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R-Diagonal Elements and Freeness With Amalgamation

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Abstract. The concept of R-diagonal element was introduced in [5], and was subsequently found to have applications to several problems in free probability. In this paper we describe a new approach to R-diagonality, which relies on freeness with amalgamation. The class of R-diagonal elements is enlarged to contain examples living in non-tracial $*$-probability spaces, such as the generalized circular elements of [7].

1 Introduction and Statement of the Results

We will consider the framework of a $*$-*probability space* $(\mathcal{A}, \varphi)$; this means that $\mathcal{A}$ is a unital $*$-algebra over $\mathbf{C}$, and that $\varphi\colon \mathcal{A} \to \mathbf{C}$ is a linear functional, normalized by $\varphi(I) = 1$ (where I is the unit of $\mathcal{A}$), and selfadjoint, in the sense that $\varphi(a^*) = \overline{\varphi(a)}$ for every $a \in \mathcal{A}$. An element $a \in \mathcal{A}$ will be occasionally referred to as a "non-commutative random variable", and $\varphi(a)$ will be called "the expectation of a". The $*$-probability space $(\mathcal{A}, \varphi)$ is said to be *tracial* if φ has the trace property ($\varphi(ab) = \varphi(ba)$, for every $a, b \in \mathcal{A}$).

If $(\mathcal{A}, \varphi)$ is a $*$-probability space and if $a \in \mathcal{A}$, then the expectations of words made with a and a^* will be called $*$-*moments* of a. The family of $*$-moments of a:

$$\big\{ \varphi(a^{s_1} \cdots a^{s_n}) \mid n \geq 1, s_1, \ldots, s_n \in \{1, *\} \big\}, \tag{1.1}$$

carries significant information about a. For instance, if $\mathcal{A}$ is a C^*-algebra and if φ is positive definite and faithful on positive elements, then the family (1.1) determines (up to isomorphism) the unital C^*-subalgebra of $\mathcal{A}$ generated by a and a^*; a similar fact is true in the framework of von Neumann algebras (see *e.g.* [11], Remark 1.8).

If $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ are $*$-probability spaces, we will say about two elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$ that they are *identically $*$-distributed* if their $*$-moments coincide:

$$\varphi(a^{s_1} \cdots a^{s_n}) = \psi(b^{s_1} \cdots b^{s_n}) \quad \text{for every } n \geq 1 \text{ and } s_1, \ldots, s_n \in \{1, *\}. \tag{1.2}$$

A fundamental concept used throughout the paper is the one of *freeness* for a family of subsets of $\mathcal{A}$ (where $(\mathcal{A}, \varphi)$ is a $*$-probability space). For the definition and basic properties of freeness, we refer the reader to [13], Chapter 2.

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The concept of R-diagonal element was introduced in [5], and was subsequently found to play an important role in several problems in free probability (see *e.g.* [6], [2], [3]). Loosely speaking, the name "R-diagonal" refers to elements which have a factorization of the form:

$$a = up, \tag{1.3}$$

where u is a unitary such that $\varphi(u^n) = 0$ for every $n \geq 1$, and where the sets $\{u, u^*\}$ and $\{p, p^*\}$ are free. A more formal definition of the fact that a is R-diagonal must amount to stating that certain equations are satisfied by the family of $*$-moments of a. (Equation (1.3) does this, in an implicit way; *i.e.*, the special properties of the family of $*$-moments of $a = up$ are implicitly contained in the equations describing the freeness of $\{u, u^*\}$ from $\{p, p^*\}$.)

The main goal of this paper is to present a new approach to R-diagonality, relying on freeness with amalgamation. For basic facts about freeness with amalgamation see *e.g.* [13] Section 3.8, or [10] (the definition of the concept is also reviewed in Section 3 below).

Several characterizations of R-diagonality being now available, it is no longer obvious which of them is the most suitable to be used as the definition of this notion. Since we could not come to an agreement on this, we just made a list of possible candidates in the Theorem-and-Definition 1.2 below, thus allowing the readers to choose their own favorite(s).

The Theorem 1.2 is at the same time the main result of the paper. The approach to R-diagonality via freeness with amalgamation is given by the characterization 5^o in this theorem. Let us mention that the characterization 1^o of Theorem 1.2 is also closely related to freeness with amalgamation. The equations in $*$-moments appearing in $1.2.1^o$ are in some sense just "an explicit spelling" of $1.2.5^o$; we felt it is worth to point them out, because they can be used as a very elementary approach to R-diagonality, which is reminiscent of how freeness itself is defined.

Notation 1.1 Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let a be an element of $\mathcal{A}$. For every $k \geq 1$ we will denote:

$$\begin{cases} P_{11;k}(a) = a^*(aa^*)^{k-1} \\ P_{12;k}(a) = (a^*a)^k - \varphi\big((a^*a)^k\big)\, I \\ P_{21;k}(a) = (aa^*)^k - \varphi\big((aa^*)^k\big)\, I \\ P_{22;k}(a) = a(a^*a)^{k-1}. \end{cases} \tag{1.4}$$

Theorem and Definition 1.2 *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let a be an element of $\mathcal{A}$. Then the following five conditions on a are equivalent. The element a is said to be R-diagonal if it satisfies one (hence all) of these conditions.*

1^o (Condition on $*$-moments) One has

$$\varphi\Big(P_{i_1 i_2;k_1}(a) P_{i_2 i_3;k_2}(a) \cdots P_{i_n i_{n+1};k_n}(a)\Big) = 0, \tag{1.5}$$

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for every $n \geq 1$, $i_1, \dots, i_n, i_{n+1} \in \{1, 2\}$ and $k_1, \dots, k_n \geq 1$, and where the elements $P_{i\,j;k}(a) \in \mathcal{A}$ are as defined in (1.4).

2º (Sufficient invariance condition) There exist an enlargement $(\tilde{\mathcal{A}}, \tilde{\varphi})$ of $(\mathcal{A}, \varphi)$ and a unitary $u \in \tilde{\mathcal{A}}$, such that: (i) $\{u, u^*\}$ is free from $\{a, a^*\}$; (ii) $\tilde{\varphi}(u) = 0$; (iii) a and ua are identically $*$-distributed.[1]

3º (Necessary Invariance condition) For every enlargement $(\tilde{\mathcal{A}}, \tilde{\varphi})$ of $(\mathcal{A}, \varphi)$ and every unitary $u \in \tilde{\mathcal{A}}$ such that $\{u, u^*\}$ is free from $\{a, a^*\}$, one has that a and ua are identically $*$-distributed.

4º (Condition on Non-Crossing Cumulants) Consider the family of non-crossing cumulants of a and a^*,

$$\left\{ \kappa_{(s_1,\dots,s_n)}(a, a^*) \mid n \geq 1, s_1, \dots, s_n \in \{1, *\} \right\}. \tag{1.6}$$

Then $\kappa_{(s_1,\dots,s_n)}(a, a^*) = 0$ whenever $(s_1, \dots, s_n)$ is not of the form $(1, *, 1, *, \dots, 1, *)$ or $(*, 1, *, 1, \dots, *, 1)$.

5º (Condition Using Freeness With Amalgamation) Let $M_2(\mathcal{A})$ be the algebra of 2×2 matrices over $\mathcal{A}$. Consider the unital subalgebras $\mathcal{D} \subset M_2(\mathbf{C}I) \subset M_2(\mathcal{A})$, where:

$$\mathcal{D} := \left\{ \begin{pmatrix} \alpha I & 0 \\ 0 & \lambda I \end{pmatrix} \,\middle|\, \alpha, \lambda \in \mathbf{C} \right\}, \quad M_2(\mathbf{C}I) := \left\{ \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \lambda I \end{pmatrix} \,\middle|\, \alpha, \beta, \gamma, \lambda \in \mathbf{C} \right\};$$

and consider the conditional expectation $E \colon M_2(\mathcal{A}) \to \mathcal{D}$ given by the formula:

$$E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \varphi(a) I & 0 \\ 0 & \varphi(d) I \end{pmatrix}, \quad a, b, c, d \in \mathcal{A}.$$

Then the matrix:

$$A := \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \in M_2(\mathcal{A}) \tag{1.7}$$

is free from $M_2(\mathbf{C}I)$, with amalgamation over $\mathcal{D}$.

The characterization originally used in [5] to define R-diagonality was the one in terms of non-crossing cumulants, 1.2.4º (the precise definition for the family (1.6) of non-crossing cumulants will be reviewed in Section 4 below). A version of the characterizations in 1.2.2º–3º appears in [6], Proposition 4.4. Note however that R-diagonality (and in particular the equivalence between the characterizations

[1] By the fact that $(\tilde{\mathcal{A}}, \tilde{\varphi})$ is an enlargement of $(\mathcal{A}, \varphi)$ we mean that $(\tilde{\mathcal{A}}, \tilde{\varphi})$ is a $*$-probability space, $\tilde{\mathcal{A}} \supset \mathcal{A}$, and $\tilde{\varphi}|\mathcal{A} = \varphi$. Note that one can always find $(\tilde{\mathcal{A}}, \tilde{\varphi})$ and $u \in \tilde{\mathcal{A}}$ such that (i) + (ii) hold. *e.g.*, one can take $(\tilde{\mathcal{A}}, \tilde{\varphi})$ to be the free product $(\mathcal{A}, \varphi) \star (L^\infty(\mathbf{T}), dz)$, where dz is the Haar measure on the torus; and one can take u to be the function $z \mapsto z$ in $L^\infty(\mathbf{T}) \subset \tilde{\mathcal{A}}$. The key condition in 1.2.2º is (iii).

which were already known) was previously discussed only in the framework of a $*$-probability space which is tracial. Removing the traciality condition is of relevance, because there exist natural examples of elements which satisfy the condition 1.2.3^o, but live in a non-tracial framework (see [7], Section 4).

We now return to make precise the "free factorization" property mentioned in Equation (1.3). As is customary in the literature on free probability, we will call *Haar unitary* an element u in a $*$-probability space $(\mathcal{A}, \varphi)$ such that u is unitary, and such that $\varphi(u^n) = 0$ for every $n \geq 1$.

Theorem 1.3 *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Let $u, p \in \mathcal{A}$ be such that u is unitary and such that $\{u, u^*\}$ is free from $\{p, p^*\}$. If either*

(A) $\varphi(u) = 0$ and $\varphi\big(p(p^*p)^{k-1}\big) = 0$, for every $k \geq 1$; or
(B) u is a Haar unitary,

then the element $a := up \in \mathcal{A}$ is R-diagonal.

There also is another side of the result presented in Theorem 1.3. Namely, given an R-diagonal element b in a $*$-probability space $(\mathcal{B}, \psi)$, we can ask: what kind of u and p can one fabricate (in some $(\mathcal{A}, \varphi)$), such that the hypothesis of 1.3 is satisfied (in either version (A) or (B)), and such that $a := up$ is identically $*$-distributed with b? We don't know of some special construction working for an arbitrary R-diagonal b—except of course the choice having $p = b$, which is guaranteed by Theorem 1.2.3^o. As the next proposition immediately implies, the free factorization game amounts essentially to finding a nice p such that p^*p, pp^* have identical moments with b^*b, bb^*, respectively. An example of how this can work is provided by the case of a tracial $*$-probability space, when selfadjoint choices of p are available—see Corollary 1.5.

