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# A NEW APPROACH TO OPERATOR SPACES

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ABSTRACT. The authors previously observed that the space of completely bounded maps between two operator spaces can be realized as an operator space. In particular, with the appropriate matricial norms the dual $V^{\dagger}$ of an operator space $V$ is completely isometric to a linear space of operators. This approach to duality enables one to formulate new analogues of Banach space concepts and results. In particular, there is an operator space version $\otimes_{\mu}$ of the Banach space projective tensor product $\hat{\otimes}$, which satisfies the expected functorial properties. As is the case for Banach spaces, given an operator space $V$, the functor $W \mapsto V \otimes_{\mu} W$ preserves inclusions if and only if $V^{\dagger}$ is an injective operator space.

1. **Introduction.** The theory of operator spaces and their completely bounded maps has provided an unexpectedly powerful tool for studying operator algebras. In particular, the Christensen-Sinclair theory of completely bounded multilinear maps [5] (see also [14]) has played a vital role in the study of Hochschild cohomology for operator algebras (see [4],[6]), completely bounded harmonic analysis (see [12],[10]), and most recently, the abstract characterization of the non-self adjoint unital operator algebras [2]. In this category, the *Haagerup tensor product* $\otimes_h$ may be used to linearize the multivariable maps, and in this sense it is analogous to the projective tensor product $\hat{\otimes}$ for normed spaces.

Despite its great utility, the category of operator spaces has some puzzling aspects which have frustrated attempts to generalize the classical theory of normed spaces (see, e.g., [13]). The most important of these is that the dual of an operator space is no longer an operator space. A related difficulty is that one does not have an analogue of the *adjoint functor* equation for the projective tensor product $\hat{\otimes}$ of Banach spaces:

$$\mathcal{B}(V\hat{\otimes}W, X) = \mathcal{B}\big(V, \mathcal{B}(W, X)\big). \tag{1.1}$$

In this note we suggest a modified category that avoids these problems. It was pointed out in [9], p. 140, that given operator spaces $V$ and $W$, the space of completely bounded maps $\mathcal{M}(V, W)$ may be provided with a natural *operator space* structure. In particular, letting $V^{\dagger}$ denote $V^* = \mathcal{M}(V, \mathbb{C})$ with this matricial structure, we show that the embedding $V \hookrightarrow V^{\dagger\dagger}$ is completely isometric. In this modified context, the completely bounded multilinear maps of Christensen and Sinclair [5] are no longer sufficiently general. Instead one must use a notion suggested by Choi [3]: we say that a bilinear map

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$\varphi: V \times W \longrightarrow X$ is *matricially bounded* if one has that $\|[\varphi(v_{ij}, w_{kl})]\| \leq c\|v\| \, \|w\|$ for all $v \in \mathbb{M}_p(V)$, $w \in \mathbb{M}_q(W)$ (see below). This is distinct from Christensen and Sinclair's notion of complete boundedness since both the product $(a,b) \longrightarrow ab$ and the reverse product $(a,b) \longrightarrow ba$ in a $C^*$-algebra are matricially bounded. This aspect of the theory is especially interesting since it would appear that matricially bounded cyclic cohomology makes sense.

In § 2 we introduce the operator space structure on mapping spaces, we prove the double dual result, and using the subscript $\mu$ to indicate the matricially bounded bilinear maps, we show that

$$\mathcal{M}_\mu(V \times W, X) \cong \mathcal{M}\big(V, \mathcal{M}(W,X)\big) \tag{1.2}$$

In § 3 we define the relevant *matricial* tensor product, which we denote by $\otimes_\mu$. We prove that such tensor products of operator spaces are again operator spaces, and that they may be used to linearize matricially bounded multilinear maps. It then follows from (1.2) that we have an analogue of (1.1):

$$\mathcal{M}(V \otimes_\mu W, X) = \mathcal{M}\big(V, \mathcal{M}(W,X)\big).$$

We show that in contrast to the Haagerup product (see [14]), the matricial product is *not injective.* In fact we show that the functor $V\otimes_\mu$ preserves inclusions if and only if $V^\dagger$ is injective, the analogue of a result of Grothendieck [11].

