.5), (5.6) and (5.7), we have

$$\|Nu\|_{X(t)} \leqslant D\varepsilon \left\|(u_0, u_1)\right\|_{H^1_{\mathcal{L}}(G) \times H^1_{\mathcal{L}}(G)} + DT\|u\|^p_{X(t)} \tag{5.8}$$

and

$$\|Nu - Nv\|_{X(T)} \leqslant DT\|u - v\|_{X(t)} \left( \|u\|^{p-1}_{X(T)} - \|v\|^{p-1}_{X(T)} \right). \tag{5.9}$$

Thus, for sufficiently small $T$, the map $N$ turns out to be a contraction in some neighbourhood of 0 in the Banach space $X(T)$. Therefore, it follows from Banach's fixed point theorem that there exists a uniquely determined fixed point $u$ of the map $N$. This fixed point $u$ is the mild solution to system (1.1) on $[0, t] \subset [0, T]$. □

**Acknowledgements**

Arun Kumar Bhardwaj acknowledges IIT Guwahati for providing financial support. The second author was supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations, the Methusalem programme of the Ghent University Special Research Fund (BOF) (grant number 01M01021) and by FWO Senior Research Grant G011522N. The third author was supported by Core Research Grant (RP03890G), Science and Engineering Research Board (SERB), DST, India.

**Data availability statement**

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

**Conflict of interest**

None.

## References

1 W. Chen. Interplay effects on blow-up of weakly coupled systems for semilinear wave equations with general nonlinear memory terms. *Nonlinear Anal.* **202** (2021), 112160.

2 R. Chill and A. Haraux. An optimal estimate for the difference of solutions of two abstract evolution equations. *J. Differ. Eq.* **193** (2003), 385–395.

3 C. R. da Luz, R. Ikehata and R. C. Charao. Asymptotic behavior for abstract evolution differential equations of second order. *J. Differ. Eq.* **259** (2015), 5017–5039.

4 M. D'Abbicco and M. R. Ebert. Diffusion phenomena for the wave equation with structural damping in the $l^p - l^q$ framework. *J. Differ. Eq.* **256** (2014), 2307–2336.

5 A. Dasgupta, V. Kumar and S. S. Mondal. Nonlinear fractional wave equation on compact Lie groups. preprint arXiv:2207.04422 (2022).

6 V. Fischer and M. Ruzhansky. *Quantization on nilpotent Lie groups*. Progress in Mathematics, vol. 314. (Birkhäuser/Springer [Cham], Springer Nature, 2016).

7 C. Garetto and M. Ruzhansky. Wave equation for sums of squares on compact Lie groups. *J. Differ. Eq.* **258** (2015), 4324–4347.

8 T. Hosono. *Asymptotic behavior of solutions for nonlinear partial differential equations with dissipation*. PhD thesis, Doctoral Thesis, Kyushu University, 2006.

9 R. Ikehata. New decay estimates for linear damped wave equations and its application to nonlinear problem. *Math. Methods Appl. Sci.* **27** (2004), 865–889.

10 R. Ikehata. Asymptotic profiles for wave equations with strong damping. *J. Differ. Eq.* **257** (2014), 2159–2177.

11 R. Ikehata. Some remarks on the asymptotic profiles of solutions for strongly damped wave equations on the 1-d half space. *J. Math. Anal. Appl.* **421** (2015), 905–916.

12 R. Ikehata, Y. Miyaoka and T. Nakatake. Decay estimates of solutions for dissipative wave equations in $\mathbb{R}^n$ with lower power nonlinearities. *J. Math. Soc. Japan* **56** (2004), 365–373.

13 R. Ikehata and A. Sawada. Asymptotic profile of solutions for wave equations with frictional and viscoelastic damping terms. *Asymptot. Anal.* **98** (2016), 59–77.

14 R. Ikehata and H. Takeda. Critical exponent for nonlinear wave equations with frictional and viscoelastic damping terms. *Nonlinear Anal.* **148** (2017), 228–253.

15 R. Ikehata, G. Todorova and B. Yordanov. Wave equations with strong damping in Hilbert spaces. *J. Differ. Eq.* **254** (2013), 3352–3368.

16 Y. Liu, Y. Li and J. Shi. Estimates for the linear viscoelastic damped wave equation on the Heisenberg group. *J. Differ. Eq.* **285** (2021), 663–685.

17 X. Lu and M. Reissig. Rates of decay for structural damped models with decreasing in time coefficients. *Int. J. Dyn. Syst. Differ. Equ.* **2** (2009), 21–55.

18 A. Matsumura. On the asymptotic behavior of solutions of semi-linear wave equations. *Publ. Res. Inst. Math. Sci.* **12** (1976), 169–189.

19 A. I. Nachman. The wave equation on the Heisenberg group. *Comm. Partial Differ. Eq.* **7** (1982), 675–714.

20 T. Narazaki. $l^p - l^q$ estimates for damped wave equations and their applications to semi-linear problem. *J. Math. Soc. Japan* **56** (2004), 585–626.

21 A. Palmieri. On the blow–up of solutions to semilinear damped wave equations with power nonlinearity in compact Lie groups. *J. Differ. Eq.* **281** (2021), 85–104.

22 A. Palmieri. Semilinear wave equation on compact Lie groups. *J. Pseudo-Differ. Oper. Appl.* **12** (2021), 43.

23 A. Palmieri. A global existence result for a semilinear wave equation with lower order terms on compact Lie groups. *J. Fourier Anal. Appl.* **28** (2022), 21.

24 G. Ponce. Global existence of small solutions to a class of nonlinear evolution equations. *Nonlinear Anal.* **9** (1985), 399–418.

25 M. Ruzhansky and C. A. Taranto. Time-dependent wave equations on graded groups. *Acta Appl. Math.* **171** (2021), 21.

26 M. Ruzhansky and N. Tokmagambetov. Nonlinear damped wave equations for the sub-laplacian on the Heisenberg group and for Rockland operators on graded Lie groups. *J. Differ. Eq.* **265** (2018), 5212–5236.

27 M. Ruzhansky and V. Turunen. *Pseudo-differential operators and symmetries: background analysis and advanced topics*. Pseudo-Differential Operators. Theory and Applications, vol. 2. (Birkhäuser Verlag, Springer Science & Business Media, 2009).

28 M. Ruzhansky and V. Turunen. Global quantization of pseudo-differential operators on compact Lie groups, su (2), 3-sphere, and homogeneous spaces. *Int. Math. Res. Not. IMRN* **2013** (2013), 2439–2496.

29 M. Ruzhansky and N. Yessirkegenov. Hardy, Hardy-Sobolev, Hardy-Littlewood-Sobolev and Caffarelli-Kohn-Nirenberg inequalities on general Lie groups. preprint arXiv:1810.08845 (2018).

30 M. Ruzhansky and N. Yessirkegenov. Very weak solutions to hypoelliptic wave equations. *J. Differ. Eq.* **268** (2020), 2063–2088.

31 Y. Shibata. On the rate of decay of solutions to linear viscoelastic equation. *Math. Methods Appl. Sci.* **23** (2000), 203–226.

32 C. A. Taranto. Wave equations on graded groups and hypoelliptic Gevrey spaces. preprint arXiv:1804.03544 (2018).

33 G. Todorova and B. Yordanov. Critical exponent for a nonlinear wave equation with damping. *J. Differ. Eq.* **174** (2001), 464–489.

https://doi.org/10.1017/prm.2023.38 Published online by Cambridge University Press