PRESENT PROBLEMS IN RELATIVISTIC CELESTIAL MECHANICS

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ABSTRACT. Review of the present problems of relativistic celestial mechanics. Advantage is taken of the method suggested earlier by the author and based on using quasi-Galilean coordinates with arbitrary coordinate functions or parameters. As compared with the previous papers the new elements are post-post-Newtonian approximation for the circular motion in the Schwarzschild problem and reduction of the artificial satellite problem including the main solar perturbations to the Schwarzschild problem. Some current questions of time scales definitions, reference frames and reduction of observations are briefly discussed.

I. INTRODUCTION

Disregarding the physical foundation of gravitation and from a purely operational point of view the distinction between relativistic and Newtonian celestial mechanics is displayed 1). mathematically by the structure of the field equations and the equations of motion and 2). physically by the way we compare the theoretical and observational data. The essence of the second problem is in the arbitrariness of the quasi-Galilean coordinates of general relativity leading to the appearance of unmeasurable coordinate-dependent quantities into astronomical practice. As stated in the previous paper by the author (Brumberg, 1986) one may suggest three possible ways to overcome the ensuing difficulties : 1). developing theoretical conclusions only in terms of measurable quantities, 2). using arbitrary coordinates and developing an unambiguous procedure for comparing measurable and calculated quantities, 3). an agreement to use one and the same specific coordinate system. The first approach is preferred by some physicists developing mathematical tools to deduce, if only locally, theoretical statements in terms of measurable quantities (Misner et al., 1970 ; Ivanitskaja, 1979 ; Vladimirov, 1982). In astronomical papers the third approach is often adopted as, for example, in Japanese Ephemeris (1985). This paper is dealing with some solved and unsolved problems of relativistic celestial mechanics on the basis of the second approach.

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2. SCHWARZSCHILD SOLUTION

For the stationary case the Schwarzschild metric has the form

$$ds^2 = p(r)c^2dt^2+2b(r)drcdt-q(r)dr^2-a^2(r)(d\theta^2+\sin^2\theta d\phi^2). \qquad (1)$$

Here

$$p(r)=1-2m/a(r), \quad q(r)=[a'^2(r)-b^2(r)]/p(r), \qquad (2)$$

$m=GM/c^2$, M is the mass of the gravitating body, G is the gravitation constant, c is the light velocity. a(r), b(r) are two arbitrary functions of r satisfying the quasi-Galilean conditions (coefficients of (1) should little differ from the Galilean values and coincide with those in the limit $r \to \infty$). A variety of the quasi-Galilean coordinates used in relativistic celestial mechanics are as follows :

	1	2	3	4	5	6
a(r)=	r	r+m	$r(1+m/2r)^2$	$a^3=r^2(a-2m)$	r	r
b(r)=	0	0	0	0	2m/r	$(2m/r)^{1/2}$

(3)

where the numbers denote standard system (I), harmonic system (2), isotropic system (3), Painlevé's static coordinates (4) (a(r) being the solution of the given cubic equation), Eddington's (5) and Painlevé's stationary (6) coordinates. It is to be noted that only for the harmonic coordinates one may explicitly indicate the appropriate coordinate conditions. The third approach mentioned in Introduction involves prescribing coordinate conditions in explicit manner but, as a rule, this is not an easy task.

To avoid any confusion let us agree to introduce three kinds of quantities : measurable, indirectly measurable and unmeasurable quantities. Measurable quantities are obtained as a result of direct astronomical measurements. Indirectly measurable quantities may be calculated on the basis of measurable quantities by means of expressions free of coordinate functions or parameters. These two kinds of quantities are coordinate-independent. Unmeasurable or coordinate-dependent quantities are related with measurable or indirectly measurable quantities by means of expressions involving coordinate functions or parameters.

