Canad. Math. Bull. Vol. 28 (4), 1985

FINITENESS OF NEGATIVE SPECTRA OF ELLIPTIC OPERATORS

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ABSTRACT. Finiteness criteria are established for the negative spectra of higher order elliptic operators on R^n. The results are obtained by establishing isomorphism theorems for elliptic operators and applying the abstract finiteness criterion of Konno-Kuroda.

The present paper is concerned with a spectral problem for the self-adjoint elliptic operator

$$H = \sum_{|\alpha|,|\beta|\leqq m} D^\alpha a_{\alpha\beta}(x) D^\beta$$

on R^n. The problem of whether the negative spectrum of H is finite has been investigated by many mathematicians. But most of the results for the problem were established in the case that $m = 1$ or $n = 1$ (see [4], [5], [7], [11], [12], and references therein), with some results also for fourth order operators in exterior domains of R^n, $n > 4$ (cf. [3]). In [9], Theorem 1.3, the author gave a finiteness criterion in the case that $2m < n$ (see also [6], Theorem 4.20). the purpose of this paper is to establish finiteness criteria in the case that $2m \geqq n$. Some isomorphism theorems on suitable function spaces, which are modifications of those in [10], play a crucial role in establishing the criteria.

1. **Main result**. We write $D_j = -i\partial/\partial x_j$, $D = (D_1, \dots, D_n)$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\langle x \rangle = (1 + |x|^2)^{1/2}$. For a real number t, $[t]$ denotes the largest integer smaller than or equal to t. We denote by $H^m(R^n)$ the usual Sobolev space of order m. Consider the sesqui-linear form

$$(1.1) \qquad h[u, v] = \sum_{|\alpha|,|\beta|\leqq m} \int a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha v}(x) dx, \qquad u, v \in H^m(R^n),$$

where the coefficients $a_{\alpha\beta}(x)$ satisfy the following condition:

(A.I) (*i*) The $a_{\alpha\beta}$ are bounded measurable functions on R^n and all $a_{\alpha\beta}$ with $|\alpha| = |\beta| = m$ are uniformly continuous on R^n; (*ii*) $a_{\alpha\beta} = \overline{a_{\beta\alpha}}$; (*iii*) there exists $\mu_0 > 0$ such that

Received by the editors January 19, 1984 and, in revised form, October 15, 1984.
AMS Subject Classification (1980): 35P99, 35B05.

https://doi.org/10.4153/CMB-1985-060-1 Published online by Cambridge University Press

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta} \geqq \mu_0|\xi|^{2m}, \qquad x, \xi \in R^n.$$

Let H be the self-adjoint operator associated with h in the sense of Friedrichs. That is, H is the Friedrichs' extension of a formal differential operator on R^n:

$$(1.2) \qquad H = \sum_{|\alpha|,|\beta| \leqq m} D^\alpha a_{\alpha\beta}(x) D^\beta.$$

Concerning the problem of whether the negative spectrum of H is finite we introduce the following conditions.

(A.II) There exist $R > 0$ and $0 < c_0 \leqq \mu_0$ such that

$$(1.3) \qquad \sum_{|\alpha|=|\beta|=m} (a_{\alpha\beta}D^\beta u, D^\alpha u) \geqq c_0((-\Delta)^m u, u)$$

for all $u \in C_0^\infty(\{x \in R^n; |x| > R\})$, where $(\ ,\ )$ is the L_2-inner product.

(A.III) There exists $\delta > 0$ such that

$$(1.4) \qquad \liminf_{|x|\to\infty} |x|^{2m} (\log |x|)^{2k(0)} a_{00}(x) > -\delta,$$

$$(1.5) \qquad \limsup_{|x|\to\infty} |x|^{2m-|\alpha+\beta|} (\log |x|)^{k(\alpha)+k(\beta)} |a_{\alpha\beta}(x)| < \delta$$

$$0 < |\alpha + \beta| < 2m,$$

where $k(\alpha)$ is given by: (i) If $2m \geqq n$ and n is even,

$$(1.6) \qquad k(\alpha) = \max\left(0, \left[\frac{2(m - |\alpha|) - n}{4}\right] + 1\right);$$

(ii) $k(\alpha) = 0$ otherwise.

