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ON THE COMPLETE INVARIANCE PROPERTY IN SOME UNCOUNTABLE PRODUCTS

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ABSTRACT. We consider uncountable products of nontrivial compact, convex subsets of normed linear spaces. We show that these products do not have the complete invariance property i.e. they include a nonempty, closed subset which is not a fixed point set (i.e. the set of all fixed points) for any continuous mapping from the product into itself. In particular we give an answer to W.Weiss' question whether uncountable powers of the unit interval have the complete invariance property.

0. Introduction. A space X is said to have the complete invariance property (C.I.P.) if and only if every nonempty closed subset $F \subseteq X$ is a fixed point set i.e., if there is a continuous function $f: X \longrightarrow X$ such that $x \in F$ if and only if $f(x) = x$ [12]. There are many classes of spaces known to possess the C.I.P., among which are all convex subsets of normed linear spaces [12], and, hence in particular, all countable products of such spaces and among them the Hilbert Cube; the countable product of unit intervals. Other examples of spaces having the C.I.P. are compact finite dimensional manifolds [11] and locally compact metric groups [7]. Also, the behaviour of the C.I.P. under products has been studied, it is known for example that there are spaces possessing the C.I.P. such that their square does not have the C.I.P. [8]. Regarding uncountable products it is known that if $(X_\alpha : \alpha < \kappa)$ is a sequence of at most countable discrete spaces, then the product $\prod_{\alpha<\kappa} X_\alpha$ does have the C.I.P. [8].

We will consider products of essentially different spaces, namely spaces X_α which are compact, convex subsets of some normed linear spaces. Any product of such spaces has the fixed point property (any continuous function from such a space into itself has a fixed point)(see Theorem 4). Our result is that $\prod_{\alpha<\kappa} X_\alpha$ does not have the C.I.P. In particular, we answer in the negative W.Weiss' question [13], whether uncountable products of the unit interval have the C.I.P. I am grateful to him for suggesting the problem to me.

The idea of the proof is as follows (see below for appropriate definitions and lemmas): for any uncountable cardinal κ we construct some special closed set $F \subseteq 2^\kappa \subseteq \prod_{\alpha<\kappa} X_\alpha$ which we call a splitting set (see Definition 11). We show that for no continuous $f: \prod_{\alpha<\kappa} X_\alpha \longrightarrow \prod_{\alpha<\kappa} X_\alpha$ is the set F its fixed point set. In order to show this, we fix f continuous as above and assume that for all $x \in F$ we have $f(x) = x$. Now we use the fact that if κ is uncountable, then there are many pairs of countable subsets A, B of κ such that $A \subset B$ and that both $f_A = \pi_A \circ f: \prod_{\alpha<\kappa} X_\alpha \longrightarrow \prod_{\alpha\in A} X_\alpha$ and

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$f_B = \pi_B \circ f\colon \prod_{\alpha<\kappa} X_\alpha \longrightarrow \prod_{\alpha\in B} X_\alpha$ depend respectively on A and B; that is for all $x, y \in \prod_{\alpha<\kappa} X_\alpha$ we have

$$(x_A = \pi_A(x) = \pi_A(y) = y_A) \Rightarrow (f_A(x) = f_A(y))$$

$$(x_B = \pi_B(x) = \pi_B(y) = y_B) \Rightarrow (f_B(x) = f_B(y))$$

(see Definition 1, Lemma 3 and the second paragraph of section 1). We will find two such B, A for which the definition of a splitting set works (i.e. $B \subset A$ such that there is $\alpha \in A - B$ such that α, B are as in Definition 11).

Now let us consider a simple example of such a situation. Let $B = \{0\}, A = \{0,1\} = \kappa, X_0 = X_1 = I, \alpha = 1$, $F = \{(x,y) \in I^2 : y \in \{0,1\}, (y = 0 \Rightarrow x \in [0, 0.5], y = 1 \Rightarrow x \in [0.5, 1])\}$ (draw a picture) where I denotes the unit interval and $f\colon I^2 \longrightarrow I^2$ is such that $f_{\{0\}}$ depends on $\{0\}$ (See definition 1).