Proposition 1.4 *Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be $*$-probability spaces, and let $a \in \mathcal{A}$, $b \in \mathcal{B}$ be R-diagonal elements. Then a and b are identically $*$-distributed if and only if:*

$$\begin{cases} \varphi\big((a^*a)^k\big) = \psi\big((b^*b)^k\big), \\ \varphi\big((aa^*)^k\big) = \psi\big((bb^*)^k\big), \end{cases} \qquad \forall k \geq 1. \tag{1.8}$$

Corollary 1.5 *Let $(\mathcal{B}, \psi)$ be a tracial $*$-probability space, and let $b \in \mathcal{B}$ be an R-diagonal element. Then one can find a tracial $*$-probability space $(\mathcal{A}, \varphi)$ and an element $a \in \mathcal{A}$ which is identically $*$-distributed with b, and which is obtained in the following way: $a = up$, where $u \in \mathcal{A}$ is a Haar unitary, $p = p^* \in \mathcal{A}$ is such that $\{u, u^*\}$ is free from $\{p\}$, and we have $\varphi(p^{2k-1}) = 0$, $\forall k \geq 1$.*

Note that the conditions on u and p appearing in Corollary 1.5 are stronger than either the versions (A) or (B) of the hypothesis of Theorem 1.3. The traciality condition in 1.5 is necessary. Indeed, φ is a trace on both the unital algebras generated by $\{u, u^*\}$ and by $\{p\}$ (these algebras being commutative); but then the freeness of $\{u, u^*\}$ from $\{p\}$ implies that φ is also a trace on the unital algebra generated by $\{up, (up)^*\}$—see [13], Proposition 2.5.3.

We next move to the discussion of the concept of "determining series" for an R-diagonal element, and to its applications to operations with free R-diagonal elements.

Definition 1.6 Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a \in \mathcal{A}$ be an R-diagonal element. Consider the family of non-crossing cumulants of a, a^* (denoted as in Equation (1.6) above), and set:

$$\begin{cases} \alpha_n := \kappa_{(\underbrace{1, *, \ldots, 1, *}_{2n})}(a, a^*) \\ \beta_n := \kappa_{(\underbrace{*, 1, \ldots, *, 1}_{2n})}(a, a^*), \end{cases} \qquad \forall n \geq 1. \tag{1.9}$$

Then the formal power series

$$f_a(z) := \sum_{n=1}^{\infty} \alpha_n z^n, \quad g_a(z) := \sum_{n=1}^{\infty} \beta_n z^n \tag{1.10}$$

are called *the determining series* of a.

The series f_a and g_a contain in a concentrated way the information about the $*$-moments of the R-diagonal element a; this is because—in view of Theorem 1.2.4^o—they determine all the non-crossing cumulants in (1.6) (and the knowledge of the non-crossing cumulants is equivalent to the one of the $*$-moments).

An example of situation when it is advantageous to use the determining series is provided by the next Corollary 1.7, concerning the addition of two free R-diagonal elements. The statement in 1.7 follows directly from the characterization 1.2.4^o of R-diagonality, combined with the additivity of non-crossing cumulants (as put into evidence in [9]).

Corollary 1.7 *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let a, b be R-diagonal elements of $\mathcal{A}$ such that $\{a, a^*\}$ is free from $\{b, b^*\}$. Then $a + b$ is also R-diagonal, and we have the relations in determining series:*

$$f_{a+b} = f_a + f_b, \quad g_{a+b} = g_a + g_b. \tag{1.11}$$

Turning to the multiplication of free R-diagonal elements, let us record the following direct consequence of the characterizations 2^o, 3^o in Theorem 1.2 (and of the obvious fact that an element is R-diagonal if and only if its adjoint is so).

Corollary 1.8 *Let $(\mathcal{A}, \varphi)$ be a a $*$-probability space. Let $a, b \in \mathcal{A}$ be such that at least one of a, b is R-diagonal, and such that $\{a, a^*\}$ is free from $\{b, b^*\}$. Then ab is also R-diagonal.*

It is then natural to ask: if a, b of Corollary 1.8 are both R-diagonal, then what is the formula for the determining series of ab, in terms of those of a and of b? In order to give some feeling of what this question is about, let us first mention that, in the case when $(\mathcal{A}, \varphi)$ is tracial, the answer can be easily read from the considerations of [5]; and the instrument used for spelling out the answer is a certain operation "$\boxed{\star}$" on formal power series. (A brief review of $\boxed{\star}$ is made in Section 5.1 below.) The extension from the tracial to the general case turns out to be really non-trivial. Based on calculations of low-order cumulants, we can guess what the formula (expressing

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f_{ab}, g_{ab} in terms of f_a, g_a, f_b, g_b, in the general case) ought to be; but we do not have at this time a proof for it. The difficulty comes from the fact that the formula is very combinatorial, and its proof probably requires some further developments in the combinatorial theory of the non-crossing cumulants (a direction which is not pursued by the present paper). A more detailed discussion of this matter is made in Section 5.

Miscellaneous Remarks 1.9 1^o An immediate consequence of the description in Theorem 1.2.1^o is that if a is R-diagonal, then a^*a and aa^* are free. Indeed, the particular case of Equation (1.5) where the sequence $i_1, i_2, \dots, i_{n+1}$ is alternating (*i.e.* is of the form $1, 2, 1, 2, \dots$ or $2, 1, 2, 1, \dots$) gives precisely the freeness of a^*a from aa^*.

2^o The concept of determining series of an R-diagonal element was introduced in [5], where however only the case of a tracial $(\mathcal{A}, \varphi)$ was considered. In this case f_a and g_a of (1.10) coincide (which is why in [5] only one determining series is considered, instead of two).

In the tracial framework one can obtain a simple formula for the determining series $f_a(= g_a)$, by using the factorization presented in Corollary 1.5. More precisely: let $(\mathcal{A}, \varphi)$ be a tracial $*$-probability space, and let $a = up \in \mathcal{A}$, with u and p as described in Corollary 1.5. Then the determining series f_a satisfies the equation:

$$f_a(z^2) = [R(p)](z), \tag{1.12}$$

where $R(p)$ is the R-transform of p. (For a selfadjoint element $x \in \mathcal{A}$, the R-transform $R(x)$ is a certain formal power series which contains exactly the same information as the family of moments $\big(\varphi(x^n)\big)_{n=1}^{\infty}$—see [13], Section 3.2.) The Equation (1.12) can be used as an alternative definition of f_a, which is however valid only in the tracial framework.

3^o The Haar unitary is one of the main examples of R-diagonal elements which motivated the work in [5]. This is the only R-diagonal unitary, but let us note that (especially if the traciality requirement is lifted) one has a larger family of R-diagonal partial isometries. To be more precise: for every $\alpha, \beta \in (0, 1]$ one can construct an R-diagonal partial isometry v in a $*$-probability space $(\mathcal{A}, \varphi)$ (where it can be arranged that $\mathcal{A}$ is a W^*-algebra and that φ is a faithful normal state of $\mathcal{A}$) such that $\varphi(v^*v) = \alpha$, $\varphi(vv^*) = \beta$. It makes sense to call such a v an "(α, β)-Haar partial isometry". It is clear from Proposition 1.4 that all the $*$-moments of an (α, β)-Haar partial isometry are completely determined by α and β; and also that every R-diagonal partial isometry in the W^*-framework has to be an (α, β)-Haar partial isometry, for some α and β.

In the case when $\alpha = 1$, a way to construct $(1, \beta)$-Haar isometries is by taking the polar decomposition of the "generalized circular elements" considered in [7]. (The generalized circular elements are R-diagonal by the Lemma 4.6 of [7]; the fact that their polar parts are also R-diagonal follows by an immediate application of 2^o–3^o in Theorem 1.2 above. Some relevant calculations concerning these polar parts appear in the Lemma 4.3 of [7].) For arbitrary α and β in $(0, 1]$, an (α, β)-Haar partial

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isometry can be obtained as $v = v_1 v_2^*$, where v_1 is an $(1,\beta)$-Haar isometry, v_2 is an $(1,\alpha)$-Haar isometry, and $\{v_1, v_1^*\}$ is free from $\{v_2, v_2^*\}$.

4^o D. Voiculescu has recently shown (see [12], Sections 5.17, 14.4) how the freeness of two subalgebras $\mathcal{A}, \mathcal{B}$, in a tracial non-commutative probability space $(\mathcal{M}, \varphi)$, can be described in terms of a certain derivation $\delta_{\mathcal{A},\mathcal{B}}$ associated to the algebras. We would like to point out that R-diagonality admits a characterization on similar lines (which seems however to work nicely only in the tracial framework).

More precisely: let $\mathbf{C}\langle X, X^*\rangle$ denote the unital $*$-algebra of non-commuting polynomials in X and X^*. On $\mathbf{C}\langle X, X^*\rangle \otimes \mathbf{C}\langle X, X^*\rangle$ we consider the natural $\mathbf{C}\langle X, X^*\rangle$-bimodule structure, determined by:

(1.13)
$$P \cdot (S \otimes T) = (PS) \otimes T, \quad (S \otimes T) \cdot Q = S \otimes (TQ), \quad P, Q, S, T \in \mathbf{C}\langle X, X^*\rangle.$$

Let $\Delta\colon \mathbf{C}\langle X, X^*\rangle \to \mathbf{C}\langle X, X^*\rangle \otimes \mathbf{C}\langle X, X^*\rangle$ denote the unique linear map which is a derivation (in the sense that $\Delta(PQ) = \Delta(P)\cdot Q + P \cdot \Delta(Q)$, $\forall P, Q \in \mathbf{C}\langle X, X^*\rangle$) and which satisfies:

$$\Delta(X) = -X \otimes 1, \quad \Delta(X^*) = 1 \otimes X^*. \tag{1.14}$$

Consider on the other hand a tracial $*$-probability space $(\mathcal{A}, \varphi)$, and an element $a \in \mathcal{A}$; and consider the linear functional $\mu\colon \mathbf{C}\langle X, X^*\rangle \to \mathbf{C}$, defined by:

$$\mu(P) := \varphi\big(P(a, a^*)\big), \quad \forall P \in \mathbf{C}\langle X, X^*\rangle \tag{1.15}$$

(μ is sometimes called the $*$-distribution of a). Then the following statement is true:

$$a \text{ is } R\text{-diagonal } \Leftrightarrow (\mu \otimes \mu) \circ \Delta = 0. \tag{1.16}$$

The equivalence (1.16) can be obtained without much difficulty from the characterization of R-diagonality in terms of $*$-moments, which is given in Theorem 1.2.1^o.