The interrelation between these kinds of quantities may be demonstrated by a simple example of the circular motion of a test particle in the static Schwarzschild field (b(r)=0). This circular motion is given by

$$r=\text{const}, \quad \phi=nt+\text{const}, \quad \theta=\Pi/2 \qquad (4)$$

with

$$n=d\phi/dt=[GM/a^3(r)]^{1/2}. \qquad (5)$$

The proper time τ of the test particle on the circular orbit is determined by an equation

$$d\tau/dt=[1-3m/a(r)]^{1/2}. \qquad (6)$$

The mean motion n' referred to the proper time τ represents a measura-

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ble quantity

$$n'=d\,\phi/d\tau = n\,[1-3m/a(r)]^{-1/2} \tag{7}$$

Introducing two auxiliary quantities r'_N , r_N by means of

$$n' = (GM/r'^{3}_N)^{1/2} \quad , \quad n=(GM/r^3_N)^{1/2} \tag{8}$$

one has

$$r'_N = r_N(1-3m/r_N)^{1/3} \quad . \tag{9}$$

Gravitational parameter GM and light velocity c being know physical constants and n' being a measurable quantity the relations (8), (9) enable us to consider r'_N, r_N, n as indirectly measurable quantities. By contrast, the radius r of the circular orbit determined by an implicit equation

$$a(r) = r_N \tag{10}$$

represents an unmeasurable, coordinate-dependent, quantity.

The function a(r) of the metric form (1) may be expanded in powers of m/r

$$a(r)=r\,[1+(1-\alpha)(m/r)+\sigma(m/r)^2+\ldots] \tag{11}$$

where parameters α, σ have the following values for the coordinate systems listed above

	1	2	3	4
α =	1	0	0	2
σ =	0	0	1/4	-3/2 .

(12)

The solution of Eq. (10) in terms of r_N or r'_N has the form

$$r=r_N\,[\,1+(\alpha-1)(m/r_N)-\sigma\,(m/r_N)^2+\ldots],\tag{13}$$

$$r=r'_N\,[1+\alpha(m/r'_N)+(1-\sigma)(m/r'_N)^2+\ldots],\tag{14}$$

Consider now the light propagation in the static Schwarzschild field. For the post-post-Newtonian approximation (11) the equations of light propagation in rectangular coordinates referred to the coordinate time t will be

$$\ddot{\underline{r}}=(m/r^3)\,[(4-2\alpha)(\underline{r}\dot{\underline{r}})\dot{\underline{r}}-(2+\alpha)\dot{\underline{r}}^2\underline{r}+3\alpha(\underline{r}\dot{\underline{r}}/r)^2\underline{r}]+(m^2/r^4)\,[(-2+8\alpha-2\alpha^2+4\sigma)(\underline{r}\dot{\underline{r}})\dot{\underline{r}}+ 2\sigma\dot{\underline{r}}^2\underline{r}+(2-4\alpha+2\alpha^2-8\sigma)(\underline{r}\dot{\underline{r}}/r)^2\underline{r}]+\ldots \tag{15}$$

At present, there are papers dealing with the light deflection

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and the Shapiro effect in the post-post-Newtonian approximation but the general solution of Eq. (15) remains to be found. In the post-Newtonian approximation this solution may be easily obtained enabling to treat different problems of relativistic astrometry (Brumberg, 1981).

3. SOLUTIONS OF KERR AND WEYL - LEVI-CIVITA

In addition to the Schwarzschild solution there exist only two rigorous solutions of the field equations used in relativistic celestial mechanics. These are the Kerr solution for a rotating spherical body and the Weyl and Levi-Civita solution for a fixed spheroid. Both solutions belong to the class of axial-symmetric metrics and have the form

$$ds^2=p(r,\theta)c^2dt^2+2b(r,\theta)drcdt+2d(r,\theta)\sin^2\theta d\phi\, cdt-q(r,\theta)dr^2-a^2(r,\theta)d\theta^2-$$
$$-f^2(r,\theta)\sin^2\theta\, d\phi^2-2g(r,\theta)\sin^2\theta\, drd\phi, \quad (16)$$

all coefficients being even functions with respect to θ . The values of these coefficients are known only for some specific coordinates. In the Kerr problem coefficients b and g may vanish but not the value of d. The most used expressions for the coefficients are given in (Carter, 1966 ; Boyer, Lindquist, 1967). In the Weyl - Levi-Cevita metric the coefficients b, d and g may vanish. The expressions for the remaining coefficients are presented in (Young, Coulter, 1969).

