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ON CLOSURE AND FACTORIZATION PROPERTIES OF SUBEXPONENTIAL AND RELATED DISTRIBUTIONS

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Abstract

For a distribution function F on $[0,\infty)$ we say $F\in\mathscr{S}$ if $\{1-F^{(2)}(x)\}/\{1-F(x)\}\to 2$ as $x\to\infty$, and $F\in\mathscr{L}_\gamma$ if for some fixed $\gamma\geqslant 0$, and for each real y, $\lim_{x\to\infty}\{1-F(x+y)\}/\{1-F(x)\}=e^{-\gamma y}$. Sufficient conditions are given for the statement $F\in\mathscr{S}\Leftrightarrow F*G\in\mathscr{S}$, and when both F and G are in $\mathscr{S}$ it is proved that $F*G\in\mathscr{S}\Leftrightarrow pF+(1-p)G\in\mathscr{S}$ for some (all) $p\in(0,1)$. The related classes $\mathscr{L}_\gamma$ are proved closed under convolutions, which implies the closure of the class of positive random variables with regularly varying tails under *multiplication* (of random variables). An example is given that shows $\mathscr{S}$ to be a *proper* subclass of $\mathscr{L}_0$.

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1. Statement of results and discussion

We consider probability distribution functions F on $[0,\infty)$ satisfying $F(0)=0$, $F(\infty)=1$, $F(x)<1$ for $x<\infty$, and will write $\bar{F}$ for the tail function $1-F$, similarly $\bar{F}^{(2)}$ for $1-F^{(2)}$, $\overline{F*G}$ for $1-F*G$, and so on, where $F^{(2)}$ denotes the convolution $F*F$.

DEFINITION. F and G are said to be *max-sum-equivalent*, written $F\sim_M G$ if

$$\overline{F*G}(x)\sim\bar{F}(x)+\bar{G}(x),\quad x\to\infty. \tag{1.1}$$

Note that $\bar{F}(x)+\bar{G}(x)\sim\bar{F}(x)+\bar{G}(x)-\bar{F}(x)\,\bar{G}(x)$, so that if X and Y are independent random variables with distribution functions F, G respectively, we may rewrite (1.1) as

$$P(X+Y>x)\sim P(\max(X,Y)>x),\quad x\to\infty.$$

Hence the terminology.

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DEFINITION. F is said to belong to the *subexponential class* $\mathscr{S}$ if $F \sim_M F$.

DEFINITION. For each $\gamma \geqslant 0$, the class $\mathscr{L}_\gamma$ is the set of F for which for each fixed real y, $\lim_{x\to\infty} \bar{F}(x+y)/\bar{F}(x) = e^{-\gamma y}$. Write $\mathscr{L}$ for $\mathscr{L}_0$.

The class $\mathscr{S}$ originated in branching processes (Chistyakov (1964)) has found applications in queues, random walks and transient renewal theory, and has recently entered the theory of infinite divisibility, where (Embrechts *et al.* (1979)) it characterizes those infinitely divisible distributions on $[0, \infty)$ whose tail functions are asymptotically equal to the tails of their Lévy measures. See the last-mentioned article for references to the other applications. Work on applications has led to interest in, and discovery of, many properties of $\mathscr{S}$, but one major conjecture remains unverified, that $\mathscr{S}$ is closed under convolutions: $F \in \mathscr{S}$, $G \in \mathscr{S} \Rightarrow_? F*G \in \mathscr{S}$. We have not solved this problem, but give some partial results about it, and relate it to the closure of $\mathscr{S}$ under finite mixing. Our results also give some sufficient conditions for the converse proposition, that $F*G \in \mathscr{S}$, $G \in \mathscr{S} \Rightarrow F \in \mathscr{S}$. In this connection it is appropriate to mention that there is no problem when $F = G$, because for any positive integer n, $F \in \mathscr{S} \Leftrightarrow F^{(n)} \in \mathscr{S}$ (see Embrechts *et al.* (1979), the left-to-right implication being Chistyakov's). This carries over to when $\bar{G}(x) \sim c\bar{F}(x)$ for some $c > 0$, because $\mathscr{S}$ is known to be closed under tail equivalence.

THEOREM 1. *Let* $F \in \mathscr{L}$, $G \in \mathscr{S}$, *and* $\sup_x \bar{G}(x)/\bar{F}(x) < \infty$. *Then* $F \in \mathscr{S} \Leftrightarrow F*G \in \mathscr{S}$.

A related result is in Pitman (1979): if F and G have densities whose ratio G'/F' is bounded, then $F \in \mathscr{S} \Rightarrow F*G \in \mathscr{S}$.

If the boundedness of $\bar{G}/\bar{F}$ in Theorem 1 is strengthened to the requirement $\lim_{x\to\infty} \bar{G}(x)/\bar{F}(x) = 0$ then the condition $F \in \mathscr{L}$ can be omitted (Embrechts *et al.* (1979), Proposition 1). However this is not so in general, for in some circumstances the conclusion $F*G \in \mathscr{S}$ *implies* $F \in \mathscr{L}$, as follows.

PROPOSITION 1. *If* $\sup_x \bar{G}(x)/\bar{F}(x) < \infty$, $F*G \in \mathscr{S}$, *and* $F \cdot G \in \mathscr{L}$, *then* $F \in \mathscr{L}$.

For the next result on convolution closure, $\mathscr{D}$ is to be the class of F with dominated-variation tails: $\sup_x \bar{F}(\tfrac{1}{2}x)/\bar{F}(x) < \infty$.

PROPOSITION 2. *If* $F \in \mathscr{D} \cap \mathscr{S}$ *and* $G \in \mathscr{D} \cap \mathscr{S}$ *then* $F*G \in \mathscr{D} \cap \mathscr{S}$.