Now we will show that f must have a fixed point outside F. Suppose the opposite i.e., that $f((x,y)) = (x,y)$ if and only if $(x,y) \in F$. Let us define for each $t \in I$ a function $f_t\colon I \longrightarrow I$ by $f_t(y) = \pi_{\{1\}}(f(t,y))$. Since $f_{\{0\}}$ depends on $\{0\}$, our assumption that F is a fixed point set of f implies that the sets of all fixed points of f_t's — $\mathrm{fix}(f_t)$ must be restricted in the following way: $\mathrm{fix}(f_t) = \{0\}$ if $t \in [0, 0.5)$, $\mathrm{fix}(f_t) \subseteq \{0,1\}$ if $t = 0.5$ and $\mathrm{fix}(f_t) = \{1\}$ if $t \in (0.5, 1]$. Since f_t's converge uniformly to $f_{0.5}$ if $t \longrightarrow 0.5$, this gives rise to a contradiction (draw a picture and look at the situation of the graphs of f_t's with respect to the main diagonal in I^2).

The last part of the argument involved the fact that at least some of fixed points of f_t's should change in a continuous way (for a more specific and general formulation, appropriate in this special case, see [1]). Hence, we must to formulate some lemmas on the "continuity of fixed points" which would work in general a setting. It appears that lemma 6 is sufficient for our purposes.

The paper is organized as follows: in section 1 we list our notational conventions; in section 2 we state basic facts about uncountable products of our spaces. In section 3 we prove the above-mentioned lemmas about the "continuity of fixed points". Section 4 is devoted to a construction of a splitting set i.e., a closed set $F \subseteq \prod_{\alpha<\kappa} X_\alpha$ witnessing the failure of the complete invariance property.

1. Notational conventions. Our set-theoretic and general topological terminology is quite standard. All unexplained symbols and notions can be found in [5] or [6]. Each X_α is assumed to be a convex, compact subspace of some normed linear space $(L_\alpha, \|\ \|_\alpha)$ (in particular, any product of such spaces has the fixed point property (see Theorem 4)), and is assumed to have more than one point. Note that our spaces X_α are always separable.

If $A \subseteq \kappa$, $x \in \prod_{\alpha<\kappa} X_\alpha$, we denote by x_A or $\pi_A(x)$ the projection of x onto coordinates belonging to A. $X^A = \prod_{\alpha\in A} X_\alpha$, hence $x_A = \pi_A(x) \in X^A$ and we of course distinguish $x_{\{\alpha\}}$ from x_α. Whenever $A \cap B = \emptyset$, $x_A \in X^A, x_B \in X^B$ we define $x_A{}^\frown x_B \in X^{A\cup B}$ by $(x_A{}^\frown x_B)_A = x_A$, $(x_A{}^\frown x_B)_B = x_B$. If $A \subseteq \kappa$ is countable, then X^A is a metric space; since any factor is metric, we will always assume that the metric on X^A is of the form

1) $$\rho^A(x,y) = \sum_{\alpha\in A} \frac{\| x(\alpha) - y(\alpha) \|_\alpha}{2^{f_A(\alpha)}}$$

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where $f_A : A \longrightarrow \omega - \{0\}$ is some 1-1 function. When we consider two $A \subseteq B \in [\kappa]^\omega$ countable subsets of κ, then we assume that ρ^A, ρ^B, and ρ^{B-A} are related in the following way: $f_B \mid A = f_A$, $f_B \mid (B - A) = f_{B-A}$, hence

2) $$\rho^A(x_A, y_A) + \rho^{B-A}(x_{B-A}, y_{B-A}) = \rho^B(x, y).$$

Of course we cannot arrange this for all $A \subseteq B \in [\kappa]^\omega$, but when we fix A, B we can construct such metrics equivalent to the topologies on X^A, X^B, X^{B-A}. By $B^A(r,a)$ $(\overline{B^A}(r,a))$ we denote a ball (a closed ball) around $a \in X^A$ of radius $r \in R_+$ with respect to ρ^A. Since each X_α is a convex subset of a linear space, we may consider a convex combination of points from X^A given by

$$[tx + (1-t)y](\alpha) = tx(\alpha) + (1-t)y(\alpha) \in X_\alpha$$

for $t \in [0,1]$, $x, y \in X^A$, $\alpha \in A$. Note that for a normed linear space we have the following relation satisfied for all α in A

$$\forall t \in [0,1] \ \| x(\alpha) - [tx(\alpha) + (1-t)y(\alpha)] \|_\alpha = (1-t) \| x(\alpha) - y(\alpha) \|_\alpha \leq \| x(\alpha) - y(\alpha) \|_\alpha .$$

Hence, by 1) it gives us

3) $$\forall t \in [0,1] \quad \rho^A(x, tx + (1-t)y) \leq \rho^A(x, y).$$

$Cl_A(F)$ for $F \subseteq X^A$ denotes a closure of F in X^A. Closures may also be denoted by $\overline{F}$ if the space in which we have taken the closure is clear from the context.