Our main result is the following

THEOREM 1.1. *Let H be the self-adjoint operator* (1.2) *satisfying* (A.I) $\sim$ (A.III). *Then there exists a positive constant δ_0 depending on c_0, m, n such that if $\delta \leqq \delta_0$, then H has at most finite number of negative eigenvalues of finite multiplicity.*

REMARK 1.2. *The optimal constant δ_0 can be calculated for a large class of operators, since the conditions* (1.4) *and* (1.5) *are given by radial functions and finiteness criteria are extensively established for ordinary differential operators (cf.* [7]*). Here we mean by "optimal" that there is an operator satisfying* (A.I) $\sim$ (A.III) *with $\delta > \delta_0$ whose negative spectrum is infinite. For example, consider the operator*

$$H = \Delta^2 - q \text{ on } R^2 \text{ or } R^4,$$

where $q(x)$ is a real-valued bounded measurable function. Then the negative spectrum of H is finite if

$$\limsup_{|x|\to\infty} |x|^4 (\log|x|)^2 q(x) < 1,$$

and is infinite if $\liminf_{|x|\to\infty}, |x|^4(\log |x|)^2 q(x) > 1$. (*See* [2], *p. 7,* [7], *Theorem 31,* *p. 40, and the proof of Theorem 1.1 below.)*

2. PROOF OF THEOREM 1.1. We shall show Theorem 1.1 only in the case that $2m > n$ and n is even, since the proof for the other case is similar.

With $k(\alpha)$ given in (1.6) we put

$$(2.1)\qquad X = \{f \in H^m_{\mathrm{loc}}(R^n);\ \|f\|_X \equiv \Big(\sum_{|\alpha|\leqq m} \int |\langle x\rangle^{|\alpha|-m}(1+\log\langle x\rangle)^{-k(\alpha)}D^\alpha f(x)|^2 dx\Big)^{1/2} < \infty\Big\}.$$

We denote by X' the dual space of the Banach space X.

LEMMA 2.1. *There exist positive constants c_1, C_1, C_2 depending only on m and n such that*

$$(2.2)\qquad \|u\|_X^2 \leqq c_1\Big\{((-\Delta)^m u, u) + C_1\int_{|x|<C_2} |u(x)|^2 dx\Big\},\qquad u \in C_0^\infty(R^n).$$

PROOF. We have that for any $0 \leqq j \leqq m$

$$\sum_{|\alpha|=j} ((-\Delta)^{m-|\alpha|}D^\alpha u, D^\alpha u) \leqq B_{j,n}((-\Delta)^m u, u),\qquad u \in C_0^\infty(R^n),$$

where $B_{j,n} = \max\{\sum_{|\alpha|=j} |\xi^\alpha|^2;\ |\xi| = 1\}$. This together with Lemma 0, Corollaries 1 and 2 in [2] shows that

$$\sum_{|\alpha|=j} \|\langle x\rangle^{|\alpha|-m}(1+\log\langle x\rangle)^{-k(\alpha)}D^\alpha f(x)\|_2^2 \leqq B_{j,n}C_{m-j,n}\|(-\Delta)^{m/2}f\|_2^2$$

for all $f \in C_0^\infty(\{x \in R^n; |x| > N\})$, where N is a sufficiently large number and $C_{m-j,n}$ are positive constants depend only on j and n with $C_{0,n} = 1$. Choosing a C^∞-function ϕ such that $0 \leqq \phi \leqq 1$, $\phi(x) = 1$ for $|x| \leqq N + 1$ and $\phi(x) = 0$ for $|x| \geqq N + 2$, we thus get

$$\|(1-\phi)u\|_X^2 \leqq \Big(\sum_{j=0}^m B_{j,n}C_{m-j,n}\Big)\|(-\Delta)^{m/2}(1-\phi)u\|_2^2,\qquad u \in C_0^\infty(R^n).$$

This together with the standard a priori estimate (see [1], Theorem 15.1′, p. 703) yields (2.2) with $c_1 < \sum_{j=0}^m B_{j,n}C_{m-j,n}$, $C_2 = N + 3$, and C_1 sufficiently large. Q.E.D.