We now give equivalent forms of convolution closure.

THEOREM 2. *Let* $F \in \mathscr{S}$, $G \in \mathscr{S}$ *and* $H = F*G$. *Then* $\limsup_{x\to\infty} \bar{H}^{(2)}(x)/\bar{H}(x) \leqslant 4$, *and the following are equivalent:*

(i) $H \in \mathscr{S}$,

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(ii) $F \sim_M G$,
(iii) $pF+(1-p)G \in \mathscr{S}$ *for some (all) p satisfying* $0<p<1$.

An immediate consequence is that the following three conjectures are equivalent (all true or all false):
(a) $\mathscr{S}$ is closed under convolutions;
(b) $\mathscr{S}$ is closed under finite mixtures;
(c) $F \in \mathscr{S},\ G \in \mathscr{S} \Rightarrow F \sim_M G$.
As regards finite mixtures, Pitman (1979) proves that under the density condition quoted above, $F \in \mathscr{S}$ implies $pF+(1-p)G \in \mathscr{S}$.

We turn to the classes $\mathscr{L}_\gamma$. It is well known that $\mathscr{S} \subset \mathscr{L}$, and $\mathscr{L}$ plays an important role in the study of $\mathscr{S}$. The classes $\mathscr{L}_\gamma$ for $\gamma>0$ bear a similar relation to the classes $\mathscr{S}_\gamma$ of Chover *et al.* (1973). We prove convolution closure. For this theorem only we drop the restriction that the distribution functions have support in $[0, \infty)$, that is we no longer insist that $F(0)=0$, $G(0)=0$. The definition of $\mathscr{L}_\gamma$ is unaltered, and γ can be any nonnegative number. Let $\mathscr{R}_\alpha$ denote the class of functions regularly varying at ∞ with exponent α.

THEOREM 3. (a) *If* $F \in \mathscr{L}_\gamma$ and $\bar{G}=o(\bar{F})$, *in particular if* $G \in \mathscr{L}_{\gamma'}$ *for* $\gamma'>\gamma$, *then* $F*G \in \mathscr{L}_\gamma$.
(b) *If* $F \in \mathscr{L}_\gamma$ *and* $G \in \mathscr{L}_\gamma$ *then* $F*G \in \mathscr{L}_\gamma$.

COROLLARY. *Let* X^* *and* Y^* *be independent positive random variables with distribution functions* F^*, G^* *respectively, and let* H^* *be the distribution function of the product* X^*Y^*. *Let* $\bar{F}^* \in \mathscr{R}_{-\gamma}$. *Then* $\bar{H}^* \in \mathscr{R}_{-\gamma}$ *if either* (a) $\bar{G}^*=o(\bar{F}^*)$, *or* (b) $\bar{G}^* \in \mathscr{R}_{-\gamma}$.

It is of interest to compare Theorem 3 with the corresponding well-knowns result in which $\mathscr{L}_\gamma$, $\mathscr{L}_{\gamma'}$ are replaced by $\mathscr{R}_{-\gamma}$, $\mathscr{R}_{-\gamma'}$. In that case one concludes not just that $\bar{F}*\bar{G} \in \mathscr{R}_{-\gamma}$ but also that $\bar{F}*\bar{G} \sim \bar{F}+\bar{G}$. However in the circumstances of Theorem 3 one does not have that sort of extra information, because if for instance $G=F$ and $\gamma=0$, then the conclusion $\bar{F}*\bar{G} \sim \bar{F}+\bar{G}$ would mean that $F \in \mathscr{S}$, and we know (see below) that $\mathscr{S}$ is a *proper* subclass of $\mathscr{L}_0$.

The classes $\mathscr{R}_{-\gamma}$, $\mathscr{S}$, $\mathscr{D}$ are closed under convolution roots (see Embrechts *et al.* (1979) for the first two; the case of $\mathscr{D}$ is elementary). We conjecture that for each $\gamma \geqslant 0$, $\mathscr{L}_\gamma$ is closed under convolution roots: $F^{(k)} \in \mathscr{L}_\gamma \Rightarrow_? F \in \mathscr{L}_\gamma$.

The corollary may be related to domains of attraction. First, $\mathscr{R}_{-\gamma}$ for $\gamma>0$ is the class of tail functions of the domain of attraction for *maxima* of Gnedenko's canonical law Φ_γ. Second, $\mathscr{R}_{-\gamma}$ for $0<\gamma<1$ is the class of tail functions of non-negative random variables in the domain of attraction for *sums* of a completely asymmetric stable law of exponent γ. Thus the corollary asserts the closure of these domains of attraction under *multiplication* by certain random variables, or (the same thing), under the taking of certain scale mixtures.

One may without difficulty relax the positivity of X^* and Y^* required by the corollary, and instead allow Y^* to take any real values and X^* to be nonnegative. The conclusion then applies to the upper tail of X^*Y^*. If the same condition as on $\bar{G}^*$ is imposed on $G^*(-x)$ as $x\to\infty$, then one also obtains a lower-tail conclusion. However, there is not necessarily any *balance* between the tails, so the extended corollary is not a domain-of-attraction theorem. Under more stringent conditions one can balance the tails; see Breiman (1965), Proposition 3.

Another view of the corollary is obtained by realizing that $\bar{H}^*$ is the Mellin–Stieltjes convolution of $\bar{G}^*$ and $\bar{F}^*$:

$$\bar{H}^*(x) = \int_0^\infty \bar{G}^*(x/t)\, F^*(dt).$$

The corollary is thus an Abelian theorem for Mellin–Stieltjes convolutions, of a rather different sort, it appears, from those mainly studied (see Bingham and Teugels (1979)).