The remaining sections of the paper are organized as follows:

- In Section 2 we collect a sequence of proofs which are "elementary", in the sense that they only use the condition in $*$-moments (1.5), and basic facts about freeness. We obtain here: the equivalences $1^o \Leftrightarrow 2^o \Leftrightarrow 3^o$ of Theorem 1.2, and the proofs of Theorem 1.3, Proposition 1.4, Corollary 1.5.
- In Section 3 we review the definition of freeness with amalgamation, and prove the equivalence $1^o \Leftrightarrow 5^o$ of Theorem 1.2.
- In Section 4 we address the relation between R-diagonality and non-crossing cumulants. This relation was first obtained in [5], via a fairly complicated argument based on the combinatorial approach to the non-crossing cumulants. A different—and sensibly shorter—route is taken in this paper; namely, the equivalence $4^o \Leftrightarrow 5^o$ of Theorem 1.3 is proved by using the "full Fock space model" approach to non-crossing cumulants (see Sections 4.4, 4.5 for details). The idea of our new proof comes from the theory of generalized free creation operators developed in [8].

- In the final Section 5 we briefly review the operation $\boxed{\star}$, then we state and make some comments around our conjectured formula for the determining series of a product of two free R-diagonal elements.

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2 Elementary Proofs

The correspondence between the sequence of propositions appearing below, and the results stated in the Introduction, goes as follows: the equivalences $1^o \Leftrightarrow 2^o \Leftrightarrow 3^o$ of Theorem 1.2 are shown in 2.7; the Theorem 1.3 is covered by 2.3–2.5; the Proposition 1.4 is covered by 2.1; the Corollary 1.5 is covered by 2.6.

Proposition 2.1 *Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be $*$-probability spaces, and consider two elements $a \in \mathcal{A}$, $b \in \mathcal{B}$. It is given that each of a and b satisfies the condition in $*$-moments described in Theorem 1.2.1^o. Then a and b are identically $*$-distributed if and only if:*

$$(2.1) \qquad \begin{cases} \varphi\big((a^*a)^k\big) = \psi\big((b^*b)^k\big), \\ \varphi\big((aa^*)^k\big) = \psi\big((bb^*)^k\big), \end{cases} \qquad \forall k \geq 1.$$

Proof The implication "$\Rightarrow$" is trivial, because the equalities in (2.1) are a subset of those appearing in (1.2).

For "$\Leftarrow$" we will assume that (2.1) hold, and we will prove by induction on n that:

$$(2.2) \qquad \varphi(a_{i_1}a_{i_2}\cdots a_{i_n}) = \psi(b_{i_1}b_{i_2}\cdots b_{i_n}), \quad \forall n \geq 1,\ \forall i_1, \dots, i_n \in \{1, 2\},$$

where we denoted $a_1 := a$, $a_2 := a^*$, $b_1 := b$, $b_2 := b^*$.

For $n = 1$, (2.2) amounts to showing that $\varphi(a) = \psi(b)$ and $\varphi(a^*) = \psi(b^*)$. This is true because all of $\varphi(a)$, $\psi(b)$, $\varphi(a^*)$, $\psi(b^*)$ are equal to 0, by (1.5) $\big(\varphi(a) = \varphi(P_{22;1}(a)\big) = 0$, *etc.*).

We consider now an $n \geq 2$. We assume that (2.2) is true for $1, 2, \dots, n-1$ and we prove it for n. Let us fix some indices $i_1, \dots, i_n \in \{1, 2\}$, about which we want to prove that (2.2) holds.

We take the product $a_{i_1}a_{i_2}\cdots a_{i_n}$, and draw a vertical bar between a_{i_m} and $a_{i_{m+1}}$ for every $1 \leq m \leq n-1$ such that $i_m = i_{m+1}$. (For instance if $a_{i_1}a_{i_2}\cdots a_{i_n}$ were to be $aa^*aaaa^*a^*a$, then our bars would look like this: $aa^*a|a|aa^*|a^*a$.) By examining the sub-products of $a_{i_1}a_{i_2}\cdots a_{i_n}$ which sit between consecutive vertical bars, we find that we have written:

$$(2.3) \qquad a_{i_1}a_{i_2}\cdots a_{i_n} = \prod_{r=1}^{s} (P_{j_r j_{r+1}; k_r}(a) + \lambda_r I)$$

for some $s \geq 1$, $j_1, \dots, j_s, j_{s+1} \in \{1, 2\}$, $k_1, \dots, k_s \geq 1$ having $k_1 + \cdots + k_s = n$, and $\lambda_1, \dots, \lambda_s \in \mathbf{C}$. The number λ_r, $1 \leq r \leq s$, is determined as follows: if $j_r = j_{r+1}$, then $\lambda_r = 0$; if $j_r = 1$, $j_{r+1} = 2$, then $\lambda_r = \varphi\big((a^*a)^{k_r}\big)$; and if $j_r = 2$, $j_{r+1} = 1$, then $\lambda_r = \varphi\big((aa^*)^{k_r}\big)$.

In a similar way we can write:

$$b_{i_1} b_{i_2} \cdots b_{i_n} = \prod_{r=1}^{s} \big(P_{j_r j_{r+1}; k_r}(b) + \lambda_r I \big); \tag{2.4}$$

and moreover, the parameters s, $j_1, \dots, j_s, j_{s+1}$, $k_1, \dots, k_s$, $\lambda_1, \dots, \lambda_s$ appearing in (2.4) coincide with those from (2.3). Indeed, the values of s, $j_1, \dots, j_{s+1}$, $k_1, \dots, k_s$ are determined solely by how the vertical bars are placed between the b_{i_m}'s in $b_{i_1} b_{i_2} \cdots b_{i_n}$, and this is identical to how the vertical bars were placed in $a_{i_1} a_{i_2} \cdots a_{i_n}$. After that, the value of every λ_r is determined as $\delta_{j_r,1}\delta_{j_{r+1},2}\psi\big((b^*b)^{k_r}\big) + \delta_{j_r,2}\delta_{j_{r+1},1}\psi\big((bb^*)^{k_r}\big)$, which is again the same as in (2.3)—due to the fact that (2.1) is assumed true.

By applying φ on both sides of (2.3) and then by expanding the product on the right-hand side, we obtain:

$$\varphi(a_{i_1} a_{i_2} \cdots a_{i_n}) = \varphi\big(P_{j_1 j_2; k_1}(a) \cdots P_{j_s j_{s+1}; k_s}(a)\big) + \sum_{\varnothing \neq S \subset \{1,\dots,s\}} \Big(\prod_{r \in S} \lambda_r \Big) \cdot \varphi\Big(\prod_{r \in \{1,\dots,s\} \setminus S} P_{j_r j_{r+1}; k_r}(a) \Big).$$

The corresponding operations done in (2.4) yield an identical formula, where we have b's instead of a's, and ψ instead of φ. But we know from (1.4) that

$$\varphi\big(P_{j_1 j_2; k_1}(a) \cdots P_{j_s j_{s+1}; k_s}(a)\big) = 0 = \psi\big(P_{j_1 j_2; k_1}(b) \cdots P_{j_s j_{s+1}; k_s}(b)\big);$$

while on the other hand the induction hypothesis gives us that:

$$\varphi\Big(\prod_{r \in \{1,\dots,s\} \setminus S} P_{j_r j_{r+1}; k_r}(a) \Big) = \psi\Big(\prod_{r \in \{1,\dots,s\} \setminus S} P_{j_r j_{r+1}; k_r}(b) \Big),$$

for every $\varnothing \neq S \subset \{1, \dots, s\}$. By combining all these equalities, we obtain (2.2). ∎

Remark 2.2 The numbers $\lambda_1, \dots, \lambda_s$ appearing in Equation (2.3) are all real (as is clear from their explicit description, made following to (2.3)). As a consequence, an induction argument similar to the one made in the preceding proof shows the following: if a satisfies the conditions in $*$-moments from 1.2.1^o, then all the $*$-moments of a are real numbers.

Proposition 2.3 *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Assume that $u, p \in \mathcal{A}$ satisfy the following conditions: (i) $\{u, u^*\}$ is free from $\{p, p^*\}$; (ii) u is unitary, with $\varphi(u) = 0$; (iii) $\varphi\big(p(p^*p)^{k-1}\big) = 0$, for every $k \geq 1$. Then the element $a := up \in \mathcal{A}$ satisfies the condition in $*$-moments of Theorem 1.2.1^o.*

Proof The expressions $P_{ij;k}(a)$ defined in (1.4) are having here the form:

$$\begin{cases} P_{11;k}(a) = p^*(pp^*)^{k-1}u^*, \quad P_{12;k}(a) = (p^*p)^k - \varphi\big((p^*p)^k\big)\, I, \\ P_{21;k}(a) = u\Big((pp^*)^k - \varphi\big((pp^*)^k\big)\, I\Big)\, u^*, \quad P_{22;k}(a) = up(p^*p)^{k-1}. \end{cases} \tag{2.5}$$

Note that in the expression for $P_{21;k}(a)$ we used the fact that $\varphi\big(u(pp^*)^k u^*\big) = \varphi\big((pp^*)^k\big)$. This follows from the general formula:

$$\varphi(x'yx'') = \varphi(x'x'')\varphi(y), \tag{2.6}$$

holding for every $x', x'', y \in \mathcal{A}$ such that $\{x', x''\}$ is free from $\{y\}$; the Equation (2.6) is in turn a direct consequence of the definition of freeness (see [13], Section 2.5).

By taking the adjoint in the hypothesis (iii), we also get that $\varphi\big(p^*(pp^*)^{k-1}\big) = 0$, for every $k \geq 1$. Hence if we denote:

$$\mathcal{W} = \{w \in \mathrm{Alg}(I, p, p^*) \mid \varphi(w) = 0\}, \tag{2.7}$$

then $p^*(pp^*)^{k-1}$, $(p^*p)^k - \varphi\big((p^*p)^k\big)\, I$, $(pp^*)^k - \varphi\big((pp^*)^k\big)\, I$, and $p(p^*p)^{k-1}$ are elements of $\mathcal{W}$, for every $k \geq 1$. From the Equations (2.5) it thus follows that every $P_{ij;k}(a)$ can be viewed as a word with 1, 2, or 3 letters over the alphabet $\{u, u^*\} \cup \mathcal{W}$; and moreover the letters which form $P_{ij;k}(a)$ always come alternatively from $\{u, u^*\}$ and $\mathcal{W}$.

Given any $n \geq 1$, $i_1, \ldots, i_n, i_{n+1} \in \{1, 2\}$ and $k_1, \ldots, k_n \geq 1$, we claim that the product:

$$w := P_{i_1 i_2;k_1}(a) P_{i_2 i_3;k_2}(a) \cdots P_{i_n i_{n+1};k_n}(a) \tag{2.8}$$

still has the same alternance property of the letters, when viewed as a word over the alphabet $\{u, u^*\} \cup \mathcal{W}$. Indeed, for every $2 \leq m \leq n$ there are two possibilities: either $i_m = 1$, in which case $P_{i_{m-1} i_m;k_m}(a)$ ends with u^* and $P_{i_m i_{m+1};k_{m+1}}(a)$ begins with a letter from $\mathcal{W}$; or $i_m = 2$, in which case $P_{i_{m-1} i_m;k_m}(a)$ ends with a letter from $\mathcal{W}$, and $P_{i_m i_{m+1};k_{m+1}}(a)$ begins with u. In both cases, the concatenation of $P_{i_{m-1} i_m;k_m}(a)$ and $P_{i_m i_{m+1};k_{m+1}}(a)$ is still alternating.