4. APPROXIMATE SOLUTIONS OF THE FIELD EQUATIONS

The most widespread method for an approximate solution of the field equations is to expand the gravitation potentials of the metric form

$$ds^2=g_{\mu\nu}dx^\mu\, dx^\nu \quad (17)$$

in powers of v^2/c^2 where v denotes the characteristic velocity of the bodies :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad (18)$$

$$\eta_{oo} = 1 \quad , \quad \eta_{oi} = 0 \quad , \eta_{ij} = -\delta_{ij} \quad . \quad (19)$$

In harmonic coordinates the initial terms of $h_{\mu\nu}$ take the form

$$\tilde{h}_{oo}=-2U/c^2+2(U^2-W,_{oo}-V)/c^4,\ \tilde{h}_{oi}=4U_i/c^3,\ \tilde{h}_{ij}=-2\,\delta_{ij}U/c^2 \quad (20)$$

where Newtonian potential U, vector-potential U_i and auxilliary potential functions V and W satisfy the Poisson equations (Brumberg, 1972).

The transformation from harmonic to arbitrary quasi-Galilean coordinates is performed by

$$\tilde{x}^o+a_o \quad , \quad \tilde{x}^i=x^i-a_i \quad (21)$$

where a_o is an arbitrary function of the third order and a_i are arbitrary functions of the second order. The gravitation potentials expressed

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in the arbitrary quasi-Galilean coordinates is performed by

$$\tilde{x}^o = x^o + a_o \quad , \quad \tilde{x}^i = x^i - a_i \tag{21}$$

where a_o is an arbitrary function of the third order and a_i are arbitrary functions of the second order. The gravitation potentials expressed in the arbitrary quasi-Galilean coordinates will be

$$\begin{aligned} h_{oo} &= \tilde{h}_{oo} + 2a_{o,o} - \tilde{h}_{oo,s}a_s - \sum_p (\partial\tilde{h}_{oo}/\partial x_p^s)(a_s)_p \quad , \\ h_{oi} &= \tilde{h}_{oi} + a_{o,i} + a_{io} \quad , \\ h_{ij} &= \tilde{h}_{ij} + a_{i,j} + a_{j,i} \end{aligned} \tag{22}$$

provided that in the post-Newtonian approximation $\tilde{h}_{oo}$ depends on the space coordinates x^s of a point of the field and the coordinates x_p^s of moving bodies. $(a_s)_p$ denotes the regular part of a_s in substituting $x^s = x_p^s$. The function a_o does not affect the post-Newtonian equations of motion.

5. EXPANSIONS IN SMALL MASSES

Expansions of the preceding section have been performed with the Galilean metric (19) as a background. If instead the Schwarzschild values are adopted for the background one may perform expansions in powers of the ratio of planetary masses to the mass of the Sun. In such a case one has to deal not with the Poisson equations but with more complicated equations in partial derivatives with variable coefficients.

Replacing (19) by the Schwarzschild solution in isotropic coordinates

$$\begin{aligned} &\eta_{oo} = A \quad , \quad \eta_{oi} = 0 \quad , \quad \eta_{ij} = -B\delta_{ij} \\ &A = (1 - m/2r)^2/(1 + m/2r)^2 \quad , \quad B = (1 + m/2r)^4 \end{aligned} \tag{23}$$

and imposing the coordinate conditions

$$h_{oo,o} + h_{ss,o} - 2h_{os,s} = 0 \quad , \quad h_{oo,m} - h_{ss,m} + 2h_{ms,s} = 0 \tag{24}$$

the field equations will become

$$\begin{aligned} &h_{oo,ss} - h_{oo,oo} = 2L_{oo} \quad , \\ &h_{om,ss} = 2L_{om} \quad , \\ &h_{mn,ss} - (B/A)h_{mn,oo} = 2L_{mn} + (B/A - 1)h_{oo,mn} - (B/A)(h_{om,on} + h_{on,om}) . \end{aligned} \tag{25}$$