Since X_α is compact, it is a bounded subset in L_α; hence, we may w.l.o.g. (i.e., still having the metric given by a norm) assume that

$$\forall \alpha \in A \forall x, y \in X^A \ \rho^{\{\alpha\}}(x(\alpha), y(\alpha)) \leq 1.$$

Thus,

$$\rho^A(x, y) \leq 1$$

holds for all countable A.

By f, g, h we will denote continuous functions. If $f : X \longrightarrow X$, then $\text{fix}(f) = \{x \in X : f(x) = x\}$; if $f : X \longrightarrow \prod_{\alpha<\kappa} X_\alpha$, $A \subseteq \kappa$, then $f_A : X \longrightarrow \prod_{\alpha \in A} X_\alpha$ is defined by $f_A(x) = (f(x))_A$. $C \subset [\kappa]^\omega$ is called a *club* set if and only if it is unbounded in $[\kappa]^\omega$ (i.e., $\forall x \in [\kappa]^\omega \exists y \in C \, x \subseteq y$) and if C is closed (i.e., if $(x_n)_{n\in\omega} \subseteq C$ is an increasing sequence of sets, then $\bigcup_{n\in\omega} x_n \in C$). $S \subseteq [\kappa]^\omega$ is called a stationary set iff it has a nonempty intersection with every *club* set (see Baumgartner's article in [6]). I will denote the unit interval.

2. Basic facts.

DEFINITION 1. *Let $A \in [\kappa]^\omega$, $f\colon \prod_{\alpha<\kappa} X_\alpha \longrightarrow X$ be a continuous function. Then we say that f depends on A, iff*

$$\forall x, y \in \prod_{\alpha<\kappa} X_\alpha (x_A = y_A \Rightarrow f(x) = f(y))$$

If $f_A\colon \prod X_\alpha \longrightarrow X^A$ depends on A, then $f^A\colon \prod_{\alpha\in A} X_\alpha \longrightarrow \prod_{\alpha\in A} X_\alpha$ is defined by $f^A(x_A) = f(x)_A$.

THEOREM 2 ([9]). *Let $f\colon \prod_{\alpha<\kappa} X_\alpha \longrightarrow X$ be a continuous function. If X is a metric space and X_α's are compact spaces, then f depends on some $A \in [\kappa]^\omega$.*

(For extensive references on the subject see [3])

LEMMA 3. *Let us suppose for each $\alpha < \kappa$, X_α is a compact, metric space, $f\colon \prod_{\alpha<\kappa} X_\alpha \longrightarrow \prod_{\alpha<\kappa} X_\alpha$ be a continuous function. Then there is a club set $C_f \subseteq [\kappa]^\omega$ such that for all $A \in C_f$, f_A depends on A.*

PROOF. First let us use Theorem 2 and for each $\alpha \in \kappa$ find A_α such that $f_{\{\alpha\}}\colon \prod_{\alpha<\kappa} X_\alpha \longrightarrow X_\alpha$ depends on A_α. Let

$$C_f = \{X \in [\kappa]^\omega : \forall\alpha \in X\, A_\alpha \subseteq X\}.$$

■

THEOREM 4. *Suppose $\forall\alpha < \kappa$ X_α is a compact, convex subset of a normed linear space, then $\prod_{\alpha<\kappa} X_\alpha$ has the fixed point property.*

PROOF. For every $F \in [\kappa]^{<\omega}$, the product $\prod_{\alpha\in F} X_\alpha$ is a convex, compact subset of a normed linear space (see [2]) so by the Schauder Theorem ([10]) every finite product of X_α's has the fixed point property. By the compactness of every factor this implies that the whole product has the fixed point property ([4]).

3. Geometric lemmas.

LEMMA 5. *Let $A \in [\kappa]^\omega$, $\alpha \in A$, $f\colon X^A \longrightarrow X^A$, $U \subseteq X_\alpha$ be open. Let us suppose that there is an $a \in U$ such that $\pi_{\{\alpha\}}\big(\mathrm{fix}(f)\big) \cap U = \{a\}$, and that we are given a family of continuous functions Γ from X^A into X^A such that*

i) $\forall\varepsilon > 0\, \exists f_\varepsilon \in \Gamma\, \forall x \in \pi^{-1}_{\{\alpha\}}(U) \rho^A\big(f(x), f_\varepsilon(x)\big) < \varepsilon$.

ii) $\forall\varepsilon > 0\ \pi_{\{\alpha\}}\big(\mathrm{fix}(f_\varepsilon)\big) \cap U = \emptyset$

then there is a function $g\colon X^A \longrightarrow X^A$ such that

iii) $\pi_{\{\alpha\}}\big(\mathrm{fix}(g)\big)\cap = \emptyset$

iv) $\forall x \in X^A\ \big(x(\alpha) \notin U \Rightarrow g(x) = f(x)\big)$.