Finally in § 4 we show that reverse multiplication is matricially bounded, and we consider the possibility of representation theorems for such maps into $\mathcal{B}(H)$.

Shortly after the first draft of this paper was completed, we received a preliminary version of [1], in which David Blecher and Vern Paulsen have independently considered the operator structure on mapping spaces. In addition to verifying its functorial properties, they have also found a number of far-reaching results regarding other tensor products, some of which we shall explore in a subsequent paper.

2. **Matricially bounded multilinear maps.** We will use the standard terminology for operator spaces (see [15],[7]). Given operator spaces $V$ and $W$ we let $\mathcal{M}(V, W)$ denote the vector space of completely bounded linear maps $\varphi: V \longrightarrow W$, on which we place the completely bounded norm $\| \quad \|_{cb}$. We regard an $n \times n$ matrix $\varphi = [\varphi_{ij}]$ of such maps as a map $\varphi: V \longrightarrow \mathbb{M}_n(W)$ by letting $\varphi(v) = [\varphi_{ij}(v)]$. We use the resulting identification

$$\mathbb{M}_n\big(\mathcal{M}(V,W)\big) \cong \mathcal{M}\big(V, \mathbb{M}_n(W)\big)$$

to define norms on the matrix spaces over $\mathcal{M}(V,W)$. As was remarked in [9], $\mathcal{M}(V,W)$ is obviously an $L^\infty$-matricially normed space, and thus it is (completely isometric to) an operator space [15]. In particular, noting that $\mathcal{M}(V, \mathbb{C})$ coincides with the dual Banach space $V^* = \mathcal{B}(V, \mathbb{C})$ we let $V^\dagger = \mathcal{M}(V, \mathbb{C})$ denote $V^*$ with this matricial structure. From [17] (see also [8]) we have that if

$$\varphi \in \mathbb{M}_n(V^\dagger) = \mathcal{M}(V, \mathbb{M}_n),$$

then $\|\varphi\|_{cb} = \|\varphi_n\|$. We let $\tau: V \longrightarrow V^{**}$ denote the usual injection, i.e., $\tau(v)(f) = f(v)$.

LEMMA 2.1 ([16], PROP. 1.2.6). *Suppose that $V$ is an operator space, and that $v_0 \in \mathbb{M}_n(V)$. Then*

$$\|v_0\| = \sup\{\,\|\varphi_n(v_0)\| : \varphi \in \mathcal{M}(V, \mathbb{M}_n), \|\varphi\|_{cb} \leq 1\}.$$

PROOF. We may assume that $V$ is acting on a Hilbert space $H$ and thus $\mathbb{M}_n(V)$ acts on $H^n$, and that $\|v_0\| = 1$. Given $\varepsilon > 0$, we may choose unit vectors $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n) \in H^n$ such that $|v_0\xi \cdot \eta| \geq 1 - \varepsilon$. We let $H_1 \subseteq H$ (resp., $H_2 \subseteq H$) be the linear span of $\xi_1, \ldots, \xi_n$ (resp., $\eta_1, \ldots, \eta_n$). Enlarging $H_1$ and $H_2$, if necessary, we may assume that $\dim H_k = n$. We let $W_k$ be an isometry of $\mathbb{C}^n$ onto $H_k$, and we define a complete contraction $\varphi: V \longrightarrow \mathbb{M}_n = \mathcal{B}(\mathbb{C}^n)$ by $\varphi(v) = W_2^* v W_1$. We then have that $W_k \otimes I$ is an isometry of $(\mathbb{C}^n)^n$ onto $H_k^n$, and

$$\begin{aligned} \|\varphi_n(v_0)\| &\geq |\varphi_n(v_0)(W_1 \otimes I)^*\xi \cdot (W_2 \otimes I)^*\eta| \\ &= |(W_2 \varphi W_1^*)_n(v_0)\xi \cdot \eta| \\ &= |v_0\xi \cdot \eta| \geq 1 - \varepsilon, \end{aligned}$$

and we are done. ∎

THEOREM 2.2. *The map $\tau: V \longrightarrow V^{\dagger\dagger}$ is a complete isometry.*