Our final result, forming Section 3, is an example of a distribution function in $\mathscr{L}\setminus\mathscr{S}$. We found this example last year, but it then seemed over-complicated in comparison with $\bar{F}_\beta(x)=\exp\{-x(\log x)^{-\beta}\}$, $x\geqslant 1$, which was thought to be an element of $\mathscr{L}\setminus\mathscr{S}$ when $0<\beta\leqslant 1$. However Pitman (1979) shows that in fact $F_\beta\in\mathscr{S}$ for each $\beta>0$. Pitman also gives a new example in $\mathscr{L}\setminus\mathscr{S}$, and his example and ours are therefore the only known elements of that class. Neither is particularly simple.

2. Proofs

Denote $\max(x,y)$ by $x\vee y$ and $\min(x,y)$ by $x\wedge y$. The operators $\vee$, $\wedge$ are to bind more tightly than $+$, $-$. Intervals of integration will include {exclude} finite right {left} endpoints, unless otherwise indicated. We always write H for $F*G$ and K for $F\cdot G$. Independent random variables X, X_1, X_2, Y, Y_1, Y_2 will be used to aid explanations, the X's each having distribution function F, the Y's each having distribution function G. Note that

$$\bar{K}(x)=P(X\vee Y>x)=\bar{F}(x)+\bar{G}(x)-\bar{F}(x)\,\bar{G}(x)\sim\bar{F}(x)+\bar{G}(x),\quad x\to\infty.$$

LEMMA 1. *If $K\in\mathscr{L}$ and $H\in\mathscr{S}$ then $F\sim_M G$.*

PROOF. Define $\hat{X}=X_1\vee X_2$, $\hat{Y}=Y_1\vee Y_2$, and denote their distribution functions by F_2, G_2 respectively, so that

$$\bar{F}_2=(2-\bar{F})\bar{F},\quad \bar{G}_2=(2-\bar{G})\bar{G}. \tag{2.1}$$

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Fix $w>0$ and $x \geqslant 2w$, then

$$(2.2) \quad \begin{aligned} \bar{H}(x) &= P(X_1 \leqslant w, X_1+Y_1>x)+P(Y_1 \leqslant w, X_1+Y_1>x) \\ &\qquad +P(X_1>w, Y_1>w, X_1+Y_1>x) \\ &\leqslant P(Y_1>x-w)+P(X_1>x-w)+P(X_1>w, Y_1>w, X_1+Y_1>x) \end{aligned}$$

Next,

$$(2.3) \quad \begin{aligned} \bar{H}^{(2)}(x) &= P(X_1+Y_1+X_2+Y_2>x) \\ &\geqslant P(\hat{X}+\hat{Y}>x) \\ &= P(\hat{X} \leqslant w, \hat{X}+\hat{Y}>x)+P(\hat{Y} \leqslant w, \hat{X}+\hat{Y}>x) \\ &\qquad +P(\hat{X}>w, \hat{Y}>w, \hat{X}+\hat{Y}>x) \\ &\geqslant P(\hat{X} \leqslant w, \hat{Y}>x)+P(\hat{Y} \leqslant w, \hat{X}>x)+P(\hat{X}>w, \hat{Y}>w, \hat{X}+\hat{Y}>x) \\ &= F_2(w)\,\bar{G}_2(x)+G_2(w)\,\bar{F}_2(x)+\int_w^\infty \bar{G}_2((x-u)\vee w)\,F_2(du), \end{aligned}$$

and then, using (2.1),

$$(2.4) \quad \begin{aligned} \int_w^\infty \bar{G}_2((x-u)\vee w)\,F_2(du) &\geqslant \{2-\bar{G}(w)\}\int_w^\infty \bar{G}((x-u)\vee w)\,F_2(du) \\ &= \{2-\bar{G}(w)\}\,P(\hat{X}>w, Y>w, \hat{X}+Y>x) \\ &= \{2-\bar{G}(w)\}\int_w^\infty \bar{F}_2((x-u)\vee w)\,G(du) \\ &\geqslant \{2-\bar{G}(w)\}\{2-\bar{F}(w)\}\int_w^\infty \bar{F}((x-u)\vee w)\,G(du) \\ &= \{2-\bar{F}(w)\}\{2-\bar{G}(w)\}\,P(X>w, Y>w, X+Y>x) \end{aligned}$$

Combining (2.2), (2.3) and (2.4),

$$\begin{aligned} \bar{H}^{(2)}(x) &\geqslant F_2(w)\,\bar{G}_2(x)+G_2(w)\,\bar{F}_2(x) \\ &\qquad +\{2-\bar{F}(w)\}\{2-\bar{G}(w)\}\{\bar{H}(x)-\bar{F}(x-w)-\bar{G}(x-w)\} \\ &\geqslant [\{F_2(w)(2-\bar{G}(x))\}\wedge\{G_2(w)(2-\bar{F}(x))\}]\{\bar{F}(x)+\bar{G}(x)\} \\ &\qquad +\{2-\bar{F}(w)\}\{2-\bar{G}(w)\}\bar{H}(x)-4\{\bar{F}(x-w)+\bar{G}(x-w)\}, \end{aligned}$$

using (2.1). Rearranging,

$$\begin{aligned} [\{2-\bar{F}(w)\}\{2-\bar{G}(w)\}-\bar{H}^{(2)}(x)/\bar{H}(x)]\,\bar{H}(x) &\leqslant 4\{\bar{F}(x-w)+\bar{G}(x-w)\} \\ &\qquad -[\{F_2(w)(2-\bar{G}(x))\}\wedge\{G_2(w)(2-\bar{F}(x))\}]\{\bar{F}(x)+\bar{G}(x)\}. \end{aligned}$$