The functions $L_{\mu\nu}$ are caused by the quantities relating to the disturbing bodies and the terms of second and higher degrees with respect

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to $h_{\mu\nu}$. At each step of approximations the right-hand members of Eqs. (25) may be regarded as known functions. The third equation of (25) is the wave equation with variable coefficients and its solution causes the main difficulty of this method (Brumberg, Tarasevich, 1983). In a similar way this method is valid to study the motion of bodies with the cosmological background. A different method of investigating the perturbations of the Schwarzschild solution has been proposed in (Peters, 1966).

The expansions of this and preceding sections are valid to represent the solar system gravitational field. In astrophysical applications relating to the massive compact rapidly rotating sources these expansions are inadequate. For these problems one cannot obtain a unified representation of the gravitational field as a whole and it is necessary to look for the solution by means of the matching procedure separating the whole space into the regions near each body and the interbody region (D'Eath, 1975).

6. PLANETARY PROBLEM

The metric for the gravitational field of a system of point masses is easily obtained on the basis of Eqs. (18) - (22) and its explicit form is given, for example, in (Brumberg, 1986). One may find different expressions of this metric for the different coordinate systems but all these expressions result from the two-parameter set of functions

$$a_o=(\nu/2c)(\partial/\partial t)\sum_i m_i\rho_i \quad , \tag{26}$$

$$a_s=\alpha\sum_i m_i(x^s-x_i^s)/\rho_i \tag{27}$$

where α, ν are arbitrary constants (coordinate parameters), the space coordinates of the body i are designated by x_i^s, ρ_i is a distance of the arbitrary point of the field from the body i and $m_i=GM_i/c^2$. As stated above, the function a_o does not affect the post-Newtonian equations of motion. But this function enters into the expression of the metric and hence it affects the interrelation between the coordinate (TDB) and the proper (TDT) time for the Earth. It is true that this influence manifests itself in terms of order v^4/c^4 while the interrelation between the coordinate and the proper time is calculated now within the accuracy of the second order (Moyer, 1981). But this is important from the theoretical point of view. In accordance with the definition adopted now TDB serves as the independent argument of the theories of barycentric motion of the solar system bodies. But in this definition there is no mention about the system of quasi-Galilean coordinates. Being the coordinate time TDB may be submitted to the arbitrary time transformation (21) with the value (26), for example. For the harmonic coordinates $\alpha=\nu=0$ and for the no less popular coordinate system of the PPN formalism $\alpha=0$, $\nu=1$.

Barycentric equations of motion of the Sun and the major planets may be presented in different forms, for example

$$\ddot{\underline{r}}_i=-\sum_{j\neq i}GM_j\underline{r}_{ij}/r_{ij}^3+\sum_{j\neq i}m_j(A_{ij}\underline{r}_{ij}+B_{ij}\dot{\underline{r}}_{ij}) \tag{28}$$

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where $\underline{r}_{ij}=\underline{r}_i-\underline{r}_j$ and the expressions for A_{ij} and B_{ij} are given in (Brumberg, Ivanova, 1982). From this, the heliocentric equations of planetary motion are easily obtained. These equations are expressed in the variables $\underline{R}_i=\underline{r}_i-\underline{r}_o$ ($\underline{r}_o$ being the barycentric position vector of the Sun) with TDB as the independent argument. Thus the heliocentric equations are not the same as the equations of motion in the solar reference frame.

A more detailed calculation of the relativistic corrections in the planetary motion has been performed in (Lestrade, Bretagnon, 1982).