But if the product w appearing in (2.8) is alternating when viewed as a word with letters from $\{u, u^*\} \cup \mathcal{W}$, then the equality $\varphi(w) = 0$ follows from the definition of freeness (since $\{u, u^*\} \cup \mathcal{W} \subset \mathrm{Ker}(\varphi)$, due to hypothesis (ii), and $\{u, u^*\}$ is free from $\{p, p^*\}$ by (i)). ■

Proposition 2.4 *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space. Assume that $u, p \in \mathcal{A}$ satisfy the following conditions:*

(i) $\{u, u^*\}$ is free from $\{p, p^*\}$;
(ii) u is a Haar unitary.

Then the element $a := up \in \mathcal{A}$ satisfies the condition in $*$-moments of Theorem 1.2.1°.

Proposition 2.4 is similar to Proposition 2.3, but its proof requires an additional argument, in order to take care of the fact that now $p(p^*p)^{k-1}$ and $p^*(pp^*)^{k-1}$ do not necessarily belong to $\mathcal{W}$ of (2.7). We will use a:

Lemma 2.5 *Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $u \in \mathcal{A}$ be a Haar unitary. Let $\mathcal{C}$ be a unital $*$-subalgebra of $\mathcal{A}$, which is free from $\{u, u^*\}$. Then $\varphi(w) = 0$ for every word of the form:*

$$w = u^{h_0} x_1 u^{h_1} \cdots x_n u^{h_n} \in \mathcal{A}, \tag{2.9}$$

where $n \geq 2$, $h_0, h_1, \dots, h_n \in \mathbf{Z}$, $x_1, \dots, x_n \in \mathcal{C}$ and such that the following conditions hold:

(i) $h_m \neq 0$, for every $1 \leq m \leq n-1$;
(ii) $h_{m-1} \cdot h_m \geq 0$, for every $1 \leq m \leq n$ such that $\varphi(x_m) \neq 0$.

Proof of Lemma 2.5 By induction on the cardinality of $\{m \mid 1 \leq m \leq n, \varphi(x_m) \neq 0\}$. If this cardinality is zero, then the equality $\varphi(w) = 0$ follows from the definition of freeness. For the induction step, we pick the smallest index $m \in \{1, \dots, n\}$ such that $\varphi(x_m) \neq 0$, and we write:

(2.10)
$$\varphi(u^{h_0} x_1 u^{h_1} \cdots x_n u^{h_n}) = \varphi\Big(u^{h_0} x_1 u^{h_1} \cdots x_{m-1} u^{h_{m-1}} \big(x_m - \varphi(x_m) I\big) u^{h_m} \cdots x_n u^{h_n}\Big)$$
$$+ \varphi(x_m) \cdot \varphi(u^{h_0} x_1 u^{h_1} \cdots x_{m-1} u^{h_{m-1}+h_m} x_{m+1} u^{h_{m+1}} \cdots x_n u^{h_n}).$$

By direct inspection, one sees that the induction hypothesis applies to both the words appearing on the right-hand side of (2.10); hence the quantity in (2.10) is equal to $0 + \varphi(x_m) \cdot 0 = 0$. ∎

Proof of Proposition 2.4 We have to show that $\varphi(w) = 0$, where w is a product of n factors $P_{ij;k}(a)$, of the special form appearing in Equation (2.8) above. The formulas for the $P_{ij;k}(a)$'s are exactly as in Equation (2.5).

If $n = 1$, then the verification of the fact that $\varphi(w) = 0$ is immediate. (*e.g.*, if w is of the form $P_{11;k}(a) = p^*(pp^*)^{k-1} u^*$, then $\varphi(w) = \varphi(p^*(pp^*)^{k-1}) \cdot \varphi(u^*) = 0$, where we first used that $\{p, p^*\}$ is free from $\{u, u^*\}$, and then the fact that $\varphi(u^*) = \overline{\varphi(u)} = 0$.) So we will assume that $n \geq 2$. In this case the argument given in the proof of Proposition 2.3 (paragraph containing Equation (2.8)) shows that we can also write:

$$w = u^{h_0} x_1 u^{h_1} \cdots x_n u^{h_n} \in \mathcal{A}, \tag{2.11}$$

with $h_1, \dots, h_{n-1} \in \{-1, 1\}$, $h_0, h_n \in \{-1, 0, 1\}$, and where every x_m $(1 \leq m \leq n)$ is of one of the forms $p^*(pp^*)^{k-1}$, $(pp^*)^k - \varphi\big((pp^*)^k\big) I$, $(p^*p)^k - \varphi\big((p^*p)^k\big) I$, or $p(p^*p)^{k-1}$, for some $k \geq 1$. Let us now make the additional remark, also following from (2.5), (2.8), that every x_m which is of the form $p^*(pp^*)^{k-1}$ "sits between two u^*'s"; or more precisely—if x_m is of the form $p^*(pp^*)^{k-1}$, then $h_m = -1$ and $h_{m-1} \in \{-1, 0\}$ (where in fact h_{m-1} can be 0 only if $m = 1$). Similarly, we remark that every x_m in (2.11) which is of the form $p(p^*p)^{k-1}$ "sits between two u's". But then the Lemma 2.4 can be applied to (2.11) (with $\mathcal{C} = \mathrm{Alg}(I, p, p^*)$), and yields that $\varphi(w) = 0$, as desired. ∎

Proposition 2.6 *Let $(\mathcal{B}, \psi)$ be a tracial $*$-probability space, and let $b \in \mathcal{B}$ be an element which satisfies the condition in $*$-moments of Theorem 1.2.1^o. Then one can find a tracial $*$-probability space $(\mathcal{A}, \varphi)$ and an element $a \in \mathcal{A}$ which is identically $*$-distributed with b, and which is obtained as $a = up$, where: $u \in \mathcal{A}$ is a Haar unitary; $p = p^* \in \mathcal{A}$ is such that $\{u, u^*\}$ is free from $\{p\}$, and such that $\varphi(p^{2k-1}) = 0$, $\forall k \geq 1$.*

Proof One can find a tracial $*$-probability space $(\mathcal{A}, \varphi)$ and a selfadjoint element $p \in \mathcal{A}$ such that:

$$\varphi(p^n) = \begin{cases} 0, & \text{if } n \text{ odd} \\ \psi\big((b^*b)^{n/2}\big), & \text{if } n \text{ even.} \end{cases} \tag{2.12}$$

For instance this can be done as follows: one takes $\mathcal{A} = \mathbf{C}[X]$ (the algebra of polynomials in an indeterminate X), endowed with the $*$-operation uniquely determined by the condition that $X^* = X$. Then one defines $\varphi \colon \mathcal{A} \to \mathbf{C}$ to be the linear functional determined by the equations:

$$\varphi(X^n) = \begin{cases} 0, & \text{if } n \text{ odd} \\ \psi\big((b^*b)^{n/2}\big), & \text{if } n \text{ even,} \end{cases}$$

and finally one chooses $p := X \in \mathcal{A}$.

We fix a tracial $*$-probability space $(\mathcal{A}, \varphi)$ containing an element $p = p^* \in \mathcal{A}$ such that (2.12) holds. By enlarging $(\mathcal{A}, \varphi)$ if necessary (via the same kind of free product construction as shown in the footnote to Theorem 1.2.2^o), we may arrange that $\mathcal{A}$ also contains a Haar unitary u such that $\{u, u^*\}$ is free from $\{p\}$.

Consider the element $a := up \in \mathcal{A}$. As implied by either Proposition 2.3 or Proposition 2.4, the element a satisfies the conditions in $*$-moments of Theorem 1.2.1^o. In addition, for every $k \geq 1$ we have:

$$\varphi\big((a^*a)^k\big) = \varphi(p^{2k}) = \psi\big((b^*b)^k\big),$$

by (2.12). Due to the traciality of φ and ψ, it is automatic that we also have $\varphi\big((aa^*)^k\big) = \psi\big((bb^*)^k\big)$, for every $k \geq 1$, and then Proposition 2.1 implies that a and b are identically $*$-distributed. ∎

2.7 Proof of Part of Theorem 1.2 (The Equivalences 1^o ⇔ 2^o ⇔ 3^o)

It is clear that $3^o \Rightarrow 2^o$ (if we also take into account the remark in the footnote to 1.2.2^o). So it is sufficient to prove that $2^o \Rightarrow 1^o \Rightarrow 3^o$.

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$2^o \Rightarrow 1^o$. Under the hypotheses of 2^o we have, for every $k \geq 1$:

$$\varphi\big(a(a^*a)^{k-1}\big) = \tilde{\varphi}\Big((ua)\cdot\big((ua)^*(ua)\big)^{k-1}\Big)$$

(because a and ua are identically $*$-distributed)

$$= \tilde{\varphi}\big(u\cdot a(a^*a)^{k-1}\big) = \tilde{\varphi}(u)\cdot\varphi\big(a(a^*a)^{k-1}\big) = 0$$

(where at the last two equalities we made use of the hypotheses (i) and (ii) of 2^o, respectively). But then the Proposition 2.3 implies that ua satisfies the conditions in $*$-moments from $1.2.1^o$. Since a and ua are identically $*$-distributed, it follows that a satisfies these conditions, too.

$1^o \Rightarrow 3^o$. Let $(\mathcal{A},\varphi)$ be a $*$-probability space, and let a be an element of $\mathcal{A}$, which satisfies the conditions in $*$-moments from $1.2.1^o$. Let $(\tilde{\mathcal{A}},\tilde{\varphi})$ be an extension of $(\mathcal{A},\varphi)$, and let $u \in \tilde{\mathcal{A}}$ be a unitary such that $\{a,a^*\}$ is free from $\{u,u^*\}$. We want to show that a and ua are identically $*$-distributed.

By replacing $(\tilde{\mathcal{A}},\tilde{\varphi})$ with the larger extension $(\tilde{\mathcal{A}},\tilde{\varphi}) \star \big(L^\infty(\mathbf{T}), dz\big)$, we may assume the existence of a Haar unitary $v \in \tilde{\mathcal{A}}$ such that all the three sets $\{a,a^*\}$, $\{u,u^*\}$, $\{v,v^*\}$ are free.

We now make the following remarks:

(a) a and va are identically $*$-distributed.

Indeed, both a and va satisfy the conditions in $*$-moments of $1.2.1^o$ (a by hypothesis, va by Proposition 2.4), and we have:

(2.13)
$$\tilde{\varphi}\Big(\big((va)^*(va)\big)^k\Big) = \tilde{\varphi}\big((a^*a)^k\big),\ \tilde{\varphi}\Big(\big((va)(va)^*\big)^k\Big) = \tilde{\varphi}\big((aa^*)^k\big) \quad \forall k \geq 1$$

(where in the second equality (2.13) we used the general fact also invoked in Equation (2.6) above). Hence the statement (a) follows from Proposition 2.1.