Divide through by $\bar{K}(x)$ and let $x\to\infty$. We have

$$\bar{F}(x-w)+\bar{G}(x-w)\sim\bar{K}(x-w)\sim\bar{K}(x) \quad \text{since } K\in\mathscr{L}.$$

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Also $\bar{H}^{(2)}(x)/\bar{H}(x)\to 2$ by our assumption that $H\in\mathscr{S}$. Hence

$$[\{2-\bar{F}(w)\}\{2-\bar{G}(w)\}-2]\limsup_{x\to\infty}\bar{H}(x)/\bar{K}(x)\leqslant 4-2\{F_2(w)\wedge G_2(w)\}.$$

Now let $w\to\infty$, and then $\limsup_{x\to\infty}\bar{H}(x)/\bar{K}(x)\leqslant 1$. However $\bar{H}\geqslant\bar{K}$ and so the lemma is proved.

LEMMA 2. *If $G\in\mathscr{L}$, $F\in\mathscr{S}$ and $c\equiv\sup_x\bar{G}(x)/\bar{F}(x)<\infty$, then $F\sim_M G$.*

PROOF. For any fixed $w>0$,

$$\int_w^\infty\{\bar{F}(x-t)/\bar{F}(x)\}F(dt)=\bar{F}^{(2)}(x)/\bar{F}(x)-1-\int_{0-}^{w}\{\bar{F}(x-t)/\bar{F}(x)\}F(dt)$$
$$\to 2-1-F(w),\quad x\to\infty,$$

the terms on the right converging by subexponentiality and dominated convergence, respectively. Then

$$\begin{aligned}\bar{F}*\bar{G}(x)&=\bar{F}(x)+\int_0^w\bar{G}(x-t)F(dt)+\int_w^x\bar{G}(x-t)F(dt)\\&\leqslant\bar{F}(x)+\bar{G}(x-w)+\bar{F}(x).c\int_w^x\{\bar{F}(x-t)/\bar{F}(x)\}F(dt)\\&\leqslant\{\bar{F}(x)+\bar{G}(x)\}.\left(\left[1+c\int_w^x\{\bar{F}(x-t)/\bar{F}(x)\}F(dt)\right]\vee\{\bar{G}(x-w)/\bar{G}(x)\}\right).\end{aligned}$$

Consequently

$$\limsup_{x\to\infty}\bar{F}*\bar{G}(x)/\{\bar{F}(x)+\bar{G}(x)\}\leqslant 1+c\bar{F}(w).$$

Let $w\to\infty$, then $\limsup_{x\to\infty}\bar{F}*\bar{G}(x)/\{\bar{F}(x)+\bar{G}(x)\}\leqslant 1$ and the lemma follows.

PROOF OF THEOREM 1. (a) *Suppose*

$$F\in\mathscr{L},\quad G\in\mathscr{L},\quad H\in\mathscr{S}\quad\textit{and}\quad c\equiv\sup_x\bar{G}(x)/\bar{F}(x)<\infty.$$

By Lemma 1, $\bar{H}(x)\sim\bar{F}(x)+\bar{G}(x)$. Since $H\in\mathscr{S}$ we have $H^{(2)}\in\mathscr{S}$; also $F^{(2)}\in\mathscr{L}$ and $G^{(2)}\in\mathscr{L}$ by Theorem 3, and so the distribution function of

$$(X_1+X_2)\vee(Y_1+Y_2),$$

whose tail is asymptotically equal to $\bar{F}^{(2)}+\bar{G}^{(2)}$, is also in $\mathscr{L}$. So Lemma 1 applies to $F^{(2)}$, $G^{(2)}$ and $H^{(2)}=F^{(2)}*G^{(2)}$, giving $\bar{H}^{(2)}(x)\sim\bar{F}^{(2)}(x)+\bar{G}^{(2)}(x)$. Since $\bar{H}^{(2)}\sim 2\bar{H}$ it follows that

$$\{\bar{F}^{(2)}(x)+\bar{G}^{(2)}(x)\}/\{\bar{F}(x)+\bar{G}(x)\}\to 2,\quad x\to\infty.$$

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So for all large x the left side is at most $2+\varepsilon$, and hence

$$\bar{F}^{(2)}(x)/\bar{F}(x) \leqslant 2+\varepsilon+\{2+\varepsilon-\bar{G}^{(2)}(x)/\bar{G}(x)\}\,\bar{G}(x)/\bar{F}(x).$$

Since $\liminf_x \bar{G}^{(2)}(x)/\bar{G}(x) \geqslant 2$ (Chistyakov (1964)) we have

$$\limsup_x \bar{F}^{(2)}(x)/\bar{F}(x) \leqslant 2+\varepsilon+(2+\varepsilon-2)\,c.$$

Hence this lim sup is at most 2, whence $F\in\mathscr{S}$.

(b) *Suppose* $F\in\mathscr{S}$, $G\in\mathscr{S}$ *and* $\sup_x \bar{G}(x)/\bar{F}(x)<\infty$. Now $F^{(2)}\in\mathscr{S}$, $G^{(2)}\in\mathscr{S}$, and $\bar{F}^{(2)}\sim 2\bar{F}$, $\bar{G}^{(2)}\sim 2\bar{G}$, whence $\sup_x \bar{G}^{(2)}(x)/\bar{F}^{(2)}(x)<\infty$. By Lemma 2, $F^{(2)}\sim_M G^{(2)}$. Then

$$\begin{aligned}\limsup_{x\to\infty} \bar{H}^{(2)}(x)/\bar{H}(x) &= \limsup_{x\to\infty}\{\bar{F}^{(2)}(x)+\bar{G}^{(2)}(x)\}/\bar{H}(x)\\ &\leqslant \limsup_{x\to\infty}\{\bar{F}^{(2)}(x)+\bar{G}^{(2)}(x)\}/\{\bar{F}(x)+\bar{G}(x)\}\\ &=2\end{aligned}$$

since $F\in\mathscr{S}$, $G\in\mathscr{S}$. Thus $H\in\mathscr{S}$, concluding the proof.