7. LUNAR MOTION

Considering the three-body problem : the Earth, the Sun and the Moon one may easily obtain on the basis of Eqs. (28) the post-Newtonian equations for the geocentric position vector $\underline{\rho}$ of the Moon and the heliocentric position vector $\underline{R}$ of the Newtonian barycentre of the Earth-Moon system (Brumberg, Ivanova, 1982). These equations may be rewritten in the Lagrangian form which was used in (Brumberg, Ivanova, 1985) to find the main relativistic perturbations in the lunar motion in the analytical form. In the semi-analytical form and by a different method this problem was solved in (Lestrade, Chapront-Touzé, 1982). Just as the heliocentric planetary equations are not the equations of motion in the solar reference frame, so the lunar geocentric equations are not the equations in the terrestrial reference frame. The components ρ in rectangular or spherical coordinates are unmeasurable quantities. However with the aid of these components one may derive the expressions for the measurable quantities such as the time delay for lunar laser ranging, the angular distance Moon-Sun or Moon-remote star. In these expressions the parameter α disappears after substituting $\underline{\rho}$ and $\underline{R}$ in functions of the measurable quantities.

At present, there are papers where the lunar motion is treated in the terrestrial reference frame. In particular, TDT but not TDB is adopted in these papers as an independent argument. In the final conclusions pertaining to measurable quantities the results of calculation in any reference frame should coincide. A significant discrepancy exists now between the papers cited above and the paper by Mashhoon (1984) where the influence of the rotation of the Sun on the variation of the Earth-Moon distance was considered and an unexpectedly large term was revealed with an amplitude independent of the light velocity c. According to Mashhoon this effect is caused by the appearance of the resonance divisor of order μ, μ denoting the relativistic small parameter. But this seems rather improbable.

On the other hand, the resonance divisors of order $\sqrt{\mu}$ might exist in some problems of relativistic celestial mechanics resulting in the power series solution with respect to $\sqrt{\mu}$. As yet, the existence of such solutions is an open question.

8. MOTION OF ARTIFICIAL SATELLITES OF THE EARTH

The equations of the lunar motion and the Lagrangian of these equations may be also used for presenting the motion of artificial Earth satellites taking into account the solar perturbations. As the ratio ρ/R for satellite is considerably smaller than for the Moon a more simple metric and more simple equations are adequate to present the satellite motion. By way of illustration let us describe the Earth-Sun field by a metric which is the linear superposition of the Schwarzschild terms due to the Earth and the Sun and perform the transformation

$$x^i=R^i(t)+\xi^i \qquad \text{or} \quad \underline{r}=\underline{R}(t)+\underline{\rho} \tag{29}$$

where $\underline{R}(t)$ is the Schwarzschild solution for the heliocentric motion of the Earth. Expanding in powers of ρ/R we obtain the equations of satellite motion as follows (Brumberg, 1986) :

$$\ddot{\underline{\rho}}=-GM_1\underline{\rho}/\rho^3+GM_2[-\underline{\rho}+3(\underline{R}\underline{\rho})\underline{R}/R^2]/R^3+\ldots+m_1\{[-(\alpha+\gamma)\dot{\underline{\rho}}^2+3\alpha(\underline{\rho}\dot{\underline{\rho}})^2/\rho^2+2(\beta+\gamma-\alpha)$$
$$GM_1/\rho]\underline{\rho}+2(\gamma-\alpha+1)(\underline{\rho}\dot{\underline{\rho}})\dot{\underline{\rho}}\}/\rho^3+m_2\{(2\gamma+1)[\dot{\underline{\rho}}.(\dot{\underline{R}}.\underline{R})]+(1-2\alpha)[(\dot{\underline{R}}\underline{\rho})\underline{R}+(\underline{R}\dot{\underline{\rho}})\dot{\underline{R}}]+[-(\alpha+\gamma)$$
$$\dot{\underline{\rho}}^2+3\alpha(\underline{R}\dot{\underline{\rho}})^2/\underline{R}^2]\underline{R}+2(\gamma-\alpha+1)(\underline{R}\dot{\underline{\rho}})\dot{\underline{\rho}}\}/R^3+\ldots \tag{30}$$