PROOF. Take $r_1 > r_2 > 0$ such that $\overline{B^{\{\alpha\}}}(r_1, a)$, $\overline{B^{\{\alpha\}}}(r_2, a) \subseteq U$. Fix $\beta, \gamma\colon I \longrightarrow I$ such that $\beta(t) + \gamma(t) = 1$ for $t \in I$, $\beta(t) = 0$ for $t \geq r_1$ and $\gamma(t) = 0$ for $t \leq r_2$.

Also consider the function $\varphi: \{x \in X^A : r_1 \geq \rho^{\{\alpha\}}(x(\alpha), a) \geq r_2\} \longrightarrow R_+$, defined by $\varphi(x) = \rho^A(x, f(x))$. φ is positive since $\pi_{\{\alpha\}}(\operatorname{dom}\varphi) \subseteq U - \{a\}$, $\pi_{\{\alpha\}}(\operatorname{fix}(f)) \cap U = \{a\}$ (by the assumptions). Since $\operatorname{dom}(\varphi) \subseteq X^A$ is closed, hence compact, there is $\delta > 0$ such that $\varphi(x) > \delta$ for all $x \in \operatorname{dom}(\varphi)$. Fix $\varepsilon \leq \delta$ and define $g: X^A \longrightarrow X^A$ as follows

$$g(x) = \beta(t) f_\varepsilon(x) + \gamma(t) f(x)$$

where $t = \rho^{\{\alpha\}}(x(\alpha), a)$. First note that by convexity of X^A, g indeed goes into X^A.

Also, since $\beta \mid [1, r_1] = 0$ and for all x such that $x(\alpha) \notin U$ $\rho^{\{\alpha\}}(x(\alpha), a) > r_1$ we have $g(x) = f(x)$ for all $x \in X^A$ such that $x(\alpha) \notin U$. So we are left with the proof that $\pi_{\{\alpha\}}(\operatorname{fix}(g)) \cap U = \emptyset$:

a) if $x(\alpha) \in U$ and $\rho^{\{\alpha\}}(a, x(\alpha)) \geq a$, then since $g(x) = f(x)$ and $\pi_{\{\alpha\}}(\operatorname{fix}(x)) \cap U = \{a\}$, we have $g(x) \neq x$;

b)if $x(\alpha) \in U$ and $\rho^{\{\alpha\}}(a, x(\alpha)) \leq r_2$, then $g(x) = f_\varepsilon(x)$ so $g(x) \neq x$ by the assumption that $\pi_{\{\alpha\}}(\operatorname{fix}(f_\varepsilon)) \cap U = \emptyset$;

c) if $x(\alpha) \in U$ and $r_2 \leq \rho^{\{\alpha\}}(a, x(\alpha)) \leq r_1$, then by 3) (Section 1) we have

$$\rho^A(f(x), \beta(t) f_\varepsilon(x) + \gamma(t) f(x)) \leq \rho^A(f(x), f_\varepsilon(x)) < \varepsilon$$

for all $x \in \pi^{-1}_{\{\alpha\}}(U)$ since $\rho^A(f(x), f_\varepsilon(x)) < \varepsilon$ (by i)). Hence $\rho^A(g(x), f(x)) < \varepsilon$, but since $x \in \operatorname{dom}\varphi$ we have $\rho^A(f(x), x) > \delta \geq \varepsilon$, so $\rho^A(g(x), x) > 0$, so $g(x) \neq x$. ∎

LEMMA 6. *Let $A \in [\kappa]^\omega$, $\alpha \in A$, $f: X^A \longrightarrow X^A$. Suppose we are given p, $q \in X_\alpha$, $p \neq q$ and two families of functions Φ, Γ from X^A into X^A such that*

i) $$\forall \varepsilon > 0\, \exists f_\varepsilon \in \Phi,\ \exists g_\varepsilon \in \Gamma\ \forall x \in X^A\ \rho^A(f(x), f_\varepsilon(x)), \rho^A(f(x), g_\varepsilon(x)) < \varepsilon$$

ii). $$\forall \varepsilon > 0\ \pi_{\{\alpha\}}(\operatorname{fix}(f_\varepsilon)) = \{p\},\ \pi_{\{\alpha\}}(\operatorname{fix}(g_\varepsilon)) = \{q\}$$

Then there exists $r \in X_\alpha$, $r \notin \{p, q\}$ such that $\pi_{\{\alpha\}}(\operatorname{fix}(f)) \ni r$.