(b) ua and uva are identically $*$-distributed.

Indeed, we have that $\{u,u^*\}$ is free from $\{a,a^*\}$, but also that $\{u,u^*\}$ is free from $\{va,(va)^*\}$, and that a and va are identically $*$-distributed. This implies that

$$\tilde{\varphi}\big(Q(u,u^*,a,a^*)\big) = \tilde{\varphi}\big(Q(u,u^*,va,(va)^*)\big), \tag{2.14}$$

for every non-commutative polynomial Q of four variables (see [13], Remark 2.5.2). The statement (b) is a direct consequence of (2.14).

(c) uva satisfies the condition in $*$-moments of $1.2.1^o$.

This follows from Proposition 2.3, because: $\{uv,(uv)^*\}$ is free from $\{a,a^*\}$; $\tilde{\varphi}(uv) = \tilde{\varphi}(u)\tilde{\varphi}(v) = 0$; and $\varphi\big(a(a^*a)^{k-1}\big) = 0$, for every $k \geq 1$, by the hypothesis on a.

(d) ua satisfies the condition in $*$-moments of $1.2.1^o$.

This is a direct consequence of (b) and (c).

(e) a and ua are identically $*$-distributed.

Indeed, the analogue of (2.13) still holds when v is replaced there by u. Since both a and ua satisfy the condition in $*$-moments from $1.2.1^o$ (a by hypothesis, ua by statement (d)), the Proposition 2.1 implies the statement (e). ■

3 The Condition in Terms of Freeness With Amalgamation

In this section we will prove the equivalence $1^o \Leftrightarrow 5^o$ of Theorem 1.2. The proof is straightforward, due to the fact that the equations in $*$-moments appearing in $1.2.1^o$ really are just an explicit spelling of the freeness condition stated in $1.2.5^o$.

Recall (*e.g.* from [13], Section 3.8) that the definition of freeness with amalgamation over a subalgebra is made as follows.

Definition 3.1 Assume that $\mathcal{M}$ is a unital algebra, that $\mathcal{B} \subset \mathcal{M}$ is a unital subalgebra, and that $E\colon \mathcal{M} \to \mathcal{B}$ is a conditional expectation (*i.e.* E is linear, satisfies $E(I) = I$, and is such that $E(b_1 x b_2) = b_1 E(x) b_2$ for every $x \in \mathcal{M}$, $b_1, b_2 \in \mathcal{B}$).

1^o Let $(\mathcal{M}_s)_{s\in S}$ be a family of subalgebras of $\mathcal{M}$, such that $\mathcal{B} \subset \mathcal{M}_s$ for every $s \in S$. We say that $(\mathcal{M}_s)_{s\in S}$ are free with amalgamation over $\mathcal{B}$ (and with respect to the conditional expectation E) if:

$$E(x_1 x_2 \cdots x_n) = 0$$

for every $n \geq 1$, every $s_1, s_2, \dots, s_n \in S$ such that $s_1 \neq s_2, s_2 \neq s_3, \dots, s_{n-1} \neq s_n$, and every $x_1 \in \mathcal{M}_{s_1}, \dots, x_n \in \mathcal{M}_{s_n}$ such that $E(x_1) = \cdots = E(x_n) = 0$.

2^o Let $(\mathcal{X}_s)_{s\in S}$ be a family of subsets of $\mathcal{M}$. We say that $(\mathcal{X}_s)_{s\in S}$ are free with amalgamation over $\mathcal{B}$ if the subalgebras $\mathcal{M}_s := \mathrm{Alg}(\mathcal{X}_s \cup \mathcal{B})$, $s \in S$, are so.

Note that the Definition 3.1 does not require that $\mathcal{M}$ and $\mathcal{B}$ are $*$-algebras (if they are, then the definition still makes sense, of course).

3.2 Proof of Part of Theorem 1.2 (The Equivalence $1^o \Leftrightarrow 5^o$)

We consider the framework of Theorem 1.2: $(\mathcal{A}, \varphi)$ is a $*$-probability space, a is an element of $\mathcal{A}$, and we denote

$$A := \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \in M_2(\mathcal{A}). \tag{3.1}$$

Let the subalgebras $\mathcal{D} \subset M_2(\mathbf{C}I) \subset M_2(\mathcal{A})$ and the conditional expectation E: $M_2(\mathcal{A}) \to \mathcal{D}$ be as described in $1.2.5^o$.

For $j \in \{1, 2\}$ we will use the notation

$$\bar{j} := 3 - j \tag{3.2}$$

(*i.e.*, $\bar{j}$ is the element of $\{1, 2\}$ which is not j).

It is immediately seen that $\mathrm{Alg}(\{A\} \cup \mathcal{D}) \subset M_2(\mathcal{A})$ is linearly spanned by the matrices of the form:

$$\begin{pmatrix} (aa^*)^k & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a(a^*a)^k \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ a^*(aa^*)^k & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & (a^*a)^k \end{pmatrix}, \quad k \geq 0.$$

This in turn implies the formula:

$$\{X \in \mathrm{Alg}(\{A\} \cup \mathcal{D}) \mid E(X) = 0\} = \mathrm{span}\,\{Z_{ij;k} \mid i, j \in \{1, 2\}, k \geq 1\}, \tag{3.3}$$

where for every $k \geq 1$ we denote:

$$
(3.4) \qquad \begin{aligned} Z_{11;k} &= \begin{pmatrix} P_{21;k}(a) & 0 \\ 0 & 0 \end{pmatrix}, \quad Z_{12;k} = \begin{pmatrix} 0 & P_{22;k}(a) \\ 0 & 0 \end{pmatrix}, \\ Z_{21;k} &= \begin{pmatrix} 0 & 0 \\ P_{11;k}(a) & 0 \end{pmatrix}, \quad Z_{22;k} = \begin{pmatrix} 0 & 0 \\ 0 & P_{12;k}(a) \end{pmatrix}, \end{aligned}
$$

and where the elements $P_{ij;k}(a) \in \mathcal{A}$ are as in Equation (1.4) of Notation 1.1.

On the other hand let us denote:

$$
(3.5) \quad V_{11} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad V_{12} = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad V_{21} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad V_{22} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix};
$$

it is clear that:

$$
(3.6) \qquad \{X \in M_2(\mathbf{C}I) \mid E_{\mathcal{D}}(X) = 0\} = \text{span}\,\{V_{12}, V_{21}\}.
$$

From (3.3) and (3.6) it follows that $\text{Alg}(\{A\} \cup \mathcal{D})$ is free from $M_2(\mathbf{C}I)$ with amalgamation over $\mathcal{D}$ if and only if:

$$
(3.7) \qquad \begin{cases} E_{\mathcal{D}}(U' Z_{j_1' j_1'';k_1} V_{i_1 \bar{i}_1} \cdots V_{i_{n-1} \bar{i}_{n-1}} Z_{j_n' j_n'';k_n} U'') = 0, \\ \forall n \geq 1, \quad \forall j_1', j_1'', \ldots, j_n', j_n'', i_1, \ldots, i_{n-1} \in \{1, 2\}, \\ \forall k_1, \ldots, k_n \geq 1, \quad \forall U', U'' \in \{V_{11} + V_{22}, V_{12}, V_{21}\}. \end{cases}
$$

The matrix product appearing in (3.7) is 0 if it is not true that $j_1'' = i_1$, $\bar{i}_1 = j_2', \ldots, j_{n-1}'' = i_{n-1}$, $\bar{i}_{n-1} = j_n'$. And consequently, (3.7) is equivalent to:

$$
(3.8) \qquad \begin{cases} E_{\mathcal{D}}(U' Z_{\bar{i}_0 i_1;k_1} V_{i_1 \bar{i}_1} Z_{\bar{i}_1 i_2;k_2} V_{i_2 \bar{i}_2} \cdots V_{i_{n-1} \bar{i}_{n-1}} Z_{\bar{i}_{n-1} i_n;k_n} U'') = 0, \\ \forall n \geq 1, \quad \forall i_0, i_1, \ldots, i_n \in \{1, 2\}, \\ \forall k_1, \ldots, k_n \geq 1, \quad \forall U', U'' \in \{V_{11} + V_{22}, V_{12}, V_{21}\}. \end{cases}
$$

But now, by taking (3.4) into account, we see that the matrix product in (3.8) has one entry equal to $P_{i_0 i_1;k_1}(a) P_{i_1 i_2;k_2}(a) \cdots P_{i_{n-1} i_n;k_n}(a)$ (which can appear on any of the four possible positions, depending on the choices of U' and U''); and has the other three entries equal to 0. This makes it immediate that the condition (3.8) is equivalent to the one presented in Equation (1.5) of Theorem 1.2.1^o. ■

4 The Condition in Terms of Non-Crossing Cumulants

In this section we will prove the equivalence $4^o \Leftrightarrow 5^o$ of Theorem 1.2.

It will be convenient that we use a version of the framework considered in the Introduction, where there is no $*$-operation.

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Definition 4.1 1^o By a *non-commutative probability space* we will understand a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra over $\mathbf{C}$, and $\varphi \colon \mathcal{A} \to \mathbf{C}$ is a linear functional, normalized by $\varphi(I) = 1$. If $a_1, a_2 \in \mathcal{A}$, then the numbers in the family:

$$\left\{ \varphi(a_{i_1} \cdots a_{i_n}) \mid n \geq 1, i_1, \ldots, i_n \in \{1, 2\} \right\}, \tag{4.1}$$

are called the *joint moments* of the pair a_1, a_2.

2^o If $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ are non-commutative probability spaces, and if $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, we say that the pairs a_1, a_2 and b_1, b_2 are *identically distributed* if:

$$\varphi(a_{i_1} \cdots a_{i_n}) = \psi(b_{i_1} \cdots b_{i_n}) \quad \forall n \geq 1,\ \forall i_1, \ldots, i_n \in \{1, 2\}. \tag{4.2}$$

3^o Let $(\mathcal{A}, \varphi)$ and $(\mathcal{A}_k, \varphi_k)$, $k \geq 1$, be non-commutative probability spaces; and consider pairs of elements $a_1, a_2 \in \mathcal{A}$ and $a_{1,k}, a_{2,k} \in \mathcal{A}_k$, $k \geq 1$. We say that the pairs $a_{1,k}, a_{2,k}$ *converge in distribution* to a_1, a_2 if

$$\lim_{k \to \infty} \varphi_k(a_{i_1,k} \cdots a_{i_n,k}) = \varphi(a_{i_1} \cdots a_{i_n}), \quad \forall n \geq 1,\ \forall i_1, \ldots, i_n \in \{1, 2\}.$$

The condition described in Theorem 1.2.5^o can be adapted to the framework without $*$-operation, as follows.