Here M_1 and M_2 are the masses of the Earth and the Sun respectively, $m_i=GM_i/c^2$, β, γ are the main PPN formalism parameters (for general relativity $\beta=\gamma=1$), α is the coordinate parameter. In relativistic terms $\underline{R}(t)$ is regarded as the heliocentric circular solution for the Earth described by relations

$$\underline{R}=R(\cos(Nt),\ \sin(Nt),0),\ \dot{\underline{R}}=NR(-\sin(Nt),\ \cos(Nt),0), \tag{31}$$

$$N^2R^3[1+(-3\alpha+2\beta+\gamma)m_2/R]=GM_2 \quad . \tag{32}$$

Not concerning with the actual satellite motion problem we have neglected in the relativistic right-hand members of Eqs. (30) the terms related to the combined action of the Earth and the Sun. Thus, these equations contain three kinds of relativistic perturbations : 1). indirect solar perturbations caused by substituting the relativistic value $\underline{R}(t)$ into the Newtonian right-hand part of Eqs. (30);2). the Schwarzschild perturbations due to the Earth ; 3). the direct solar perturbations surviving when $\rho/R \to 0$. The approximate analytical solution of these equations may be achieved without any difficulties in three steps. At the first step one performs the coordinate transformation to the coordinates $\underline{r}^*$ with the value $\alpha=1/2$.For the heliocentric coordinates of the Earth and the geocentric coordinates of the satellite one obtains :

$$\underline{R}=\underline{R}^*+(\alpha-1/2)m_2\underline{R}/R+\ldots \quad , \tag{33}$$

$$\underline{\rho}=\underline{\rho}^*+(\alpha-1/2)\{m_1\underline{\rho}/\rho+(m_2/R)[\underline{\rho}-(\underline{R}\underline{\rho})\underline{R}/R^2-(\underline{R}\underline{\rho})\underline{\rho}/R^2+(-\underline{\rho}^2+3(\underline{R}\underline{\rho})^2$$
$$/R^2)\underline{R}/(2R^2)+\ldots]\} \quad . \tag{34}$$

The equations of motion in these coordinates have the form (30) with the value $\alpha=1/2$ and replacing $\underline{\rho}$, $\underline{R}$ by ρ^*, $\underline{R}^*$ respectively. Thus in the solar right-hand member of (30) the term with the coefficient $1-2\alpha$ is removed. At he second step one eliminates the term with the coefficient $2\gamma+1$ describing the geodesic precession. This is achieved by introducing the rotating coordinate system

$$\underline{\rho}^*=A(t)\underline{\rho}',A(t)\begin{pmatrix}\cos\ \omega t & -\sin\ \omega t & 0\\ \sin\ \ t & \cos\ \ t & 0\\ 0 & 0 & 1\end{pmatrix},\omega=(\gamma+1/2)Nm_2/R. \quad (35)$$

The equations for the coordinates $\underline{\rho}'$ are obtained from Eqs. (30) substituting $\underline{\rho}^*$, $\underline{R}^*$ by ρ', $\underline{R}'$ respectively, $\underline{R}'$ resulting from $\underline{R}^*$ after changing the frequancy $\bar{N}$ of trigonometric arguments by N-ω. Hence, the solar part of Eqs. (30) will not contain any terms linear with respect to the satellite velocity. To remove the remaining quadratic terms it is sufficient to perform the transformation

$$\underline{\rho}'=\underline{\rho}''+m_2\{[-(2\gamma+1)\rho'^2+3(\underline{R}'\underline{\rho}')^2/R'^2]\underline{R}'+2(2\gamma+1)(\underline{R}'\underline{\rho}')\underline{\rho}'\}/4R'^3 . \quad (36)$$

The equations for the coordinates $\underline{\rho}''$ result from Eqs. (30) substituting $\underline{\rho}'$ by $\underline{\rho}''$. These equations will contain only the Newtonian and Schwarzschild terms. Consequently, the solution for $\underline{\rho}''$ will consist of the Newtonian terms with the transformation $\underline{R}\rightarrow\underline{R}^*\rightarrow\underline{R}'$ and of the Schwarzschild terms. Then the solution for $\underline{\rho}$ may be derived with the aid of Eqs. (34)-(36). Needless to say, that this method is valid within the accuracy of Eqs. (30) themselves.