PROOF. Given $v \in \mathbb{M}_n(V)$, we have that

$$\tau_n(v) = [\tau(v_{ij})] \in \mathbb{M}_n(V^{\dagger\dagger}) = \mathcal{M}(V^\dagger, \mathbb{M}_n)$$

and thus

$$\begin{aligned} \|\tau_n(v)\|_{cb} &= \|\tau_n(v)_n\| \\ &= \sup\{\,\|\tau_n(v)_n(f)\| : f \in \mathbb{M}_n(V^\dagger),\ \|f\| = 1\} \\ &= \sup\{\,\|[\tau_n(v)(f_{kl})_{kl=1}^n]\| : f \in \mathbb{M}_n(V^\dagger),\ \|f\| = 1\} \\ &= \sup\Big\{\Big\|\big[[\tau(v_{ij})(f_{kl})]_{ij=1}^n\big]_{kl=1}^n\Big\| : f \in \mathbb{M}_n(V^\dagger),\ \|f\| = 1\Big\} \\ &= \sup\Big\{\Big\|\big[[\tau(v_{ij})(f_{kl})]_{kl=1}^n\big]_{ij=1}^n\Big\| : f \in \mathbb{M}_n(V^\dagger),\ \|f\| = 1\Big\} \\ &= \sup\Big\{\Big\|\big[[f_{kl}(v_{ij})]_{kl=1}^n\big]_{ij=1}^n\Big\| : f \in \mathbb{M}_n(V^\dagger),\ \|f\| = 1\} \\ &= \sup\{\,\|[f(v_{ij})]_{ij=1}^n\| : f \in \mathcal{M}(V, \mathbb{M}_n),\ \|f\| = 1\} \\ &= \sup\{\,\|f_n(v)\| : f \in \mathcal{M}(V, \mathbb{M}_n),\ \|f\| = 1\} \\ &= \|v\|_{cb}, \end{aligned}$$

where the last equality follows from Lemma 2.1. ∎

Given positive integers $p_1, \ldots, p_r$, (resp., $q_1, \ldots, q_r$), we order the $r$-tuples $i = (i_1, \ldots, i_r)$, $1 \leq i_k \leq p_k$, (resp., $j = (j_1, \ldots, j_r)$, $1 \leq j_k \leq q_k$) lexicographically. We may then regard an array $[w_{ij}]$ with $i$ and $j$ such $r$-tuples and $w_{ij} \in W$ as an element of

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$\mathbb{M}_{p,q}(W)$, where $p = p_1 \cdots p_r$, $q = q_1 \cdots q_r$. Given operator spaces $V_1, \ldots, V_r$, and a multilinear map $\varphi: V_1 \times \cdots \times V_r \longrightarrow W$, we define

$$\varphi_{p_1,q_1|\cdots|p_r,q_r}: \mathbb{M}_{p_1,q_1}(V_1) \times \cdots \times \mathbb{M}_{p_r,q_r}(V_r) \longrightarrow \mathbb{M}_{p,q}(W)$$

by letting $\varphi_{p_1,q_1|\cdots|p_r,q_r}(v_1, \ldots, v_r)$ be the $p \times q$ matrix with entries

$$\varphi_{p_1,q_1|\cdots|p_r,q_r}(v_1, \ldots, v_r)_{ij} = \varphi(v_{1,i_1,j_1}, \ldots, v_{r,i_r,j_r}).$$

Following Choi [3], we say that $\varphi$ is *matricially bounded* (resp., *matricially contractive*) if there exists a constant $K$ (resp., with $K \leq 1$) such that for all subscripts

$$\|\varphi_{p_1,q_1|\cdots|p_r,q_r}\| \leq K,$$

where we use the usual multilinear norm, i.e., $\|\varphi_{p_1q_1|\cdots|p_rq_r}\| \leq K$ if and only if

$$\|\varphi_{p_1,q_1|\cdots|p_r,q_r}(v_1, \ldots, v_r)\| \leq K\|v_1\| \cdots \|v_r\|, \quad \big(\text{for all } v_k \in \mathbb{M}_{p_k}(V_k)\big).$$