PROOF. Suppose the opposite i.e. $\pi_{\{\alpha\}}(\operatorname{fix}(f)) \subseteq \{p, q\}$. Choose two open sets U_p, U_q such that $p \in U_p \subseteq X_\alpha$, $q \in U_q \subseteq X_\alpha$ with disjoint closures. First apply Lemma 5 for f, Φ, U_q, by *i*)and *ii*) the assumptions are satisfied. We obtain $g: X^A \longrightarrow X^A$ such that $\pi_{\{\alpha\}}(\operatorname{fix}(g)) \cap U_q = \emptyset$ and the following holds: for all x such that $x(\alpha) \notin U_q$ we have $g(x) = f(x)$, so

iii) $$\forall x\ s.t.\ x(\alpha) \in \overline{U_p}\ \ g(x) = f(x),\ \pi_{\{\alpha\}}(\operatorname{fix}(g)) \subseteq \{p\}.$$

Since if $r \in \pi_{\{\alpha\}}(\operatorname{fix}(g))$ then $r \notin U_q$ but then $r \in \pi_{\{\alpha\}}(\operatorname{fix}(f)) \subseteq \{p, q\}$, hence $r = p$.

Now apply Lemma 5 for g, Γ, U_p, by *i*), *ii*), *iii*), *iv*). We get $h: X^A \longrightarrow X^A$ such that $\pi_{\{\alpha\}}(\operatorname{fix}(h)) \cap U_p = \emptyset$ and the following holds: for all x such that $x(\alpha) \notin U_p$ $h(x) = g(x)$. Now note that $\operatorname{fix}(h) = \emptyset$: if $r \in \pi_{\{\alpha\}}(\operatorname{fix}(h))$, then $r \notin U_p$ so $r \in \pi_{\{\alpha\}}(\operatorname{fix}(g)) \cap (X^A - U_p) = \emptyset$ by *iv*); hence, we arrive at a contradiction with the fixed point property for X^A which follows from the assumptions about X_α's and Theorem 4. ∎

DEFINITION 7. Let $B \subseteq A$, $x_B \in X^B$, $f: X^A \longrightarrow X^A$, then $f_{x_B}: X^{A-B} \longrightarrow X^{A-B}$ is defined as follows:

$$f_{x_B}(y_{A-B}) = (f(x_B {}^\frown y_{A-B}))_{A-B}.$$

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LEMMA 8. *Let $B \subseteq A$, $A, B \in [\kappa]^\omega$, $f: X^A \longrightarrow X^A$, such that f_B depends on B and that there is a fixed point $x \in X^A$ of f. If y_{A-B} is a fixed point of f_{x_B}, then $x_B {}^\frown y_{A-B}$ is a fixed point of f*

PROOF. $f(x_B {}^\frown y_{A-B})_B = x_B$ since f_B depends on B and x is a fixed point of f. Also, $(f(x_B {}^\frown y_{A-B}))_{A-B} = f_{x_B}(y_{A-B}) = y_{A-B}$, which completes the proof. ∎

LEMMA 9. *Let $A, B \in [\kappa]^\omega$, $B \subseteq A$, $\alpha \in A - B$, $F \subseteq X^A$ be closed , $p_0, p_1 \in X_\alpha$, $p_0 \neq p_1$, $F_{p_0}, F_{p_1} \subseteq F$, $cl_B(\pi_B(F_{p_0})) \cap cl_B(\pi_B(F_{p_1})) \neq \emptyset$. Suppose also*

i) $\pi_{\{\alpha\}}(\pi_B^{-1}(\pi_B(F_{p_i})) \cap F) = \{p_i\}$ for $i = 0, 1$

ii) $\pi_{\{\alpha\}}(F) \subseteq \{p_0, p_1\}$.

Suppose that $f: X^A \longrightarrow X^A$, f_B depends on B and that for all $x \in F$ we have $f(x) = x$. Then there is $x \in X^A - F$ such that $f(x) = x$.