NOTE. In (a), that is, the '$\Leftarrow$' statement of Theorem 1, we used only $G\in\mathscr{L}$ rather than the stronger assumption $G\in\mathscr{S}$.

PROOF OF PROPOSITION 1. By Lemma 1, $F\sim_M G$. Suppose $F\notin\mathscr{L}$, then for some $t_0>0$ there exists $x_n\to\infty$ such that $\bar{F}(x_n-t_0)/\bar{F}(x_n)\to l>1$ as $n\to\infty$. Passing to a subsequence, we may take it that also $\bar{G}(x_n)/\bar{F}(x_n)\to c<\infty$. Define

$$\mu(t)=\liminf_n \bar{F}(x_n-t)/\bar{F}(x_n)\geqslant 1,$$

then $\mu(t)\geqslant l$ for all $t\geqslant t_0$, so $\int \mu(t)\,G(dt)>1$. We have

$$\bar{H}(x_n)=\bar{G}(x_n)+\bar{F}(x_n)\int_{0-}^{x_n}\{\bar{F}(x_n-t)/\bar{F}(x_n)\}\,G(dt)$$

whence

$$\begin{aligned}&\liminf_n \bar{H}(x_n)/\{\bar{F}(x_n)+\bar{G}(x_n)\}\\ &\qquad = c/(1+c)+(1+c)^{-1}\liminf_n\int_{0-}^{x_n}\{\bar{F}(x_n-t)/\bar{F}(x_n)\}\,G(dt)\end{aligned}$$

and by Fatou's lemma the right side is at least

$$c/(1+c)+(1+c)^{-1}\int \mu(t)\,G(dt)>1.$$

This contradicts $F\sim_M G$. So it must be the case that $F\in\mathscr{L}$, proving the proposition.

PROOF OF PROPOSITION 2. First,

$$P(X+Y>x)/P(X+Y>2x) \leqslant P(X>\tfrac{1}{2}x)/P(X+Y>2x)+P(Y>\tfrac{1}{2}x)/P(X+Y>2x)$$
$$\leqslant P(X>\tfrac{1}{2}x)/P(X>2x)+P(Y>\tfrac{1}{2}x)/P(Y>2x)$$

which is bounded above as $x\to\infty$, and so $H\in\mathscr{D}$. Therefore to show $H\in\mathscr{S}$ we have only, by Goldie (1978), Theorem 1, to show $H\in\mathscr{L}$. Now

$$\bar{H}(x)=P(X+Y>x, X\leqslant \tfrac{1}{2}x)+P(X+Y>x, Y\leqslant \tfrac{1}{2}x)+P(X>\tfrac{1}{2}x)P(Y>\tfrac{1}{2}x). \tag{2.5}$$

Taking the last two terms on the right,

$$\{P(X+Y>x, Y\leqslant \tfrac{1}{2}x)+P(X>\tfrac{1}{2}x)P(Y>\tfrac{1}{2}x)\}/\bar{F}(x) \tag{2.6}$$
$$=\int_{0-}^{\frac{1}{2}x}\{\bar{F}(x-u)/\bar{F}(x)\}\,G(du)+\{\bar{F}(\tfrac{1}{2}x)/\bar{F}(x)\}\,\bar{G}(\tfrac{1}{2}x)$$

and the integrand is bounded by a constant and converges pointwise to 1, so that the integral of (2.6) converges to 1. The other term on the right of (2.6) is $O(1).o(1)$. Thus the right side of (2.6) tends to 1. Similarly,

$$P(X+Y>x, X\leqslant \tfrac{1}{2}x)/\bar{G}(x)\to 1.$$

Returning to (2.5), we conclude $\bar{H}(x)\sim\bar{F}(x)+\bar{G}(x)$. Since $F\in\mathscr{L}$ and $G\in\mathscr{L}$, the conclusion $H\in\mathscr{L}$ follows easily, which is enough to complete the proof as indicated earlier.

PROOF OF THEOREM 2. Assume $F\in\mathscr{S}$ and $G\in\mathscr{S}$. We prove

$$\bar{H}^{(2)}(x)/\bar{H}(x)=4-2\{\bar{F}(x)+\bar{G}(x)\}/\bar{H}(x)+o(1), \quad x\to\infty. \tag{2.7}$$

Thus, let $\bar{f}(y)=\sup\{\bar{F}^{(2)}(x)/\bar{F}(x);\, x\geqslant y\}$, $\underline{f}(y)=\inf\{\bar{F}^{(2)}(x)/\bar{F}(x);\, x\geqslant y\}$, and define $\bar{g}$, $\underline{g}$ similarly in terms of G. For brevity, write $X^{(2)}$ for X_1+X_2, and $Y^{(2)}$ for Y_1+Y_2. Fix $w>0$, then for $x\geqslant 2w$,

$$\bar{H}^{(2)}(x)=P(X^{(2)}\leqslant w, X^{(2)}+Y^{(2)}>x)+P(Y^{(2)}\leqslant w, X^{(2)}+Y^{(2)}>x)$$
$$+P(X^{(2)}>w, Y^{(2)}>w, X^{(2)}+Y^{(2)}>x). \tag{2.8}$$