We define the *matricial norm* of $\varphi$ by

$$\|\varphi\|_\mu = \sup\{\|\varphi_{p_1,q_1|\cdots|p_r,q_r}\| : p_g, q_h \in \mathbb{N}\ (1 \leq g, h \leq r) \text{ arbitrary}\},$$

and we let $\mathcal{M}_\mu(V_1 \times \cdots \times V_r, W)$ denote the corresponding normed space of matricially bounded maps. We define an $L^\infty$-matricial structure on $\mathcal{M}_\mu(V_1 \times \cdots \times V_r, W)$ by using the identification

$$\mathbb{M}_n \mathcal{M}_\mu(V_1 \times \cdots \times V_r, W) \cong \mathcal{M}_\mu\big(V_1 \times \cdots \times V_r, \mathbb{M}_n(W)\big).$$

For $r = 1$ it is evident that $\mathcal{M}_\mu(V, W) = \mathcal{M}(V, W)$.

Given a matricially bounded bilinear map $\varphi: V \times W \longrightarrow X$, we define $T_\varphi: V \longrightarrow \mathcal{M}(W, X)$ by letting $T_\varphi(v)(w) = \varphi(v, w)$. This determines a map

$$T: \mathcal{M}_\mu(V \times W, X) \longrightarrow \mathcal{M}\big(V, \mathcal{M}(W, X)\big).$$

THEOREM 2.3. *The map $T$ is a completely isometric surjection.*

PROOF. It suffices to prove that $T$ is isometric, since we then have a commutative diagram of isometries

$$\begin{array}{ccc}
\mathbb{M}_n \mathcal{M}_\mu(V \times W, X) & \xrightarrow{T_n} & \mathbb{M}_n \mathcal{M}\big(V, \mathcal{M}(W, X)\big) \\
\downarrow & & \downarrow \\
\mathcal{M}_\mu(V \times W, \mathbb{M}_n X) & \xrightarrow{T} & \mathcal{M}\big(V, \mathcal{M}(W, \mathbb{M}_n X)\big).
\end{array}$$

We have that

$$\begin{aligned}
\|T_\varphi\|_{cb} &= \sup\{\|(T_\varphi)_p(v)\|_{cb} : v \in \mathbb{M}_p(V),\ \|v\| \leq 1,\ p \in \mathbb{N}\} \\
&= \sup\{\|(T_\varphi)_p(v)_q(w)\|_{cb} : v \in \mathbb{M}_p(V),\ w \in \mathbb{M}_q(W),\ \|v\|, \|w\| \leq 1, p, q \in \mathbb{N}\} \\
&= \sup\{\|[\varphi(v_{ij}, w_{kl})]\| : v \in \mathbb{M}_p(V),\ w \in \mathbb{M}_q(W),\ \|v\|, \|w\| \leq 1, p, q \in \mathbb{N}\} \\
&= \|\varphi\|_\mu,
\end{aligned}$$

i.e., $T$ is isometric. It is a simple matter to verify that if $\theta: V \longrightarrow \mathcal{M}(W, X)$ is completely bounded, then $\theta = T_\varphi$, where $\varphi(v, w) = \theta(v)(w)$ is matricially bounded, and thus $T$ is surjective. ∎

3. **The matricial tensor product.** Given operator spaces $V$, $W$ and elements $v \in \mathbb{M}_{p,q}(V)$, $w \in \mathbb{M}_{r,s}(W)$, then we define

$$v \otimes w \in \mathbb{M}_{pr,qs}(V \otimes W)$$

by

$$(v \otimes w)_{ij} = v_{i_1 j_1} \otimes w_{i_2 j_2}.$$

(see the matrix convention in § 2). Given integers $p, q, r$ and $v \in \mathbb{M}_{p,q}(V)$, $w \in \mathbb{M}_{q,r}(W)$, we define

$$v \odot w \in \mathbb{M}_{p,r}(V \otimes W)$$

by

$$(v \odot w)_{ij} = \Sigma v_{i,k} \otimes w_{k,j}.$$

We note that if $v \in \mathbb{M}_{p,q}(V)$, and $w \in \mathbb{M}_{r,s}(W)$, then $\Gamma_r(v) = [v_{ij} \oplus \cdots \oplus v_{ij}] \in \mathbb{M}_{pr,qr}(V)$, and $\Delta_q(w) = w \oplus \cdots \oplus w \in \mathbb{M}_{qr,qs}(W)$ satisfy

$$v \otimes w = \Gamma_r(v) \odot \Delta_q(w).$$

Given scalar matrices $\alpha \in \mathbb{M}_{t,p}$ and $\beta \in \mathbb{M}_{r,u}$

$$\alpha(v \odot w)\beta = (\alpha v) \odot (w\beta)$$

whereas for $\alpha \in \mathbb{M}_{t,pr}$ and $\beta \in \mathbb{M}_{qs,u}$ we have

$$\alpha(v \otimes w)\beta \neq (\alpha v) \otimes (w\beta)$$

since the right side does not make sense.