PROOF. Suppose the opposite i.e. $f(x) = x$ if and only if $x \in F$. Let $x \in F$. Consider the set of all fixed points of f_{x_B}. By lemma 8, if $y_{B-A} \in \text{fix}(f_{x_B})$, then $(x_B {}^\frown y_{A-B}) \in \text{fix}(f)$. So by the assumption that $f(x) = x$ iff $x \in F$, we obtain that $x_B {}^\frown y_{A-B} \in F$. Hence, by *i)* we obtain that for each $x \in F_{p_i}$ we have $\pi_{\{\alpha\}}(\text{fix}(f_{x_B}) \subseteq \{p_i\}$. Generally by *ii)* for each $x \in F$ we have $\pi_{\{\alpha\}}(\text{fix}(f_{x_B})) \subseteq \{p_0, p_1\}$.

Let $z \in F$ be such that $z_B \in cl_B(\pi_B(F_{p_0})) \cap cl_B(\pi_B(F_{p_1}))$. Now in order to arrive at a contradiction, we are going to apply lemma 6 for $f = f_{z_B}: X^{A-B} \longrightarrow X^{A-B}$, $\Phi = \{f_{x_B} : x \in F_{p_0}\}$, $\Gamma = \{f_{x_B} : x \in F_{p_1}\}$ in the following way: lemma 6 implies that there is an $r \in X_\alpha$, $r \notin \{p_0, p_1\}$ such that $r \in \pi_{\{\alpha\}}(\text{fix}(f_{z_B})$. But if $y_{A-B} \in \text{fix}(f_{z_B})$ and $\pi_{\{\alpha\}}(z_B {}^\frown y_{A-B}) = r$ then $z_B {}^\frown y_{A-B} \notin F$ by *ii)*. On the other hand by lemma 8, $z_B {}^\frown y_{A-B}$ is a fixed point of f. It contradicts the assumption that $f(x) = x$ if and only if $x \in F$.

So let us check if the assumptions of lemma 6 are satisfied. The assumption *ii)* is satisfied by our above observations. For the assumption *i)* fix $\varepsilon > 0$. Since X^A is compact, $f: X^A \longrightarrow X^A$ is uniformly continuous. Hence there is $\delta > 0$ such that if $\rho^A(x, x') < \delta$, then $\rho^A(f(x), f(x')) < \varepsilon$. Since $z_B \in cl_B(\pi_B(F_{p_0}))$, there is an $x \in F_{p_0}$ such that $\rho^B(z_B, x_B) < \delta$. We claim that $D^{A-B}(f_{z_B}, f_{x_B}) < \varepsilon$. This is because for any $y_{A-B} \in X^{A-B}$ we have

$$\rho^A(z_B {}^\frown y_{A-B}, x_B {}^\frown y_{A-B}) = \rho^B(z_B, x_B) < \delta$$

(see 2) section 1). Hence $\rho^A(f(z_B {}^\frown y_{A-B}), f(x_B {}^\frown y_{A-B})) < \varepsilon$. So (by 1) section1)

$$\rho^{A-B}(f_{z_B}(y_{A-B}), f_{x_B}(y_{A-B})) < \varepsilon$$

for any $y_{A-B} \in X^{A-B}$. Hence $f_{x_B} \in \Phi$ and $D^{\{\alpha\}}(f_{z_B}, f_{x_B}) < \varepsilon$ (again by the above and 1) section 1)). So, the assumption *i)* is satisfied for Φ. Checking *i)* for Γ is similar. ∎

LEMMA 10. *Let $A \in [\kappa]^\omega$. Suppose that $F \subseteq X^\kappa$ is closed, that $f: X^\kappa \longrightarrow X^\kappa$ and that f_A depends on A. If there is $x_A \in (X^A - \pi_A(F))$ such that $f^A(x_A) = x_A$, then there exists $x \in (X^\kappa - F)$ such that $x = f(x)$.*

PROOF. Take $f_{x_A}: X^{\kappa-A} \longrightarrow X^{\kappa-A}$. It has a fixed point, say $y_{\kappa-A}$ by theorem 4. Since $f^A(x_A) = x_A$, as in lemma 8, we obtain that $x_A {}^\frown y_{\kappa-A}$ is a fixed point of f and that it is not in F since its projection on A, x_A is not in the projection of F on A. ∎

DEFINITION 11. $F \subseteq 2^\kappa$ is called a *splitting* set iff F is closed and there is a stationary set $S \subseteq [\kappa]^\omega$ such that for each $B \in S$ there is an $\alpha \in \kappa - B$, $F_0, F_1 \subseteq F$ such that

i) $cl_B(\pi_B(F_0)) \cap cl_B(\pi_B(F_1)) \neq \emptyset$

ii) $\pi_{\{\alpha\}}(\pi_B^{-1}(\pi_B(F_i)) \cap F) = \{i\}$ *for* $i = 0, 1$.