The sum of the first two terms on the right is at least

$$\bar{G}^{(2)}(x)\,F^{(2)}(w)+\bar{F}^{(2)}(x)\,G^{(2)}(w),$$

which is bounded below by

$$\underline{g}(x)\,\bar{G}(x)\,F^{(2)}(w)+\underline{f}(x)\,\bar{F}(x)\,G^{(2)}(w).$$

Similarly, an upper bound for the same quantity is

$$\bar{g}(x-w)\,\bar{G}(x-w)\,F^{(2)}(w)+\bar{f}(x-w)\,\bar{F}(x-w)\,G^{(2)}(w).$$

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The third term on the right of (2.8) is equal to

$$\int_w^\infty \bar{G}^{(2)}((x-u)\vee w)\,F^{(2)}(du) \leqslant \bar{g}(w)\int_w^\infty \bar{G}((x-u)\vee w)\,F^{(2)}(du)$$
$$= \bar{g}(w)\,P(X^{(2)}>w,\, Y>w,\, X^{(2)}+Y>x)$$
$$= \bar{g}(w)\int_w^\infty \bar{F}^{(2)}((x-u)\vee w)\,G(du)$$
$$\leqslant \bar{g}(w)\bar{f}(w)\int_w^\infty \bar{F}((x-u)\vee w)\,G(du)$$
$$= \bar{f}(w)\,\bar{g}(w)\left\{\bar{H}(x) - \int_0^w \bar{G}(x-u)\,F(du) - \int_0^w \bar{F}(x-u)\,G(du)\right\}$$
$$\leqslant \bar{f}(w)\,\bar{g}(w)\,\{\bar{H}(x) - \bar{G}(x)\,F(w) - \bar{F}(x)\,G(w)\}.$$

Similarly, a lower bound for the third term on the right of (2.8) is

$$\underline{f}(w)\,\underline{g}(w)\,\{\bar{H}(x) - \bar{G}(x-w)\,F(w) - \bar{F}(x-w)\,G(w)\}.$$

Replacing the right side of (2.8) by its upper bounds we calculate

$$\bar{H}^{(2)}(x) + 2\bar{F}(x) + 2\bar{G}(x) \leqslant \bar{f}(w)\,\bar{g}(w)\,\bar{H}(x)$$
$$+ [2 - \bar{f}(w)\,\bar{g}(w)\,\{F(w)\wedge G(w)\}\,]\,\{\bar{F}(x) + \bar{G}(x)\}$$
$$+ [\{\bar{g}(x-w)\,F^{(2)}(w)\}\vee\{\bar{f}(x-w)\,G^{(2)}(w)\}]\,\{\bar{F}(x-w) + \bar{G}(x-w)\}.$$

Divide by $\bar{H}(x)$ and take $\limsup_{x\to\infty}$. We know $F\in\mathscr{L}$, $G\in\mathscr{L}$, so that

$$\bar{F}(x-w) + \bar{G}(x-w) \sim \bar{F}(x) + \bar{G}(x).$$

Further, $\limsup(\bar{F}+\bar{G})/\bar{H} \leqslant 1$. Hence

$$\limsup_{x\to\infty}\{\bar{H}^{(2)}(x) + 2\bar{F}(x) + 2\bar{G}(x)\}/\bar{H}(x)$$
$$\leqslant \bar{f}(w)\,\bar{g}(w) + |\,2 - \bar{f}(w)\,\bar{g}(w)\,\{F(w)\wedge G(w)\} + 2\{F^{(2)}(w)\vee G^{(2)}(w)\}\,|$$
$$\to 4+0, \quad w\to\infty.$$

Similarly, using the lower bounds, the lim inf is at least 4. So (2.7) is proved. This immediately gives the first conclusion of the theorem, and also $H\in\mathscr{S}\Leftrightarrow\bar{H}\sim\bar{F}+\bar{G}$, that is, (i) $\Leftrightarrow$ (ii). To complete the theorem we show that if $F\sim_M G$ then $pF+qG\in\mathscr{S}$ for *all* p, $0<p<1$, $q=1-p$, and conversely if $pF+qG\in\mathscr{S}$ for *some* $p\in(0,1)$ then $F\sim_M G$.

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First assume $F \sim_M G$, then

$$\begin{aligned}\overline{(pF+qG)}^{(2)}/\overline{(pF+qG)} &= \{p^2\bar{F}^{(2)}+2pq\bar{F}*\bar{G}+q^2\bar{G}^{(2)}\}/(p\bar{F}+q\bar{G})\\ &\sim \{p^2\bar{F}^{(2)}+2pq(\bar{F}+\bar{G})+q^2\bar{G}^{(2)}\}/(p\bar{F}+q\bar{G})\\ &\sim \{2p^2\bar{F}+2pq(\bar{F}+\bar{G})+2q^2\bar{G}\}/(p\bar{F}+q\bar{G})\\ &= 2.\end{aligned}$$

Thus $pF+qG\in\mathscr{S}$.

For the converse, suppose it is *not* true that $F\sim_M G$, so that there exists $\varepsilon>0$ such that on an unbounded set of x, $\bar{F}*\bar{G}\geqslant(1+\varepsilon).(\bar{F}+\bar{G})$. Then

$$\begin{aligned}\limsup_{x\to\infty}\overline{(pF+qG)}^{(2)}/\overline{(pF+qG)} &= \limsup\{p^2\bar{F}^{(2)}+2pq\bar{F}*\bar{G}\\ &\qquad\qquad +q^2\bar{G}^{(2)}\}/(p\bar{F}+q\bar{G})\\ &\geqslant \limsup\{2p^2\bar{F}+2pq(1+\varepsilon)(\bar{F}+\bar{G})\\ &\qquad\qquad +2q^2\bar{G}\}/(p\bar{F}+q\bar{G})\\ &= 2+2pq\varepsilon.\limsup(\bar{F}+\bar{G})/(p\bar{F}+q\bar{G})\\ &\geqslant 2+2pq\varepsilon/(p\vee q)>2.\end{aligned}$$

and so $pF+qG\notin\mathscr{S}$. Theorem 2 is proved.