Definition 4.2 Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Consider the unital subalgebras $\mathcal{D} \subset M_2(\mathbf{C}I) \subset M_2(\mathcal{A})$, where:

$$\mathcal{D} := \left\{ \begin{pmatrix} \alpha I & 0 \\ 0 & \lambda I \end{pmatrix} \middle| \ \alpha, \lambda \in \mathbf{C} \right\}, \quad M_2(\mathbf{C}I) := \left\{ \begin{pmatrix} \alpha I & \beta I \\ \gamma I & \lambda I \end{pmatrix} \middle| \ \alpha, \beta, \gamma, \lambda \in \mathbf{C} \right\};$$

and consider the conditional expectation $E \colon M_2(\mathcal{A}) \to \mathcal{D}$ given by the formula:

$$E\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \varphi(a)I & 0 \\ 0 & \varphi(d)I \end{pmatrix}, \quad a, b, c, d \in \mathcal{A}.$$

We say that a pair of elements $a_1, a_2 \in \mathcal{A}$ satisfy the *(RDA) condition*[2] if the matrix

$$A := \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \in M_2(\mathcal{A})$$

is free from $M_2(\mathbf{C}I)$, with amalgamation over $\mathcal{D}$.

Proposition 4.3 *Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be non-commutative probability spaces, and let $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ be such that the pairs a_1, a_2 and b_1, b_2 are identically distributed. If one of the pairs a_1, a_2 and b_1, b_2 satisfies (*RDA*), then the other pair also satisfies (*RDA*).*

[2] "RD" and "A" are meant to remind of the words "R-diagonal" and "amalgamation".

Proof The statement of the proposition is an immediate consequence of the equality:

$$E(X_0AX_1 \cdots AX_n) = E(X_0BX_1 \cdots BX_n), \tag{4.3}$$

for $n \geq 1$, $X_0, \ldots, X_n \in M_2(\mathbf{C}I)$, and where we denoted:

$$A := \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix} \in M_2(\mathcal{A}), \quad B := \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} \in M_2(\mathcal{B}).$$

(The notations in (4.3) were slightly abused, in the respect that "$M_2(\mathbf{C}I)$" is a subalgebra of $M_2(\mathcal{A})$ on the left-hand side and is a subalgebra of $M_2(\mathcal{B})$ on the right-hand side; similarly with the meaning of "E".) On the other hand, (4.3) is in turn an immediate consequence of the fact that the pairs a_1, a_2 and b_1, b_2 are identically distributed. ■

We now turn to the concept of *non-crossing cumulants*. These were introduced in [9] by combinatorial methods. We will use here an alternative approach, observed in [1], and called "modeling on the full Fock space".

Notation 4.4 We will denote by $\mathcal{T}$ the full Fock space over $\mathbf{C}^2$, *i.e.* the Hilbert space

$$\mathcal{T} := \mathbf{C} \oplus \bigoplus_{n=1}^{\infty} (\underbrace{\mathbf{C}^2 \otimes \cdots \otimes \mathbf{C}^2}_{n}). \tag{4.4}$$

The number 1 in the first summand $\mathbf{C}$ on the right-hand side of (4.4) is denoted by Ω ($\in \mathcal{T}$), and is called the "vacuum-vector". We will denote by $\varphi_{\mathrm{vac}} \colon B(\mathcal{T}) \to \mathbf{C}$ the linear functional defined by

$$\varphi_{\mathrm{vac}}(x) := \langle x\Omega \mid \Omega \rangle, \quad x \in B(\mathcal{T}).$$

We will denote by $l_1, l_2 \in B(\mathcal{T})$ the "left-creation" operators determined by the formula:

$$l_i(\xi_1 \otimes \cdots \otimes \xi_k) = e_i \otimes \xi_1 \otimes \cdots \otimes \xi_k, \quad \forall k \geq 0,\ \forall \xi_1, \ldots, \xi_k \in \mathbf{C}^2, \tag{4.5}$$

where e_1, e_2 is the canonical basis of $\mathbf{C}^2$. The operators l_1, l_2 form a family of Cuntz isometries, *i.e.* they satisfy the the relations

$$l_i^* l_j = \delta_{i,j} I, \quad i, j \in \{1, 2\}. \tag{4.6}$$

By using (4.6), it is easily seen that every monomial in l_1, l_1^*, l_2, l_2^* either is equal to 0 or can be brought to the form—called *Wick-ordered form*:

$$l_{i_1} \cdots l_{i_p} l_{j_1}^* \cdots l_{j_q}^* \quad \text{(for some } p, q \geq 0 \text{ and } i_1, \ldots, i_p, j_1, \ldots, j_q \in \{1, 2\}). \tag{4.7}$$

The Wick-ordered monomials listed in (4.7) form a linear basis for the sub–$*$–algebra of $B(\mathcal{T})$ generated by l_1 and l_2. Note that the action of φ_{vac} on this $*$-algebra is given by the formula:

$$\varphi_{\mathrm{vac}}(l_{i_1} \cdots l_{i_p} l_{j_1}^* \cdots l_{j_q}^*) = \begin{cases} 1, & \text{if } p = q = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{4.8}$$

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4.5 Review of the Non-Crossing Cumulants

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let a_1, a_2 be a pair of of elements of $\mathcal{A}$. The non-crossing cumulants of a_1, a_2 are a family of complex numbers:

$$\{ \kappa_{(i_1,\dots,i_n)}(a_1, a_2) \mid n \geq 1, i_1, \dots, i_n \in \{1, 2\} \}, \tag{4.9}$$

which was defined by Speicher in [9], via a combinatorial recipe involving lattices of non-crossing partitions. Out of the combinatorial definition for the family in (4.9) we will only retain the following fact: for every $n \geq 1$ and $i_1, \dots, i_n \in \{1, 2\}$ there exist universal polynomials with integer coefficients, P, Q (depending on $i_1, \dots, i_n$ but not on $(\mathcal{A}, \varphi)$ and a_1, a_2), such that:

(4.10)
$$\begin{cases} \varphi(a_{i_1} \cdots a_{i_n}) = \kappa_{(i_1,\dots,i_n)}(a_1, a_2) \\ \qquad\qquad\qquad + P\big(\{ \kappa_{(j_1,\dots,j_m)}(a_1, a_2) \mid m < n, j_1, \dots, j_m \in \{1, 2\} \} \big), \\ \kappa_{(i_1,\dots,i_n)}(a_1, a_2) = \varphi(a_{i_1} \cdots a_{i_n}) \\ \qquad\qquad\qquad + Q\big(\{ \varphi(a_{j_1} \cdots a_{j_m}) \mid m < n, j_1, \dots, j_m \in \{1, 2\} \} \big). \end{cases}$$

In order to compensate for the fact that an explicit characterization for P, Q of (4.10) is not given, we indicate a concrete construction for a pair a_1, a_2 in $\big(B(\mathcal{T}), \varphi_{\text{vac}}\big)$, which has a prescribed finitely supported family of non-crossing cumulants. This goes as follows. Let $N \geq 1$ and a family of complex numbers $\{ \gamma_{(i_1,\dots,i_n)} \mid n \leq N, i_1, \dots, i_n \in \{1, 2\} \}$ be given. Consider the operator:

$$x = I + \sum_{n=1}^{N} \sum_{i_1,\dots,i_n=1}^{2} \gamma_{(i_1,\dots,i_n)} l_{i_n} \cdots l_{i_1} \in B(\mathcal{T}), \tag{4.11}$$

and set

$$a_1 := l_1^* x, \quad a_2 := l_2^* x \tag{4.12}$$

(where l_1, l_2 are the left-creation operators from (4.5)). Then the non-crossing cumulants of the pair a_1, a_2, in $\big(B(\mathcal{T}), \varphi_{\text{vac}}\big)$, are:

$$\kappa_{(i_1,\dots,i_n)}(a_1, a_2) = \begin{cases} \gamma_{(i_1,\dots,i_n)} & \text{if } n \leq N \\ 0 & \text{if } n > N. \end{cases}$$

It is easy to see that the construction described in (4.11–12) can be used (together with the general fact stated in (4.10)) in order to provide a consistent definition of the non-crossing cumulants. The equivalence between this and the original definition of Speicher was shown in [1].

From (4.10) it is immediate that every family $\{\gamma_{(i_1,\dots,i_n)} \mid n \geq 1, i_1, \dots, i_n = 1, 2\}$ can appear as the family of non-crossing cumulants of a pair of elements in some non-commutative probability space. Moreover, if the family of $\gamma_{(i_1,\ldots,i_n)}$'s has the additional property that:

$$\gamma_{(\bar{i}_n,\ldots,\bar{i}_1)} = \overline{\gamma_{(i_1,\ldots,i_n)}}, \quad \forall n \geq 1,\ \forall i_1,\ldots,i_n \in \{1,2\} \tag{4.13}$$

(where we used the convention $\bar{i} := 3 - i$, same as in Equation (3.2) of Section 3.2), then one can find a $*$-probability space $(\mathcal{B}, \psi)$ and an element $b \in \mathcal{B}$, such that:

$$\kappa_{(i_1,\ldots,i_n)}(b, b^*) = \gamma_{(i_1,\ldots,i_n)}, \quad \forall n \geq 1,\ \forall i_1,\ldots,i_n \in \{1,2\}.$$

We conclude this review of non-crossing cumulants by stating a fact which follows by elementary considerations when the explicit formulas of P, Q in (4.10) (in terms of non-crossing partitions) are written down—see [5], Section 5.

Proposition 4.6 *Let $(\mathcal{A}, \varphi)$ and $(\mathcal{B}, \psi)$ be non-commutative probability spaces, and let $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ be such that:*

$$\begin{cases} \varphi\big(a_1(a_2a_1)^k\big) = \varphi\big(a_2(a_1a_2)^k\big) = 0 \\ \psi\big(b_1(b_2b_1)^k\big) = \psi\big(b_2(b_1b_2)^k\big) = 0, \end{cases} \qquad \forall k \geq 1 \tag{4.14}$$

and such that

$$\begin{cases} \kappa_{(\underbrace{1,2,\ldots,1,2}_{2k})}(a_1,a_2) = \kappa_{(\underbrace{1,2,\ldots,1,2}_{2k})}(b_1,b_2) \\ \kappa_{(\underbrace{2,1,\ldots,2,1}_{2k})}(a_1,a_2) = \kappa_{(\underbrace{2,1,\ldots,2,1}_{2k})}(b_1,b_2), \end{cases} \qquad \forall k \geq 1. \tag{4.15}$$

Then we also have:

$$\begin{cases} \varphi\big((a_1a_2)^k\big) = \psi\big((b_1b_2)^k\big) \\ \varphi\big((a_2a_1)^k\big) = \psi\big((b_2b_1)^k\big), \end{cases} \qquad \forall k \geq 1. \tag{4.16}$$

The condition on non-crossing cumulants which is of interest in this paper is the following:

Definition 4.7 Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space, and let a_1, a_2 be a pair of elements of $\mathcal{A}$. We say that a_1, a_2 satisfy the *(RDC) condition*[3] if $\kappa_{(i_1,\ldots,i_n)}(a_1, a_2) = 0$ whenever $(i_1, \ldots, i_n)$ is not of the form $(1,2,1,2,\ldots,1,2)$ or $(2,1,2,1,\ldots,2,1)$.