We define the *matricial* norm on $\mathbb{M}_{m,n}(V \otimes W)$ by

$$\|u\|_\mu = \inf\{ \|\alpha\| \, \|v\| \, \|w\| \, \|\beta\| : u = \alpha(v \otimes w)\beta \}, \tag{3.1}$$

where we select $\alpha \in \mathbb{M}_{m,pr}$, $v \in \mathbb{M}_{p,q}(V)$, $w \in \mathbb{M}_{r,s}(W)$, $\beta \in \mathbb{M}_{qs,n}$. We let $V \otimes_\mu W$ be the vector space tensor product together with these norms.

THEOREM 3.1. *Given operator spaces $V$ and $W$, $V \otimes_\mu W$ is again an operator space.*

PROOF. Let us suppose that $u_k = \alpha_k(v_k \otimes w_k)\beta_k$, $(k = 1, 2)$ where

$$\|\alpha_k\| \, \|v_k\| \, \|w_k\| \, \|\beta_k\| \leq \|u_k\|_\mu + \varepsilon.$$

We may assume that $\|v_k\| = \|w_k\| = 1$, and that $\|\alpha_k\| = \|\beta_k\| \leq (\|u_k\|_\mu + \varepsilon)^{1/2}$. We let

$$\alpha = [\, \alpha_1 \quad 0 \quad 0 \quad \alpha_2 \,], \quad \beta = \begin{bmatrix} \beta_1 \\ 0 \\ 0 \\ \beta_2 \end{bmatrix}.$$

It follows that

$$u_1 + u_2 = \alpha(v_1 \oplus v_2) \otimes (w_1 \oplus w_2)\beta,$$

and thus,

$$\begin{aligned}\|u_1+u_2\|_\mu &\le \|\alpha\|\,\|\beta\|\\ &\le \frac{1}{2}[\|\alpha\|^2+\|\beta\|^2]\\ &= \frac{1}{2}[\|\alpha\alpha^*\|+\|\beta^*\beta\|]\\ &\le \frac{1}{2}(\|\alpha_1\|^2+\|\alpha_2\|^2+\|\beta_1\|^2+\|\beta_2\|^2)\\ &\le \frac{1}{2}[2(\|u_1\|_\mu+\varepsilon)+2(\|u_2\|_\mu+\varepsilon)]\\ &= \|u_1\|_\mu+\|u_2\|_\mu+2\varepsilon,\end{aligned}$$

and since $\varepsilon$ is arbitrary, $\|\ \ \|_\mu$ is subadditive. That $\|cu\| = |c|\,\|u\|$ for $c \in \mathbb{C}$ follows from the more general inequality proved for matrix scalars below.

We recall that the Haagerup norm is defined by

$$\|u\|_h = \inf\{\,\|v\|\,\|w\| : u = v \odot w\}.$$

Given $u = \alpha(v \otimes w)\beta$ with $\|\alpha\|\,\|v\|\,\|w\|\,\|\beta\| \le \|u\|_\mu + \varepsilon$, it follows that

$$u = \alpha\Gamma_r(v) \odot \Delta_q(w)\beta,$$

and thus $\|u\|_h \le \|u\|_\mu + \varepsilon$ for all $\varepsilon > 0$, i.e., $\|u\|_h \le \|u\|_\mu$. It follows that $\|\ \ \|_\mu$ is non-degenerate, and thus is a norm.