Choosing a C^∞-function q such that $q \geqq 0$, $q(x) = C_1$ for $|x| \leqq C_2$, and $q(x) = 0$ for $|x| \geqq C_2 + 1$, put

$$(2.3)\qquad A = (-\Delta)^m + q.$$

Then we have

LEMMA 2.2. *The operator A from X to X' is an isomorphism satisfying*

$$(2.4)\qquad \|u\|_X \leqq c_1\|Au\|_{X'},\qquad u \in X.$$

PROOF. The limiting argument shows that (2.2) holds for all u in X. Thus

$$(2.5)\qquad \|u\|_X^2 \leqq c_1|(Au,u)| \leqq c_1\|Au\|_{X'}, \|u\|_X,$$

which yields (2.4). The estimate (2.4) implies that A is injective and has closed range. Since $A^* = A$, we thus obtain that A is an isomorphism. Q.E.D.

Since A is a nonnegative operator in $L_2(R^n)$, we see that the resolvent $R(\lambda) = (A - \lambda)^{-1}$ of A exists for any $\lambda < 0$ and is a bounded operator from $H^{-m}(R^n)$ to $H^m(R^n)$. Thus $R(\lambda)$ is a bounded operator from X' to X, for $X' \subset H^{-m}(R^n)$ and $H^m(R^n) \subset X$.

LEMMA 2.3. *(i)* $\|R(\lambda)\|_{X'\to X} \leqq c_1^{-1}$ *for all* $\lambda < 0$. *(ii) For any* $f \in X'$, $R(\lambda)f \to A^{-1}f$ *in* X *as* $\lambda \to 0$.

PROOF. The assertion *(i)* can be shown in the same way as (2.4). For f in X' with $A^{-1}f \in C_0^\infty(R^n)$, we have that

$$R(\lambda)f - A^{-1}f = \lambda R(\lambda)(A^{-1}f) \to 0 \text{ in } X \text{ as } \lambda \to 0.$$

This together with *(i)* implies *(ii)*, since the set $\{f \in X'; A^{-1}f \in C_0^\infty(R^n)\}$ is dense in X'. Q.E.D.

LEMMA 2.4. *Let* $\delta_0 = c_0c_1^{-1}$ *and* δ *be the constant in* (A.III). *If* $\delta \leqq \delta_0$, *then there exist* $\mu > 0$ *and* $\phi \in C_0^\infty(R^n)$ *such that*

$$(2.6)\qquad (Hf,f) \geqq \mu((A - \phi^2)f, f), \qquad f \in H^m(R^n).$$

PROOF. We have by (A.I) and (A.II) that

$$(2.7)\qquad H \geqq c_0B \equiv c_0(-\Delta)^m + \sum_{0<|\alpha+\beta|<m} D^\alpha a_{\alpha\beta}(x)D^\beta + \min(0, a_{00}(x)).$$

By the assumption (A.III) and (2.5), for any $\epsilon > 0$ there exist C_ϵ and N_ϵ such that

$$(2.8)\qquad (Bf,f) \geqq \{1 - (\delta - \epsilon)c_0^{-1}c_1 - \epsilon\}(Af, f) - C_\epsilon \int_{|x|\leqq N_\epsilon} |f(x)|^2\, dx.$$

Choosing ϵ so small that $1 - (\delta_0 - \epsilon)\, c_0^{-1}c_1 - \epsilon > 0$, we thus get the lemma from (2.7) and (2.8). Q.E.D.