Remark 4.8 Our next goal is to prove the implication "(RDC) $\Rightarrow$ (RDA)", with (RDC) and (RDA) defined as in 4.7 and 4.2. It turns out that the proof can be made by putting together two facts about the left-creation operators $l_1, l_2 \in B(\mathcal{T})$ considered in the Notation 4.4. Roughly speaking, the two facts in question are that: (a) the

[3] "RD" and "C" are meant to remind of the words "R-diagonal" and "cumulants".

pair l_2, l_1 satisfies a strengthened form of (RDA); and (b) the pair l_2, l_1 generates, in a certain sense, all the pairs which satisfy (RDC). The precise statement of the fact (a) will be given in the next Proposition 4.9, while the fact (b) will appear in the proof of Proposition 4.10. The idea for this line of proof comes from the general theory of free creation operators developed in [8].

Proposition 4.9 *Consider the non-commutative probability space $\big(B(\mathcal{T}), \varphi_{\mathrm{vac}}\big)$ described in the Notation 4.4. For this particular space, consider the subalgebras $\mathcal{D} \subset M_2(\mathbf{C}I) \subset M_2\big(B(\mathcal{T})\big)$ and the expectation $E\colon M_2\big(B(\mathcal{T})\big) \to \mathcal{D}$, as described in Definition 4.2. Consider moreover the left-creation operators $l_1, l_2 \in B(\mathcal{T})$ (as in (4.5)), and form the matrix:*

$$L := \begin{pmatrix} 0 & l_2 \\ l_1 & 0 \end{pmatrix} \in M_2\big(B(\mathcal{T})\big).$$

Then $\{L, L^*\}$ is free from $M_2(\mathbf{C}I)$, with amalgamation over $\mathcal{D}$.

Proof Let $\eta\colon \mathcal{D} \to \mathcal{D}$ be the automorphism defined by:

$$\eta\left(\begin{pmatrix} \alpha I & 0 \\ 0 & \lambda I \end{pmatrix}\right) = \left(\begin{pmatrix} \lambda I & 0 \\ 0 & \alpha I \end{pmatrix}\right), \quad \alpha, \lambda \in \mathbf{C}. \tag{4.17}$$

It is immediately verified that:

$$L^* A L = \eta\big(E(A)\big), \quad \forall A \in M_2(\mathbf{C}I). \tag{4.18}$$

The desired freeness with amalgamation follows from (4.18), by the virtue of Theorem 2.3 in [8]. For the reader's convenience, the next two paragraphs show how this argument goes.

By using the relation $L^* D L = \eta(D)$, $D \in \mathcal{D}$ (which is a particular case of (4.18)), we see that $\mathrm{Alg}(\{L, L^*\} \cup \mathcal{D})$ is the linear span of the elements of the form:

$$W = D_0 L D_1 \cdots L D_p L^* D'_1 \cdots L^* D'_q, \tag{4.19}$$

where $D_0, D_1, \ldots, D_p, D'_1, \ldots, D'_q \in \mathcal{D}$. Moreover, by also using (4.8) and the formula for the expectation E, it is easily seen that $\mathrm{Alg}(\{L, L^*\} \cup \mathcal{D}) \cap \mathrm{Ker}(E)$ is the linear span of those words W as in (4.19) for which the non-negative integers p, q (appearing in (4.19)) are not both equal to zero. In view of this description of $\mathrm{Alg}(\{L, L^*\} \cup \mathcal{D}) \cap \mathrm{Ker}(E)$, the statement of the proposition amounts to showing that:

$$E(A_0 W_1 A_1 \cdots W_r A_r) = 0 \tag{4.20}$$

whenever $r \geq 1, A_1, \ldots, A_{r-1} \in M_2(\mathbf{C}I) \cap \mathrm{Ker}(E)$, each of A_0, A_r is either in $M_2(\mathbf{C}I) \cap \mathrm{Ker}(E)$ or is $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, and each of $W_1, \ldots, W_r$ is as in (4.19), with the corresponding p, q satisfying $p + q > 0$.

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Finally, (4.20) is proved as follows. Let us denote $A_0W_1A_1\cdots W_rA_r =: X$. When each of $W_1,\dots,W_r$ is replaced (in X) by its form as in (4.19), X gets the expression:

$$X = B_0L_{i_1}B_1\cdots L_{i_s}B_s, \tag{4.21}$$

where $s \geq 1$, $L_{i_1},\dots,L_{i_s} \in \{L, L^*\}$, $B_0,B_1,\dots,B_s \in M_2(\mathbf{C}I)$. If in the sequence $L_{i_1},\dots,L_{i_s}$ there is no L^* followed by an L, then the fact that $E(X)=0$ is an immediate consequence of (4.8). In the opposite case, let us pick an m, $1 \leq m \leq s-1$, such that $L_{i_m} = L^*$, $L_{i_{m+1}} = L$. It is then immediately checked that the matrix B_m appearing in (4.21) has $E(B_m)=0$. (Indeed, B_m must be of the form $B_m = D'A_kD''$, with $1 \leq k \leq r-1$, and where A_k is taken from (4.20); hence $E(B_m) = D'E(A_k)D'' = 0$.) But then (4.18) gives us:

$$L^*B_mL = \eta\big(E(B_m)\big) = 0,$$

and it follows that X itself is equal to zero. ■

Proposition 4.10 *Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and let a_1, a_2 be a pair of elements of $\mathcal{A}$, which satisfies the condition* (RDC)*. Then the pair a_1, a_2 also satisfies the condition* (RDA)*.*

Proof We denote:

$$\kappa_{(\underbrace{1,2,\dots,1,2}_{2n})}(a_1,a_2) =: \alpha_n,\ \kappa_{(\underbrace{2,1,\dots,2,1}_{2n})}(a_1,a_2) =: \beta_n, \quad \forall n \geq 1.$$

It is easily verified that both the conditions (RDC) and (RDA) are preserved under convergence in distribution (as defined in 4.1.3°). Due to this fact (and since pairs of elements can be constructed with truncated families of non-crossing cumulants–*cf.* Section 4.5) we may assume that there exists $N \geq 1$ such that $\alpha_n = 0 = \beta_n$ for $n > N$.

Furthermore, since (RDC) and (RDA) depend only on the joint moments of a_1 and a_2, we may (and will) assume that $(\mathcal{A},\varphi) = \big(B(\mathcal{T}),\varphi_{\mathrm{vac}}\big)$ and that a_1, a_2 are as described in the Equations (4.11–12) of Section 4.5:

$$a_i = l_i^*\Big(I + \sum_{n=1}^{N}\beta_n(l_1l_2)^n + \alpha_n(l_2l_1)^n\Big), \quad i = 1,2. \tag{4.22}$$

We can rewrite (4.22) as

$$\begin{cases} a_1 = l_1^* + \sum_{n=1}^{N}\beta_n l_2(l_1l_2)^{n-1} \\ a_2 = l_2^* + \sum_{n=1}^{N}\alpha_n l_1(l_2l_1)^{n-1}; \end{cases}$$

or in matrix form:

$$A = L^* + \sum_{n=1}^{N}\begin{pmatrix}\beta_n & 0\\ 0 & \alpha_n\end{pmatrix}L^{2n-1}, \tag{4.23}$$

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with

$$A := \begin{pmatrix} 0 & a_1 \\ a_2 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 0 & l_2 \\ l_1 & 0 \end{pmatrix}. \tag{4.24}$$

From (4.23) it follows that $A \in \mathrm{Alg}(\{L, L^*\} \cup \mathcal{D})$, hence that $\mathrm{Alg}(\{A\} \cup \mathcal{D}) \subset \mathrm{Alg}(\{L, L^*\} \cup \mathcal{D})$. The latter algebra is free from $M_2(\mathbf{C}I)$, with amalgamation over $\mathcal{D}$ (by Proposition 4.9); hence $\mathrm{Alg}(\{A\} \cup \mathcal{D})$ is also free from $M_2(\mathbf{C}I)$, with amalgamation over $\mathcal{D}$. ■

4.11 Proof of Part of Theorem 1.2 (The Equivalence $4^o \Leftrightarrow 5^o$)

The particularization of Proposition 4.10 to the case of a $*$-probability space gives the implication $4^o \Rightarrow 5^o$, so we only need to discuss $5^o \Rightarrow 4^o$. We will show that $5^o \Rightarrow 4^o$ actually follows from the combination of $4^o \Rightarrow 5^o$ with an exhaustion argument.

So, let $(\mathcal{A}, \varphi)$ be a tracial $*$-probability space and let $a \in \mathcal{A}$ be an element which satisfies the condition 5^o. By what was proved in Section 3, we know that a also satisfies the condition 1^o of Theorem 1.2.

We denote:

$$\alpha_n := \kappa_{(\underbrace{1,2,\ldots,1,2}_{2n})}(a, a^*), \ \beta_n := \kappa_{(\underbrace{2,1,\ldots,2,1}_{2n})}(a, a^*), \quad \forall n \geq 1. \tag{4.25}$$

Remark that $\alpha_n, \beta_n \in \mathbf{R}$, for every $n \geq 1$. Indeed, all the $*$-moments of a are real numbers (Remark 2.2); hence the non-crossing cumulants of a, a^* must also be real numbers, since the polynomials P, Q of (4.10) have integer (in particular real) coefficients.

Consider the family of real numbers $\{\gamma_{(i_1,\ldots,i_n)} \mid n \geq 1, i_1, \ldots, i_n = 1, 2\}$ defined as follows:

$$\gamma_{(i_1,\ldots,i_n)} = \begin{cases} \alpha_k & \text{if } (i_1, \ldots, i_n) = (\underbrace{1,2,\ldots,1,2}_{2k}) \\ \beta_k & \text{if } (i_1, \ldots, i_n) = (\underbrace{2,1,\ldots,2,1}_{2k}) \\ 0 & \text{otherwise.} \end{cases}$$

These $\gamma_{(i_1,\ldots,i_n)}$'s clearly satisfy the Equation (4.13) in Section 4.5. Consequently (by the remark made there) one can find a $*$-probability space $(\mathcal{B}, \psi)$ and an element $b \in \mathcal{B}$ such that:

$$\kappa_{(i_1,\ldots,i_n)}(b, b^*) = \gamma_{(i_1,\ldots,i_n)}, \quad \forall n \geq 1, \ \forall i_1, \ldots, i_n \in \{1, 2\}.$$

By construction, the element $b \in \mathcal{B}$ satisfies the condition 4^o of Theorem 1.2; hence b also satisfies 1^o of 1.2—because $4^o \Rightarrow 5^o \Leftrightarrow 1^o$ were proved above.