Given scalar matrices $\gamma$ and $\delta$, we have that

$$\gamma u\delta = \gamma\alpha(v \otimes w)\beta\delta,$$

and thus $\|\gamma u\delta\|_\mu \le \|\gamma\alpha\|\,\|v\|\,\|w\|\,\|\beta\delta\| \le \|\gamma\|\,\|\delta\|(\|u\|_\mu + \varepsilon)$. Finally given $u_k = \alpha_k(v_k \otimes w_k)\beta_k$ as above, we may assume that $\|v_k\| = \|w_k\| = \|\beta_k\| = 1$, and thus that $\|\alpha_k\| \le \|u_k\| + \varepsilon$. We have that

$$u_1 \oplus u_2 = \tilde{\alpha}\big((v_1 \oplus v_2) \otimes (w_1 \oplus w_2)\big)\tilde{\beta},$$

where

$$\tilde{\alpha} = \begin{bmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & 0 & 0 & \alpha_2\end{bmatrix}, \quad \tilde{\beta} = \begin{bmatrix}\beta_1 & 0\\ 0 & 0\\ 0 & 0\\ 0 & \beta_2\end{bmatrix},$$

and thus since the other factors have norm equal to 1,

$$\|u_1 \oplus u_2\|_\mu \le \|\tilde{\alpha}\| = \max\{\,\|\alpha_k\|\} \le \max\{\,\|u_k\|\} + \varepsilon.$$

It follows from [15] that with these matrix norms, $V \otimes_\mu W$ is an operator space. ∎

Given operator space $V$, $W$, $X$ and a map $\varphi: V \times W \longrightarrow X$, we define $L(\varphi): V \otimes W \longrightarrow X$ by $L(\varphi)(v \otimes w) = \varphi(v, w)$. We thus obtain the *linearization* map

$$L: \mathcal{M}_\mu(V \times W, X) \longrightarrow \mathcal{M}(V \otimes_\mu W, X)$$

THEOREM 3.2. *Linearization provides a complete isometry*

$$\mathcal{M}_\mu(V \times W, X) \cong \mathcal{M}(V \otimes_\mu W, X).$$

PROOF. It suffices to show that

$$L: \mathcal{M}_\mu(V \times W, X) \longrightarrow \mathcal{M}(V \otimes_\mu W, X)$$

is isometric, since we then have a diagram of isometries

$$\begin{array}{ccc} \mathbb{M}_n\mathcal{M}_\mu(V \times W, X) & \xrightarrow{L_n} & \mathbb{M}_n\mathcal{M}(V \otimes_\mu W, X). \\ \downarrow & & \downarrow \\ \mathcal{M}_\mu(V \times W, \mathbb{M}_n X) & \xrightarrow{L} & \mathcal{M}(V \otimes_\mu W, \mathbb{M}_n X). \end{array}$$

Given $u = \alpha(v \otimes w)\beta \in \mathbb{M}_{m,n}(V \otimes_\mu W)$ (see (3.1)), we have that

$$\|L(\varphi)_{m,n}\big(\alpha(v \otimes w)\beta\big)\| = \|\alpha\,\varphi_{p,q|r,s}(v, w)\beta\| \leq \|\alpha\|\,\|\varphi_{p,q|r,s}\|\,\|v\|\,\|w\|\,\|\beta\|,$$

and thus $\|L(\varphi)\|_{cb} \leq \|\varphi\|_\mu$. On the other hand

$$\|\varphi_{p,q|r,s}(v, w)\| = \|L(\varphi)_{pr,qs}(v \otimes w)\| \leq \|L(\varphi)\|_{cb}\,\|v\|\,\|w\|,$$

and thus $\|\varphi\|_\mu \leq \|L(\varphi)\|_{cb}$. It is a trivial matter to see that $L$ is surjective. ∎

Given linear maps $\varphi_k: V_k \longrightarrow W_k (k = 1, 2)$, it is immediate that the linear map

$$\varphi_1 \otimes \varphi_2: V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2$$

satisfies $\|\varphi_1 \otimes \varphi_2\|_{cb} \leq \|\varphi_1\|_{cb}\|\varphi_2\|_{cb}$. However, in contrast to the Haagerup tensor product [13], the matricial product is not *injective*. In fact we have by analogy with [11]:

PROPOSITION 3.3. *If $V$ is an operator space, then the following are equivalent:*

*(1) Given an arbitrary inclusion of operator spaces $W_1 \hookrightarrow W_2$, the induced map $V \otimes_\mu W_1 \longrightarrow V \otimes_\mu W_2$ is completely isometric.*

