Then

$$a^2 = (d - p)^2 + q^2$$
$$b^2 = f^2$$
$$c^2 = (d - p)^2 + (f - q)^2$$
$$r^2 = d^2 = p^2 + q^2.$$

Then, $c^2 = a^2 + b^2 - 2fq$, $\dfrac{c^2 - a^2 - b^2}{-2f} = q$, $\dfrac{(c^2 - a^2 - b^2)^2}{4b^2} = q^2$.
Also $|q|$ is the height of B from OC in the triangle OBC.

Let A_1 be the area of the isosceles triangle OBC with base BC, as shown in Figure 1, so that

$$A_1 = \frac{a\sqrt{r^2 - \frac{1}{4}a^2}}{2}.$$

Let A_2 be the area of the triangle OBC with base OC and height $|q|$, so

$$A_2 = \tfrac{1}{2}r\,|q|\,.$$

Then,

$$\frac{r\,|q|}{2} = \frac{a\sqrt{r^2 - \frac{1}{4}a^2}}{2} \Rightarrow |q| = \frac{a\sqrt{4r^2 - a^2}}{2r} \Rightarrow q^2 = a^2 - \frac{a^4}{4r^2}.$$

Hence,

$$\frac{(c^2 - a^2 - b^2)^2}{4b^2} = a^2 - \frac{a^4}{4r^2}.$$

And so,

$$r^2 = \frac{a^4b^2}{2\left(a^2b^2 + a^2c^2 + b^2c^2\right) - a^4 - b^4 - c^4}.$$

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107.10 Trapezia whose side-lengths form an arithmetic progression

Everyone can easily verify that there are quadrangles with side-lengths forming an arithmetic progression. Are there trapeziums among such quadrangles? The following theorem answers this question.

Theorem

There is no trapezium for which the lengths of consecutive sides form an arithmetic progression.

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Proof

Let a, $a + d$, $a + 2d$, $a + 3d$ be the lengths of consecutive sides of trapezium $ABCD$. Without loss of generality, we can assume that $d > 0$.

Case 1:

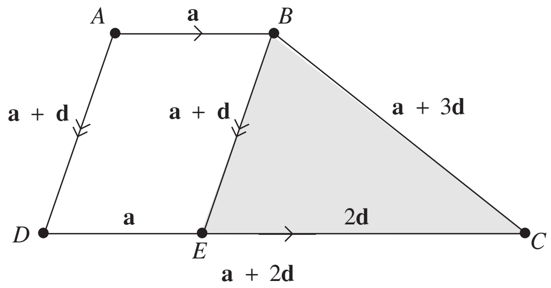

FIGURE 1

Case 2.

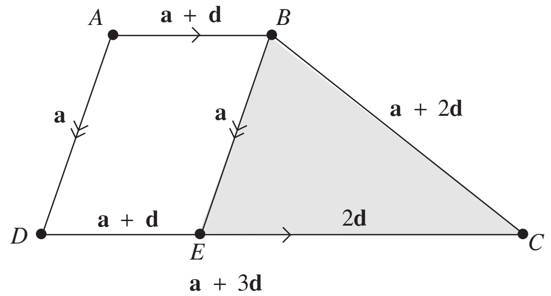

FIGURE 2

From Figures 1 and 2, one can see that triangle EBC does not exist $(BE + EC = BC)$.

Note: If we abandon the requirement that these are the lengths of *consecutive* sides, then such a trapezium exists (see Figure 3).

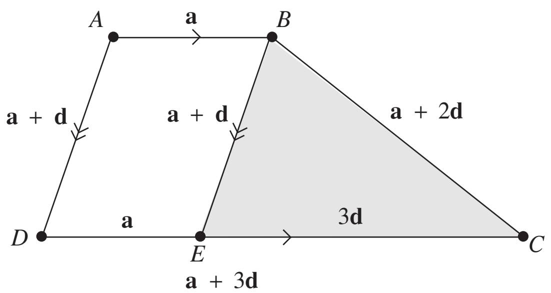

FIGURE 3

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This figure also gives an idea of how such a trapezium with given side-lengths a, $a + d$, $a + 3d$, $a + 2d$ can be constructed with the help of a compass and a ruler.

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107.11 The Steiner-Lehmus Theorem à la Ceva

The Steiner-Lehmus theorem (SL) states that *if the internal angle bisectors of two angles of a triangle are equal, then the corresponding sides are equal*. Despite being easy to state, it is by no means trivial to prove. One of the scores of proofs of this theorem can be found on p. 396 in [1].

In this Note, we first use the SL to prove a theorem, Theorem 1, that we call *the Steiner-Lehmus theorem inside out* (SLIO). We then show in Theorem 2 that the SL itself is a consequence of SLIO. Thus the SL and the SLIO are equivalent. In Theorem 3, we give a fairly simple proof of (a stronger form of) the SLIO using Ceva's theorem and what is often referred to as *the open mouth theorem*. The latter is illustrated in Figure 3 below. Thus we essentially have a proof of SL based on Ceva's theorem. This, hopefully, justifies the title of this Note. The Note ends with a stronger form of the SL, and with a description of the context that led to the SLIO.

Theorem 1 (The Steiner-Lehmus theorem inside out)

Let O be the circumcentre of the acute-angled triangle ABC, and let BE and CF be the cevians through O, as in Figure 1. If $AE = AF$, then $AB = AC$.

Proof

Let the line through A parallel to FC meet the extension of BE at N, and let the line through O parallel to BA meet AN at M; see Figure 1.

By elementary angle-chasing, AE is the bisector of $\angle NAO$ and OM is the bisector of $\angle AON$. By construction, $AFOM$ is a parallelogram and so $OM = AF = AE$. Hence by SL on triangle NAO we see that $NO = NA$ and so $\angle NOA = \angle NAO$. It follows that $\angle FCA = \angle EBA$ and so $\angle ACB = \angle ABC$ and $AB = AC$ as desired.

The proof above also reveals that the Steiner-Lehmus theorem follows from Theorem 1. For the convenience of the reader, we formulate this in the next theorem.