We show that $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are identically $*$-distributed. We first observe that, by the construction of b, we have:

$$\begin{cases} \kappa_{(\underbrace{1,2,\ldots,1,2}_{2k})}(a,a^*) = \kappa_{(\underbrace{1,2,\ldots,1,2}_{2k})}(b,b^*) \\ \kappa_{(\underbrace{2,1,\ldots,2,1}_{2k})}(a,a^*) = \kappa_{(\underbrace{2,1,\ldots,2,1}_{2k})}(b,b^*) \end{cases} \qquad \forall k \geq 1.$$

But then the Proposition 4.6 implies that:

$$\varphi\big((a^*a)^k\big) = \psi\big((b^*b)^k\big),\ \varphi\big((aa^*)^k\big) = \psi\big((bb^*)^k\big), \quad \forall k \geq 1$$

(note that the condition (4.14) required in the hypothesis of Proposition 4.6 is automatically fulfilled, due to the fact that both a and b satisfy 1^o of Theorem 1.2). It only remains to invoke Proposition 2.1; this can be done, again because a and b satisfy 1^o of Theorem 1.2, and gives us that a and b are identically $*$-distributed.

But from Equation (4.10) it is clear that if a and b are identically $*$-distributed, then a, a^* and b, b^* must have identical non-crossing cumulants. Hence for every $n \geq 1$ and every $(i_1,\ldots,i_n) \in \{1,2\}^n$ which is not of the form $(1,2,1,2,\ldots,1,2)$ or $(2,1,2,1,\ldots,2,1)$ we obtain:

$$\kappa_{(i_1,\ldots,i_n)}(a,a^*) = \kappa_{(i_1,\ldots,i_n)}(b,b^*) = 0.$$

This means that a satisfies the condition 4^o of Theorem 1.2. ■

5 Conjectured Formulas for the Determining Series of a Product

5.1 Review of the Operation $\boxed{\star}$

Let Θ denote the set of all formal power series of the form $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, where $\alpha_1, \alpha_2, \alpha_3, \cdots \in \mathbf{C}$. Then $\boxed{\star}$ is a binary operation on Θ, determined by the formula:

$$R(a) \boxed{\star} R(b) = R(ab), \tag{5.1}$$

holding whenever a and b are free elements in some non-commutative probability space $(\mathcal{A}, \varphi)$. The Equation (5.1) does really define $\boxed{\star}$, in a coherent way, because:

(a) given $f, g \in \Theta$, one can always construct a non-commutative probability space $(\mathcal{A}, \varphi)$ and two free elements a, b in $\mathcal{A}$, with R-transforms $R(a) = f$, $R(b) = g$;

(b) if a and b are free in $(\mathcal{A}, \varphi)$, then the moments of ab (and hence $R(ab)$) are completely determined by the moments of a and of b (hence by $R(a)$ and $R(b)$)—see [13], Proposition 2.5.5.

The operation $\boxed{\star}$ can also be given a direct combinatorial definition. That is: if $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$, $g(z) = \sum_{n=1}^{\infty} \beta_n z^n$, then the coefficient of order n of $f \boxed{\star} g$ can be defined via a certain summation formula over non-crossing partitions of the set

$\{1,\dots,n\}$, which involves $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$. For instance, the formulas for the first three coefficients $\gamma_1,\gamma_2,\gamma_3$ of $f \boxed{\star}\, g$ come out as:

$$\begin{cases}\gamma_1 = \alpha_1\beta_1,\\ \gamma_2 = \alpha_2\beta_1^2 + \alpha_1^2\beta_2,\\ \gamma_3 = \alpha_3\beta_1^3 + 3\alpha_1\alpha_2\beta_1\beta_2 + \alpha_1^3\beta_3.\end{cases}$$

We do not insist here on the combinatorics of $\boxed{\star}$, the interested reader can find a detailed presentation in [4].

It is immediate from (5.1) that the operation $\boxed{\star}$ is associative. Less obvious, but nevertheless true is that $\boxed{\star}$ is also commutative (see [13], Remark 3.6.2; or Proposition 1.4.2 of [4], in the combinatorial approach to $\boxed{\star}$). The series:

$$\operatorname{id}(z) := z \tag{5.2}$$

is the unit for $\boxed{\star}$. Two other important series which appear in the considerations about $\boxed{\star}$ are the Zeta and the Moebius series:

$$\begin{cases}\operatorname{Zeta}(z) = \sum_{n=1}^{\infty} z^n\\ \operatorname{Moeb}(z) = \sum_{n=1}^{\infty}[(-1)^{n+1}(2n-2)!/n!\,(n-1)!]z^n.\end{cases} \tag{5.3}$$

Moeb and Zeta are inverse to each other with respect to $\boxed{\star}$ (*i.e.*, Moeb $\boxed{\star}$ Zeta $=$ id $=$ Zeta $\boxed{\star}$ Moeb).

5.2 Review of a Result From [5]

Let $(\mathcal{A},\varphi)$ be a tracial $*$-probability space. If $a \in \mathcal{A}$ is R-diagonal, then the determining series f_a can be calculated as:

$$f_a = R(a^*a) \boxed{\star} \operatorname{Moeb} \tag{5.4}$$

(see [5], Proposition 1.7). An equivalent way of writing (5.4) is:

$$R(a^*a) = f_a \boxed{\star} \operatorname{Zeta} \tag{5.5}$$

(since Zeta is the inverse of Moeb with respect to $\boxed{\star}$).

From (5.4), (5.5), one can easily obtain a formula for f_{ab}, where a, b are R-diagonal elements in a tracial $*$-probability space $(\mathcal{A},\varphi)$, such that $\{a,a^*\}$ is free from $\{b,b^*\}$. Indeed, note first that:

$$\begin{aligned} R\big((ab)^*(ab)\big) &= R(b^*a^*ab)\\ &= R(a^*abb^*) \quad (\text{because } (\mathcal{A},\varphi) \text{ is tracial})\\ &= R(a^*a) \boxed{\star} R(bb^*) \quad (\text{by } (5.1))\\ &= R(a^*a) \boxed{\star} R(b^*b) \quad (\text{because } (\mathcal{A},\varphi) \text{ is tracial}).\end{aligned}$$

Therefore one has:

$$
\begin{aligned}
f_{ab} &= R\big((ab)^*(ab)\big) \boxed{\star} \text{Moeb} \quad \text{(by (5.4))}\\
&= R(a^*a) \boxed{\star} R(b^*b) \boxed{\star} \text{Moeb}\\
&= (f_a \boxed{\star} \text{Zeta}) \boxed{\star} (f_b \boxed{\star} \text{Zeta}) \boxed{\star} \text{Moeb} \quad \text{(by (5.5))}\\
&= f_a \boxed{\star} f_b \boxed{\star} \text{Zeta}. \qquad (5.6)
\end{aligned}
$$

We believe that the Equations (5.5) and (5.6) have the following generalizations to the situation of an arbitrary (not necessarily tracial) $*$-probability space.

5.3 Conjectured Formulas

Let $(\mathcal{A}, \varphi)$ be a $*$-probability space.

1^o Let $a \in \mathcal{A}$ be an R-diagonal element, and assume that $\varphi(a^*a) \neq 0 \neq \varphi(aa^*)$. The determining series f_a and g_a are then invertible under the operation of composition, "$\circ$", for power series; we denote the inverses of f_a and g_a (with respect to $\circ$) by $f_a^{\langle -1\rangle}$ and $g_a^{\langle -1\rangle}$, respectively. Then we have:

$$
\begin{cases} R(aa^*) = f_a \circ g_a^{\langle -1\rangle} \circ (g_a \boxed{\star} \text{Zeta}) \\ R(a^*a) = g_a \circ f_a^{\langle -1\rangle} \circ (f_a \boxed{\star} \text{Zeta}). \end{cases} \qquad (5.7)
$$

2^o Let $a, b \in \mathcal{A}$ be R-diagonal elements, such that $\{a, a^*\}$ is free from $\{b, b^*\}$, and such that $\varphi(a^*a) \neq 0 \neq \varphi(aa^*)$, $\varphi(b^*b) \neq 0 \neq \varphi(bb^*)$. Then we have:

$$
\begin{cases} f_{ab} = f_a \circ g_a^{\langle -1\rangle} \circ (g_a \boxed{\star} f_b \boxed{\star} \text{Zeta}) \\ g_{ab} = g_b \circ f_b^{\langle -1\rangle} \circ (f_b \boxed{\star} g_a \boxed{\star} \text{Zeta}). \end{cases} \qquad (5.8)
$$

Remarks 5.4 1^o If one wants a version of the Equations (5.8) where the operation $\boxed{\star}$ does not appear in an explicit way, then the following can be used. Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a, b \in \mathcal{A}$ be R-diagonal elements such that $\{a, a^*\}$ is free from $\{b, b^*\}$. In an enlargement of $(\mathcal{A}, \varphi)$ construct selfadjoint elements p, q such that: $\{p\}$ is free from $\{a, a^*\}$ and $R(p) = f_b$; $\{q\}$ is free from $\{b, b^*\}$ and $R(q) = g_a$ (finding such p and q is always possible). Then:

$$
f_{ab} = R(apa^*), \quad g_{ab} = R(b^*qb). \qquad (5.9)
$$

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The Equation (5.9) can be derived from the conjectures made in 5.3. While less precise than (5.8), this equation would still provide a "concrete" way of calculating the determining series of ab, which only uses the 1-dimensional R-transform.

2^o Let $(\mathcal{A}, \varphi)$ be a $*$-probability space, and let $a \in \mathcal{A}$ be an R-diagonal element, such that $\varphi(a^*a) \neq 0 \neq \varphi(aa^*)$. The series $f_a \circ g_a^{\langle -1 \rangle}$ can be used for measuring "how far is φ from being a trace" on the unital $*$-algebra $\mathrm{Alg}(I, a, a^*)$ generated by a (at least in the sense that $\varphi | \mathrm{Alg}(I, a, a^*)$ is a trace $\Leftrightarrow f_a = g_a \Leftrightarrow f_a \circ g_a^{\langle -1 \rangle} = \mathrm{id}$). The Equations (5.8) would imply a "multiplicativity property" for $f_a \circ g_a^{\langle -1 \rangle}$, or more precisely that:

$$f_{ab} \circ g_{ab}^{\langle -1 \rangle} = (f_a \circ g_a^{\langle -1 \rangle}) \circ (f_b \circ g_b^{\langle -1 \rangle}), \tag{5.10}$$

whenever a, b are R-diagonal, with $\varphi(a^*a) \neq 0 \neq \varphi(aa^*)$, $\varphi(b^*b) \neq 0 \neq \varphi(bb^*)$, and such that $\{a, a^*\}$ is free from $\{b, b^*\}$. The Equation (5.10) is a direct consequence of (5.8) and of the fact that $\boxed{\star}$ is commutative.

Note added in proof The formulae conjectured in Section 5.3 were proved by B. Krawczyk and R. Speicher, in the paper *Combinatorics of free cumulants*, Journal of Combinatorial Theory Series A, **90**(2000), 267–292. An alternative proof for these formulae was also found by U. Haagerup and F. Larsen (paper in preparation; see also Chapter 5 in PhD Thesis of F. Larsen, University of Odense, Denmark, October 1999).

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