The components $\underline{\rho}$ have no physical meaning but as for the case of the lunar problem they serve for calculating measurable quantities.

Recently Ashby and Bertotti (1984) have derived the equations of the satellite motion in the terrestrial reference frame. The use of such a frame permits to remove the main solar relativistic perturbations, so that the remaining relativistic terms are the Schwarzschild terms due to the Earth and the tidal solar perturbations of the order $m_2\rho^2/R^3$. But this metric gives rise to some questions.

First of all, it is possible to introduce a local inertial reference frame along the world-line of a test particle not possessing its own gravitational field. In the Earth satellite problem it is necessary to take into account the gravitational field of the Earth which produces inevitable coordinate arbitraryness into the metric. Secondly, the use of the local system is convenient in case of observations within this system (laser or Doppler satellite observations for example). If the observations related with the distant sources are also used then the advantages of the local system are somewhat lost. Finally, the local reference frame involves using its own system of units (Fukushima et al., 1986) while the motion of the perturbing bodies (the Sun, the Moon) is referred to the barycentric frame. This leads to some logical inconsistency though it is practically quite negligible at the present time.

9. GRAVITATIONAL RADIATION

Just quite recently the problem of motion of celestial bodies taking into account the gravitational radiation was solved in terms of celestial mechanics (Damour, 1983 ; Grishchuk, Kopejkin, 1983 ; Kopejkin, 1985). In these papers the equations of the two-body problem including the gravitational radiation terms are given which permitted to draw in rigorous manner the conclusions deduced earlier from the quadrupole radiation formula of the linearized theory. On the other hand these papers demonstrate that much remains to be done for revealing the structure of the equations of motion in relativistic celestial mechanics (the rigorous form of equations, the influence of coordinate conditions, the convergence of subsequent approximations, etc.).

10. ROTATIONAL MOTION

So far the problem of rotational motion has had no satisfactory solution in relativistic celestial mechanics. Some problems of this kind such as the elaboration of the relativistic theory of the Earth rotation, the study of the gyroscopic effects in the vicinity of the Earth, etc., have attained the level of experimental accuracy which is stimulatory to further investigations in this field. The difficulties of these problems consist in the definitions of rotating body, its spin, its angular velocity. The equations of rotational motion derived by different methods coincide in their main terms linear in angular velocities but disagree in terms of higher orders. A fascinating problem is the interrelation of the rotational and translatory motion of the celestial bodies having in the Newtonian approximation a spherical shape. Some details of the present status in this field may be found in (Brumberg, 1972 ; Barker, O'Connell, 1976 ; Will, 1981).

11. RELATIVISTIC ASTROMETRY

One has to deal in astrometry with such measurable quantities as time, mutual angular distances and light or radio freaquancies. The relativistic reduction of astrometric measurements is to take into account the dependence of measurable quantities on the observer's velocity and the value of the gravitation potentials at the point of observation. The velocity reduction taken into consideration in the first approximation even in Newtonian physics is to re-calculate the measurable quantities for some fictitious point of observation which in the problem at hand may be regarded as fixed. The gravitation potential reduction is to re-calculate the measurable quantities for the fictitious observer at infinite distance from the gravitating bodies where one may neglect the influence of these bodies and consider a flat space-time. For the relative measurements such as the angular distances between two light sources, for example, the problem of reduction may be easily solved and the corresponding formulae are given in many papers. For example, in (Brumberg, 1986) the gravitational field of any specific problem is represented by metric (17)-(19) and the origin of the corresponding quasi-Galilean coordinate system is chosen at the point taken to be at rest for this problem. The motion $\underline{R}=\underline{R}(t)$ of the observer and the trajectories of the light rays are calculated in this metric. Then one performs the transformation (29) and the local splitting of the obtained metric at the point of observation as follows