*(2) $V^\dagger$ is an injective operator space.*

PROOF. From above, the induced map $V \otimes_\mu W_1 \longrightarrow V \otimes_\mu W_2$ is completely contractive. If it is isometric, then from the classical Hahn-Banach Theorem, any $f \in (V \otimes_\mu W_1)^*$ has a contractive extension $F \in (V \otimes_\mu W_2)^*$ of the same norm. Since the norm and complete bounded norms of scalar functionals coincide, we see that any $\varphi: W_1 \longrightarrow V^\dagger$ has an extension $\Phi: W_2 \longrightarrow V^\dagger$ with the same completely bounded norm, and thus $V^\dagger$ is an injective operator space. Conversely, let us suppose that $V^\dagger$ is injective. Then $\mathbb{M}_n(V^\dagger)$ is injective, since given $V^\dagger \subseteq \mathcal{B}(H)$ and a completely contractive projection $\Phi$ of $\mathcal{B}(H)$ onto $V^\dagger$, $\Phi_n$ is a completely contractive projection of the injective $\mathcal{B}(H^n)$ onto $\mathbb{M}_n(V^\dagger)$. Given $u \in \mathbb{M}_n(V \otimes_\mu W_1)$, we may use Lemma 2.1 to choose a complete contraction $\varphi: V \otimes_\mu W_1 \longrightarrow \mathbb{M}_n$ such that $\|\varphi_n(u)\| \geq \|u\| - \varepsilon$. We have that

$$\varphi \in \mathbb{M}_n\big((V \otimes_\mu W_1)^\dagger\big) = \mathbb{M}_n\big(\mathcal{M}(W_1, V^\dagger)\big) = \mathcal{M}\big(W_1, \mathbb{M}_n(V^\dagger)\big),$$

and thus $\varphi$ has an extension to a complete contraction $\varphi': W_2 \longrightarrow \mathbb{M}_n(V^\dagger)$. Regarding $\varphi'$ as a functional on $\mathbb{M}_n(V \otimes_\mu W_2)$ and letting $\|\ \ \|'$ be the norm on the latter space, we see that $\|u\|' \geq \|\varphi'(u)\| \geq \|u\| - \varepsilon$, and we are done. ∎

4. **Regarding representation theorems.** As we remarked in the introduction, given a $C^*$-algebra $A$, the reverse multiplication map $R: A \times A \longrightarrow A: (a,b) \longrightarrow ba$ is matricially contractive. To see this, simply note that given $a \in \mathbb{M}_{p,q}(A)$ and $b \in \mathbb{M}_{r,s}(A)$, we have that

$$\|R_{p,q|r,s}(a,b)\| = \|\Delta_p(b)\Gamma_s(a)\| \leq \|a\|\,\|b\|.$$

An interesting consequence of this result is that if $F: A \times A \longrightarrow \mathbb{C}$ is a bilinear function satisfying

$$|F(a,b)| \leq p_1(aa^*)^{1/2}q_1(b^*b)^{1/2} + p_2(a^*a)^{1/2}q_2(bb^*)^{1/2}$$

for states $p_i, q_j$ then $F$ is matricially bounded. It would seem likely that the converse is also true. More generally one might conjecture that there is a representation theorem for matricially bounded maps $\varphi: A \times A \longrightarrow \mathcal{B}(H)$ involving expressions of the form $R\pi(a)S\theta(b)T$ and $R'\theta'(b)S'\pi'(a)T'$. The Haagerup form of the Grothendieck Theorem seems to say that all bounded bilinear functionals on $A$ are constructed from completely bounded functionals on $A \otimes A$ and $A \otimes A^{\mathrm{op}}$. The operator system approach of Paulsen and Smith [14] does not appear to work, since despite the fact that $A \otimes_\mu A$ is an $A$-bimodule, the operations are presumably not completely contractive in the sense of Christensen and Sinclair.

Finally we observe that the matricial category seems more suitable to cyclic cohomology than the completely bounded category, since cyclic theory involves reversed products. A new representation theorem might conceivably enable one to reduce the relevant calculations to algebraic manipulations, as was the case for completely bounded cohomology.

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