COMPLETION OF THE PROOF OF THEOREM 1.1. We see that the multiplication operator $\phi\cdot$ is a compact operator from X (or L_2) to L_2 (or X'). Thus Lemma 2.3 shows that *(i)* $\phi R(\lambda)\phi$ is a compact operator from L_2 to L_2 for each $\lambda < 0$; *(ii)* $\phi R(\lambda)\phi \to \phi A^{-1}\phi$ in the operator norm as $\lambda \to 0$. Hence Corollary in [8], p. 57, shows that the negative spectrum of $A - \phi^2$ is finite, which together with Lemma 2.4 proves the theorem. Q.E.D.

3. **Concluding remarks**.

REMARK 3.1. It is easily seen from the proof of Theorem 1.1 that the formal differential operator H satisfying (A.I) ~ (A.III) with $\delta \leqq \delta_0$ is non-oscillatory. That is, there exists $R > 0$ such that for any bounded domain $\Omega \subset \{|x| > R\}$ there are no nontrivial solutions of the equation $Hu = 0$, $u \in H_0^m(\Omega)$ (cf. [2]).

REMARK 3.2. Let Ω be an exterior domain of R^n. Let H be the self-adjoint operator (1.2) in $L_2(\Omega)$ associated with Dirichlet boundary condition, where $a_{\alpha\beta}$ satisfy (A.I) ~ (A.III). Then the conclusion of Theorem 1.1 clearly holds also for H.

REMARK 3.3. The same argument as in Section 2 shows that the conclusion of Theorem 1.1 is valid also for H satisfying (A.I) and the following conditions:

(A.II′). There exist positive numbers R, c_0, c_1 and an integer $0 < \ell < m$ such that

$$\sum_{|\alpha|,|\beta| \geqq 2\ell} \int a_{\alpha\beta}(x) D^\beta u(x) \overline{D^\alpha u}(x)\,dx \geqq \int \bar{u}(x)\{c_0(-\Delta)^\ell + c_1(-\Delta)^m\} u(x)\,dx$$

for all $u \in C_0^\infty(\{x \in R^n; |x| > R\})$.

(A.III′). The condition (A.III) with m replaced by ℓ holds.

REMARK 3.4. Theorem 1.1 can be extended to such elliptic systems as Dirac operators (cf. [10]).

ACKNOWLEDGEMENT. The author would like to express his cordial thanks to the referee for informing him of Birman-Solomjak's work [6].

REFERENCES

1. S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, I, Comm. Pure Appl. Math. XII (1959), pp. 623–727.

2. W. Allegretto, *A kneser type theorem for higher order elliptic equations*, Canad. Math. Bull. **20**, (1977), pp. 1–8.

3. ———, *Finiteness of lower spectra of a class of higher order elliptic operators*, Pacific J. Math. **83**, (1979), pp. 303–309.

4. ———, *Positive solutions and spectral properties of second order elliptic operators*, Pacific J. Math. **92**, (1981), pp. 15–25.

5. M. Š. Birman, *The spectrum of singular boundary problems*, Amer. Math. Soc. Translations, Ser. 2, **53**, (1966), pp. 23–80.

6. M. Š. Birman and M. Z. Solomjak, *Quantitative analysis in Soblev imbedding theorems and applications to spectral theory*, Amer. Math. Soc. Translations, Ser. 2, **114**, (1980).

7. I. M. Glazman, *Direct methods of qualitative spectral analysis of singular differential operators*, Israel Program of Scientific Translations, Jerusalem, 1965.

8. R. Konno and S. T. Kuroda, *On the finiteness of perturbed eigenvalues*, J. Fac. Sci. Univ. Tokyo Sec. I, **13**, (1966), pp. 55–63.

9. M. Murata, *Finiteness of eigenvalues of self-adjoint elliptic operators*, J. Fac. Sc. Univ. Tokyo Sec. IA, **25**, (1978), pp. 205–218.

10. ———, *Isomorphism theorems for elliptic operators in R^n*, Comm. in Partial Differential Equations, **9**, (1984), in press.

11. J. Piepenbrink, *A conjecture of Glazman*, J. Diff. Eq. **24**, (1977), pp. 173–177.

12. M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV.* Analysis of Operators, Academic Press, New York, 1978.

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