THEOREM 12. *Suppose that* κ *is an uncountable cardinal and that* (X_α)*'s for* $\alpha < \kappa$ *are compact and convex subsets of some normed linear spaces and have at least two points. If there exists a splitting set* $F \subseteq 2^\kappa$ *then* $\prod_{\alpha<\kappa} X_\alpha$ *does not have the complete invariance property.*

PROOF. Choose distinct $p_0^\alpha, p_1^\alpha \in X_\alpha$ distinct. We may now find a homeomorphic copy (using the coordinatewise homeomorphism between 2^κ and $\prod_{\alpha<\kappa}\{p_0^\alpha, p_1^\alpha\}$) of a splitting set F in $\prod_{\alpha<\kappa}\{p_0^\alpha, p_1^\alpha\}$ which is a closed subset of $\prod_{\alpha<\kappa} X_\alpha$. We will show that for this copy denoted also by F there is no continuous function $f: \prod_{\alpha<\kappa} X_\alpha \longrightarrow \prod_{\alpha<\kappa} X_\alpha$ such that $f(x) = x$ if and only if $x \in F$.

Let $C_f = \{A \in [\kappa]^\omega : f_A \text{ depends on } A\}$. Since C_f is a club set by lemma 3, so take $B \in C_f \cap S$, $\alpha \in \kappa - B$ where S and α are taken from the definition of a splitting set (definition 11). Also fix $A \in [\kappa]^\omega \cap C_f$ such that $B \subseteq A$, $\alpha \in A$ (by cofinality of club sets in $[\kappa]^\omega$). Now apply lemma 9 for $F_{p_i} = \pi_A(F_i)$ and $f = f^A$. By definition the assumptions of the lemma are satisfied; in particular, *ii*) follows from the fact that $F \subseteq \prod_{\alpha<\kappa}\{p_0^\alpha, p_1^\alpha\}$. So by lemma 9 there is an $x_A \notin \pi_A(F)$ such that $f^A(x_A) = x_A$. Now apply lemma 10 to get $x \notin F$ such that $f(x) = x$. ∎

4. A construction of a closed set.

THEOREM 13. *There is a splitting set* $F \subseteq 2^{\omega_1}$

PROOF. We will construct closed $\pi_\alpha(F)$'s by induction on α such that $\omega < \alpha \leq \omega_1$. Of course we will require that

i) $\forall \beta < \alpha \ \pi_\beta(F) = \pi_\beta(\pi_\alpha(F))$

ii) If α is a limit ordinal then $\pi_\alpha(F) = \{x \in 2^\alpha : \forall \beta < \alpha \ x_\beta \in \pi_\beta(F)\}$

i.e. the construction is an inverse limit construction. For $\alpha = \omega$ we put $\pi_\alpha(F) = 2^\omega$. For any α such that $\omega \leq \alpha \leq \omega_1$ we will ensure that 1_α is a non-isolated point of $\pi_\alpha(F)$, where $1 \in 2^{\omega_1}$ is such that $1(\beta) = 1$ for all $\beta < \omega_1$. Note that if 1_β is a non-isolated point of $\pi_\beta(F)$ for any $\beta < \alpha$ and α is a limit ordinal, then according to *ii*), 1_α is a non-isolated point in $\pi_\alpha(F)$; also, *ii*) implies that $\pi_\alpha(F)$ is closed in 2^α and that for each $\beta < \alpha \ \pi_\beta(F) = \pi_\beta(\pi_\alpha(F))$ in the case of α limit.

So now let us assume that $\alpha = \beta + 1$ and that we are given $(\pi_\gamma(F))_{\gamma \leq \beta}$ satisfying *i*) with a non-isolated 1_β. The above and the fact that β has to be countable in this case and that 2^β is zero-dimensional, implies that there is a sequence $(U_n^\beta)_{n\leq\omega}$ of clopen subsets of $\pi_\beta(F)$ such that

iii) $(U_n^\beta)_{n\leq\omega}$ is a nested neighbourhood base at 1^β;

iv) $\forall n \, (U_n^\beta - U_{n+1}^\beta) \neq \emptyset$;

v) $U_0^\beta = \pi_\beta(F)$.

Now define $\pi_\alpha(F)$: $x \in \pi_\alpha(F)$ if and only if $x \in 2^\alpha$ and the following conditions are satisfied:

vi) $x_\beta \in \pi_\beta(F)$;
vii) $(x(\beta) = i)$ iff $(x_\beta \in (U_n^\beta - U_{n+1}^\beta)$ and $n = 2k + i$ for some $k \in \omega)$ or $(1_\beta = x_\beta)$.