PROOF OF THEOREM 3. (a) *Assume* $F\in\mathscr{L}_\gamma$ *and* $\lim_{x\to\infty}\bar{G}(x)/\bar{F}(x)=0$. Fix $t>0$ and $v>t$. We have

$$\bar{H}(x)=P(X+Y>x)=P(X+Y>x,\ Y\leqslant x-v)+P(X+Y>x,\ Y>x-v)$$

and

$$\begin{aligned}0 &\leqslant P(X+Y>x,\ Y>x-v)/P(X+Y>x,\ Y\leqslant x-v)\\ &\leqslant P(Y>x-v)/P(X>x, 0<Y\leqslant x-v)\\ &= \bar{G}(x-v)/[\bar{F}(x)\{G(x-v)-G(0)\}]\\ &= \{\bar{G}(x-v)/\bar{F}(x-v)\}\{\bar{F}(x-v)/\bar{F}(x)\}\{G(x-v)-G(0)\}^{-1}\\ &\to 0.e^{\gamma v}.1/\bar{G}(0)=0.\end{aligned}$$

Thus

$$\bar{H}(x)\sim\int_{-\infty}^{x-v}\bar{F}(x-u)\,G(du),\quad x\to\infty. \tag{2.9}$$

Altering v to $v-t$ and x to $x-t$,

$$\bar{H}(x-t)\sim\int_{-\infty}^{x-v}\{\bar{F}(x-u-t)/\bar{F}(x-u)\}\,\bar{F}(x-u)\,G(du). \tag{2.10}$$

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Defining

$$M_F(y)=\sup\{\bar{F}(x-t)/\bar{F}(x), x\geqslant y\},\quad m_F(y)=\inf\{\bar{F}(x-t)/\bar{F}(x), x\geqslant y\},$$

the right side of (2.10) is between

$$m_F(v)\int_{-\infty}^{x-v}\bar{F}(x-u)\,G(du)\quad\text{and}\quad M_F(v)\int_{-\infty}^{x-v}\bar{F}(x-v)\,G(du).$$

Hence, using (2.9)

$$m_F(v)\leqslant\liminf_{x\to\infty}\bar{H}(x-t)/\bar{H}(x)\leqslant\limsup_{x\to\infty}\bar{H}(x-t)/\bar{H}(x)\leqslant M_F(v).$$

Let $v\to\infty$ and (a) follows.

(b) *Assume* $F\in\mathscr{L}_\gamma$ *and* $G\in\mathscr{L}_\gamma$. Define m_G and M_G for G, exactly like m_F and M_F for F. This time

$$\bar{H}(x)=\int_{-\infty}^{x-v}\bar{F}(x-u)\,G(du)+\int_{-\infty}^{v}\bar{G}(x-u)\,F(du)+\bar{F}(v)\,\bar{G}(x-v),\tag{2.11}$$

and altering x to $x-t$ and v to $v-t$,

$$\begin{aligned}\bar{H}(x-t)&=\int_{-\infty}^{x-v}\{\bar{F}(x-u-t)/\bar{F}(x-u)\}\,\bar{F}(x-u)\,G(du)\\&\quad+\int_{-\infty}^{v-t}\{\bar{G}(x-u-t)/\bar{G}(x-u)\}\,\bar{G}(x-u)\,F(du)\\&\quad+\{\bar{F}(v-t)/\bar{F}(v)\}\,\bar{F}(v)\,\bar{G}(x-v)\\&\leqslant M_F(v)\int_{-\infty}^{x-v}\bar{F}(x-u)\,G(du)+M_G(x-v+t)\int_{-\infty}^{v}\bar{G}(x-u)\,F(du)\\&\quad+M_F(v)\,\bar{F}(v)\,\bar{G}(x-v)\\&\leqslant\{M_F(v)\vee M_G(x-v+t)\}\,\bar{H}(x),\end{aligned}\tag{2.12}$$

hence

$$\limsup_{x\to\infty}\bar{H}(x-t)/\bar{H}(x)\leqslant M_F(v)\to e^{\gamma t},\quad v\to\infty.$$

Now

$$\begin{aligned}&\liminf_{x\to\infty}\bar{H}(x)\Big/\int_{v-t}^{v}\bar{G}(x-u)\,F(du)\\&\qquad\geqslant\liminf_{x\to\infty}\int_{0}^{v}\bar{G}(x-u)\,F(du)/\{\bar{G}(x-v+t)\,\bar{F}(v-t)\},\end{aligned}$$

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which by Fatou's lemma is at least

$$\{\bar{F}(v-t)\}^{-1}\int_0^v \liminf_{x\to\infty}\{\bar{G}(x-u)/\bar{G}(x-v+t)\}\,F(du)$$
$$=\int_0^v e^{\gamma u}\,F(du)/\{e^{\gamma(v-t)}\bar{F}(v-t)\}.$$

As $v\to\infty$ the last term tends to ∞, because for $v\geqslant nt$,

$$\int_0^v e^{\gamma u}F(du)/\{e^{\gamma(v-t)}\bar{F}(v-t)\}\geqslant \sum_{k=1}^n e^{\gamma(v-kt)}\{\bar{F}(v-kt)$$
$$-\bar{F}(v-(k-1)t)\}/\{e^{\gamma(v-t)}\bar{F}(v-t)\}$$
$$\to \sum_{k=1}^n e^{-(k-1)\gamma t}(e^{\gamma(k-1)t}-e^{\gamma(k-2)t})$$
$$=n(1-e^{-\gamma t}).$$