$$ds^2=c^2d\tau^2-dl^2 \quad , \quad dl^2 = \gamma_{ij}d\xi^i \, d\xi^j \tag{37}$$

where

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$$c d\tau=(1+h_{oo}/2-\dot{\underline{R}}^2/2c^2+h_{oi}\dot{R}^i/c+h_{ij}\dot{R}^i\dot{R}^j/2c^2-h^2_{oo}/8+h_{oo}\dot{\underline{R}}^2/4c^2-\dot{\underline{R}}^4/8c^4)c\,dt+$$

$$+(-\dot{R}^i/c+h_{oi}+h_{ij}\dot{R}^j/c+h_{oo}\dot{R}^i/2c-\dot{\underline{R}}^2\dot{R}^i/2c^3)d\xi^i \quad , \tag{38}$$

$$\gamma_{ij}=\delta_{ij}-h_{ij}+\dot{R}^i\dot{R}^j/c^2 \quad . \tag{39}$$

$d\tau$ represents the element of the proper time of the observer on his world-line ($d\xi^i=0$). The calculation of all angles is performed at the point of observation in the three-dimensional space determined by the metric form dl^2 and all reductions in the relative measurements may be easily derived.

To take into account the relativistic reduction of absolute measurements performed in some astronomical reference frame is a more complicated problem. First of all, it is necessary to have a rigorous definition of the reference frame at hand based on some specific system of operations. The operational definition of the reference frame will permit to determine unambiguously the necessary relativistic reductions. In physical problems the observer's proper reference frame is often used. This frame is determined by the observer's proper time and three space directions fixed by the gyroscopes. The expression of the metric and the equations of motion of test particles in such a frame are given in (Ni, Zimmermann, 1978). But to define the real astronomical kinematic or dynamic frames of reference this approach may be inadequate. There are different descriptions of relativistic reference frames for astronomy and corresponding reduction procedures (Murray, 1983 ; Pavlov, 1984 ; Japanese Ephemeris, 1985) but this question requires further investigation.

The problem of the time reduction has been solved for practical purposes in (Moyer, 1981). But from the theoretical point of view the definitions of TDB, TDT and TAI are not quite convincing. As stated in Section 6 TDB represents the coordinate time and to be determined rigorously it should be fixed by specific coordinate conditions. TAI is generally the proper time but averaged over the terrestrial observatories. TDT manifests itself as the proper time with respect to TDB and as the coordinate time with respect to TAI. The role of TDT and TAI for the global time scale for the Earth has been discussed in (Ashby, Allan, 1979). The definitions of TDB, TDT and TAI are closely connected with the definitions of the geocentric and topocentric reference frames and should be refined.

The transformations between the reference frames involve the problem of the astronomical system of units. One may consider the units of time and length as dependent on the reference frame (Fukushima et al., 1986). This means a new, substantially relativistic, element in the astronomical practice when a single defined physical quantity such as the geocentric gravitation constant, for example, has different numerical values in different reference frames.

12. CONCLUSION

In conclusion let us list again some problems of relativistic celestial mechanics which seem to be especially relevant here.

1. Kerr and Weyl - Levi-Civita metrics in the arbitrary quasi-Galilean coordinates.

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2. Existence of solutions in powers of the square root of the relativistic small parameter (relativistic resonance).
3. Perturbations of the Schwarzschild metric expanded in powers of the small planetary masses.
4. Qualitative study of the equations of motion in higher orders with respect to v/c (the rigorous form of the equations, the influence of coordinate conditions, the convergence of subsequent approximations).
5. Relativistic theory of rotation of the Earth.
6. General theory of interrelation between translatory and rotational motion of celestial bodies.
7. Operational definitions and transformations between different reference frames (barycentric, geocentric, topocentric, satellite frames).
8. Operational treatment of astronomical measurements.
9. Logically consistent time scales.
10. Relativistic system of astronomical constants.

Some of these problems are becoming practically important which is beneficial for their ultimate solution.

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