So 1_β is split in both possible directions 0 and 1, while the points from $\bigcup_{k\in\omega}(U_{2k+i}^\beta - U_{2k+i+1}^\beta)$ have only extensions with projections on $\{\beta\}$ equal to i. Now we have to check that $\pi_\alpha(F)$ is closed , with non-isolated 1_α and satisfies *i*). By *v*) we get $\pi_\beta(\pi_\alpha(F)) \supseteq \pi_\beta(F)$, by *vi*) $\pi_\beta(\pi_\alpha(F)) \subseteq \pi_\beta(F)$; hence, by the induction assumptions, *i*) is satisfied. Suppose now that 1_α were isolated. Then thate would be $U \subseteq \pi_\beta(F)$ open such that $1_\beta \in U$ and $(\pi_\alpha(F) - \{1_\alpha\}) \cap (U \times \{1\}) = \emptyset$. But, by *iii*) there is $n \in \omega$ such that $U_n^\beta \times \{1\} \subseteq U \times \{1\}$. Let $y \in (U_{2n+1}^\beta - U_{2n+2}^\beta)$ (by *iv*), by *vii*) $y^\frown 1 \in \pi_\alpha(F)$ and $y^\frown 1 \in U \times \{1\}$, which gives us a contradiction. Also, $\pi_\alpha(F)$ is closed in 2^α because if $x \in (2^\alpha - \pi_\alpha(F))$, then in the case $x_\beta \notin \pi_\beta(F)$ we are done because $\pi_\beta(F)$ is closed. If $x_\beta \in \pi_\beta(F)$, then $x_\beta \in (U_{2k+i}^\beta - U_{2k+i+1}^\beta)$ and $x(\beta) = 1 - i$, for some $i = 0, 1$ but then $(U_{2k+i}^\beta - U_{2k+i+1}^\beta) \times \{1 - i\}$ is clopen and isolates x_β from $\pi_\alpha(F)$.

Now let us check that $F = \pi_{\omega_1}(F)$ is a splitting set. Put $S = \{B \in [\omega_1]^\omega : B \in \omega_1\}$, fix $B \in S$ and let $\alpha = B$. Let

$$F_i = \{x \in F : \ \pi_\beta(x) \in \bigcup_{k\in\omega}(U_{2k+i}^\beta - U_{2k+i+1}^\beta)\}$$

Let us check *i*) of the definition 11. We claim that

$$1_\beta \in cl_\beta(\pi_\beta(F_0)) \cap cl(\pi_\beta(F_1))$$

which follows from *iii*). For *ii*) of the definition 11, let $x \in \pi_\beta^{-1}(\pi_\beta(F_i)) \cap F$, which means that $\pi_\beta(x) = x_\beta \in \pi_\beta(F_i)$ and $x \in F$. This implies that $\pi_\beta(x) \in \bigcup_{k\in\omega}(U_{2k+i}^\beta - U_{2k+i+1}^\beta)$ and that $\pi_{\beta+1}(x) \in \pi_{\beta+1}(F)$, but this means (by *vii*) that $\pi_{\{\alpha\}}(x) = i$. This completes the proof of the theorem. ∎

LEMMA 14. *If there is a splitting $F \subseteq 2^{\omega_1}$, then there is a splitting $H \subseteq 2^\kappa$ for any uncountable ordinal κ.*

PROOF. Let S_F be a stationary set ensuring that F is a splitting set. If $B \in S_F$ let $\alpha_B \in (\omega_1 - B)$, F_0^B, F_1^B be such that *i*), *ii*) of the definition 11 are satisfied. Let us check that $H = F \times 2^{\kappa - \omega_1}$ is a splitting subset of 2^κ. Of course H is closed. Moreover $S = \{B \in [\kappa]^\omega : B \cap \omega_1 \in S_F\}$ is stationary. Take $B \in S$, such that $B \cap \omega_1 = B' \in S_F$ and take $F_0^{B'} \times 2^{B-B'}$, $F_1^{B'} \times 2^{B-B'}$. Then *i*), *ii*) of the definition 11 are satisfied. ∎

As corollaries from the above lemmas we obtain the following theorems.

THEOREM 15. *For any uncountable κ there is a splitting set $F \subset 2^{\kappa}$*

THEOREM 16. *Suppose each X_α has at least two points and is a compact, convex subset of a normed linear space. Suppose κ is an uncountable cardinal, then $\prod_{\alpha<\kappa} X_\alpha$ does not have the complete invariance property.*

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