Thus by taking v sufficiently large we may ensure that

$$\limsup_{x\to\infty}\int_{v-t}^v \bar{G}(x-u)\,F(du)/\bar{H}(x)\leqslant \tfrac{1}{2}\varepsilon.$$

For this v it will then be true for all large x that, from (2.11),

$$\int_{-\infty}^{x-v}\bar{F}(x-u)\,G(du)+\int_{-\infty}^{v-t}\bar{G}(x-u)\,F(du)+\bar{F}(v)\,\bar{G}(x-v)\geqslant \bar{H}(x)(1-\varepsilon).$$

Then from (2.12),

$$\bar{H}(x-t)\geqslant m_F(v)\int_{-\infty}^{x-v}\bar{F}(x-u)\,G(du)+m_G(x-v+t)\int_{-\infty}^{v-t}\bar{G}(x-u)\,F(du)$$
$$+m_F(v)\,\bar{F}(v)\,\bar{G}(x-v)$$
$$\geqslant\{m_F(v)\wedge m_G(x-v+t)\}\,\bar{H}(x)(1-\varepsilon).$$

Hence

$$\liminf_{x\to\infty}\bar{H}(x-t)/\bar{H}(x)\geqslant m_F(v)(1-\varepsilon)\to e^{\gamma t}(1-\varepsilon),\quad v\to\infty.$$

Since $\varepsilon>0$ was arbitrary the above lim inf is at least $e^{\gamma t}$, and (b) is proved.

Proof of Corollary. Set $X^*=e^X$, $Y^*=e^Y$, so that $F^*(x)=F(\log x)$, $G^*(x)=G(\log x)$, and the corollary becomes just a re-statement of the theorem.

3. A distribution function in $\mathscr{L}\setminus\mathscr{S}$

Let a_n be a sequence of positive numbers satisfying $a_n\to\infty$, $a_n<\frac{1}{2}(n+1)!$. We specify the sequence fully later. Define F by its tail function:

$$\begin{aligned}\bar{F}(x) &= 1, \quad -\infty < x \leqslant 2,\\ &= 1/(n+1)!, \quad (n+1)!+na_n \leqslant x \leqslant (n+2)!, \quad n=1,2,\dots,\\ \bar{F}((n+1)!+u) &= (1+n-u/a_n)/(n+1)!, \quad 0 \leqslant u \leqslant na_n, \quad n=1,2,\dots.\end{aligned}$$

The idea is that $\bar{F}$ decreases to zero by a sequence of slowly-flattening linear slopes, with increasingly long flats in between. We first show $F \in \mathscr{L}$. Fix t and let n be large enough for $na_n > t$. Then within the interval

$$(n+1)! \leqslant x \leqslant (n+2)!,$$

$\bar{F}(x-t)/\bar{F}(x)$ will be greatest in the sub-interval

$$(n+1)!+t \leqslant x \leqslant (n+1)!+na_n.$$

For such $x=(n+1)!+u$,

$$\begin{aligned}\bar{F}(x-t)/\bar{F}(x) &= \{1+n-(u-t)/a_n\}/(1+n-u/a_n)\\ &= 1+(t/a_n)/(n+1-u/a_n) \leqslant 1+t/a_n \to 1.\end{aligned}$$

Thus $F \in \mathscr{L}$.

Now we show $F \notin \mathscr{S}$. Let X, X' be independent, each with distribution function F. Then for $0<u<na_n$, using the fact that $2\{n!+(n-1)a_{n-1}\}<(n+1)!$,

$$\begin{aligned}\bar{F}^{(2)}((n+1)!+u) &= P(Y>(n+1)!+u)+P(Y'>(n+1)!+u)+P((n+1)! \leqslant Y\\ &\quad \leqslant (n+1)!+u<Y+Y')+P((n+1)! \leqslant Y' \leqslant (n+1)!+u\\ &\quad <Y+Y')-P(Y>(n+1)!)P(Y'>(n+1)!)\\ &= 2\bar{F}((n+1)!+u)+\{2a_n^{-1}/(n+1)!\}\int_0^u \bar{F}(u-x)\,dx\\ &\qquad -\{\bar{F}((n+1)!)\}^2.\end{aligned}$$

So

$$\bar{F}^{(2)}((n+1)!+u)/\bar{F}((n+1)!+u) = 2+2\int_0^u \bar{F}(x)\,dx/(a_n+na_n-u)+o(1).$$

Therefore

$$\bar{F}^{(2)}(b_n)/\bar{F}(b_n) = 2+(2/a_n)\int_0^{na_n} \bar{F}(x)\,dx+o(1), \tag{3.1}$$

where $b_n=(n+1)!+na_n$. To prevent $F \in \mathscr{S}$ we must ensure the right side of (3.1) does not converge to 2, and so intuitively a_n must tend to ∞ very slowly. Set

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$k_n = \max\{k: (k+1)! \leqslant na_n\}$, then

$$\bar{F}^{(2)}(b_n)/\bar{F}(b_n) \geqslant 2+(2/a_n)\sum_{k=1}^{k_n}\int_{k!}^{(k+1)!}\bar{F}(x)\,dx$$
$$\geqslant 2+(2/a_n)\sum_{k=1}^{k_n}\{(k+1)!-k!\}/k!$$
$$=2+(2/a_n)\sum_{k=1}^{k_n}k > 2+k_n^2/a_n.$$

So $F\notin\mathscr{S}$ if $k_n^2/a_n \nrightarrow 0$. This can be achieved by setting $a_n = \max\{k: (k+1)! \leqslant n\}$. For then $a_n\to\infty$, and on comparing the definitions of k_n and a_n we see that $k_n > a_n$, which suffices.

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