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Entire functions mapping countable dense subsets of the reals onto each other monotonically

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It is shown that for arbitrary countable dense subsets A and B of the real line, there exists a transcendental entire function whose restriction to the real line is a real-valued strictly monotone increasing surjection taking A onto B. The technique used is a modification of the procedure Maurer used to show that for countable dense subsets A and B of the plane, there exists a transcendental entire function whose restriction to A is a bijection from A to B.

1. Introduction

The following problem was posed by Erdös [2; p. 297, Problem 24] in 1957:

Does there exist an entire function f not of the form $a_0 + a_1 z$, such that the number $f(x)$ is rational or irrational according as x is rational or irrational? More generally, if A and B are two denumerable dense sets, does there exist an entire function which maps A onto B?

A solution to the first part of the problem is to be found in Neumann and Rado [4], while if one interprets the second part of the problem to mean countable dense subsets A and B of the plane, then a solution to this part is given by Maurer [3], who establishes the existence of a

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transcendental entire function whose restriction to A is a bijection from A to B. In fact, the authors [5] have shown that such a function can be constructed so that each of its derivatives has this property as well.

On the other hand, if A and B are considered to be countable dense subsets of the real line, then the following theorem, due to Barth and Schneider [1], solves the problem:

THEOREM. *Let A and B be two countable dense subsets of the real line. Then there exists an entire transcendental function f such that $f(z) \in B$ iff $z \in A$.*

It is still unresolved whether such a statement holds if A and B are countable dense subsets of the plane. By the Picard Theorem, the function Maurer constructs cannot possible satisfy this condition. The purpose of this paper is to show that a modification of Maurer's technique gives a straightforward proof of the theorem that Barth and Schneider actually proved.

2. Monotone generalized interpolation

THEOREM 1. *Let A and B be countable dense subsets of the real line. Then there exists a transcendental entire function f such that f restricted to the real line is a real homeomorphism, and $f(A) = B$.*

Proof. Suppose that we have both A and B enumerated. Choose the first element in the enumeration of A and of B, say a_1 and b_1 respectively. Define $f_0(z) = (z-a_1) + b_1$, and $A_0 = \{a_1\}$, $B_0 = \{b_1\}$. Suppose at the nth stage we have sets $A_{n-1} = \{a_1, a_2, \ldots, a_{2n-1}\} \subset A$ and $B_{n-1} = \{b_1, b_2, \ldots, b_{2n-1}\} \subset B$ and a monotone increasing polynomial f_{n-1} such that $f_{n-1}(a_i) = b_i$ for $i = 1, 2, \ldots, 2n-1$. Construct A_n, B_n and f_n as follows:

(i) Let $h_n(z) = \prod_{i=1}^{2n-1} (z-a_i)$ and choose the first element remaining in the enumeration of A after A_{n-1} is removed. Denote this element by a_{2n}. Let C_{2n} be the intersection of all closed intervals containing

$A_{n-1} \cup \{a_{2n}\}$. There exists a real $\delta > 0$ such that for each real k satisfying $0 < k < \delta$, the polynomial $f_{n-1}(x) + kh_n(x)$ is monotone increasing on C_{2n} , and hence on the whole line. Since $B \sim B_{n-1}$ is dense, we may choose such a k , say k_n , as small as we like so that $f_{n-1}(a_{2n}) + k_n h_n(a_{2n}) = b_{2n} \in B \sim B_{n-1}$. Let $g_n(z) = f_{n-1}(z) + k_n h_n(z)$.

(ii) Now choose the first element remaining in the enumeration of B after $B_{n-1} \cup \{b_{2n}\}$ is removed. Denote this element by b_{2n+1} . Let C_{2n+1} be the intersection of all closed intervals containing $C_{2n} \cup \left[g_n^{-1}(b_{2n+1}) - \frac{1}{2}, g_n^{-1}(b_{2n+1}) + \frac{1}{2}\right]$. Let $F(z, w) = g_n(z) + w(z-a_{2n})^2 h_n(z) - b_{2n+1}$. Since the restriction of g_n to the reals is a surjective real-valued function, there is a real x_0 such that $g_n(x_0) = b_{2n+1}$. Thus $F(x_0, 0) = 0$, while $\frac{\partial f}{\partial w}(x_0, 0) \neq 0$, and so the implicit function theorem asserts that for arbitrarily small $\varepsilon > 0$, there exists a real l_n , satisfying $0 < l_n < \varepsilon$, and $a_{2n+1} \in A \sim (A_{n-1} \cup \{a_{2n}\})$ with $F(a_{2n+1}, l_n) = 0$. Define $f_n(z) = f_{n-1}(z) + k_n h(z) + l_n(z-a_{2n})^2 h(x)$, $A_n = A_{n-1} \cup \{a_{2n}, a_{2n+1}\}$ and $B_n = B_{n-1} \cup \{b_{2n}, b_{2n+1}\}$. Then $f_n|A_n$ is a bijection from A_n to B_n . As in case (i), f_n can be made monotone increasing on C_{2n+1} by choosing l_n sufficiently small and thus f_n can be made monotone increasing on the whole line. Moreover, at each stage we need only consider a finite number of conditions to obtain an upper bound for k_n and l_n so that $f(z) = \sum_{n=1}^{\infty} f_n(z)$ converges to a transcendental entire function. By the construction of each f_n , the restriction of f to the reals is a monotone increasing surjective real-valued function taking A onto B .

References

[1] K.F. Barth and W.J. Schneider, "Entire functions mapping countable dense subsets of the reals onto each other monotonically", *J. London Math. Soc.* (2) **2** (1970), 620-626.

[2] Paul Erdős, "Some unsolved problems", *Michigan Math. J.* **4** (1957), 291-300.

[3] W.D. Maurer, "Conformal equivalence of countable dense sets", *Proc. Amer. Math. Soc.* **18** (1967), 269-270.

[4] B.H. Neumann and R. Rado, "Monotone functions mapping the set of rational numbers onto itself", *J. Austral. Math. Soc.* **3** (1963), 282-287.

[5] Daihachiro Sato and Stuart Rankin, "Highly conformal equivalence of countable dense sets", submitted.

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