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SEMIPRIME NEAR-RINGS

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Abstract

Some properties of ν-semiprime ($\nu = 0, 1, 2$) near-rings are pointed out. In particular ν-semiprime near-rings which contain nil non-nilpotent ideals are studied.

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1. Preliminaries

Throughout this paper, N will denote a right zerosymmetric near-ring and terminology and notation will agree with those introduced by Pilz in [4]. In particular, for any two sets A and B, the product AB will be the set of the products ab with a in A and b in B.

Let I be a two-sided ideal of N. As Pilz suggests in [4, 2.108], the following definitions can be given:

(a) I is 0-*semiprime* if every two-sided ideal A of N, such that A^2 is contained in I, is contained in I;

(b) I is 1-*semiprime* if every left ideal L of N, such that L^2 is contained in I, is contained in I;

(c) I is 2-*semiprime* if every N-subgroup S of N, such that S^2 is contained in I, is contained in I.

Being N zerosymmetric, every 2-semiprime ideal is 1-semiprime too and every 1-semiprime ideal is 0-semiprime too. Moreover, the 0-semiprime ideals are the semiprime ideals in the usual sense [4, Definition 2.82].

Adapting the proof for 0-semiprime ideals, one proves that for $\nu = 0, 1, 2$ the following conditions are equivalent:

(i) I is a ν-semiprime two-sided ideal;

(ii) if x does not belong to I, then $(x)_\nu^2$ is not contained in I, where $(x)_0$, $(x)_1$, $(x)_2$ mean the two-sided ideal, the left ideal and the N-subgroup respectively, generated by x;

(iii) if X_ν properly contains I, the product X_ν^2, is not contained in I, where X_0, X_1 and X_2 respectively denote a two-sided ideal, a left ideal and an N-subgroup of N.

It is an immediate consequence of condition (ii) that I is a ν-semiprime ideal ($\nu = 0, 1, 2$) if and only if $N \backslash I$ is an sp_ν-system, that is, a set S such that if $s \in S$ there exist two elements s_1, s_2 of $(s)_\nu$ whose product $s_1 s_2$ belongs to S.

Observe that, for $\nu = 0, 1, 2$, any intersection of ν-semiprime ideals is ν-semiprime. In particular, this applies to ν-prime ideals [4, 2.108]: so, for every ideal I of N, the ν-prime radical $P_\nu(I)$ (that is, the intersection of the ν-prime ideals containing I) is ν-semiprime.

2. ν-semiprime near-rings

A near-ring N will be called *ν-semiprime* if (0) is ν-semiprime ($\nu = 0, 1, 2$).

For instance, for every near-ring N and every ν-semiprime ideal B, the near-ring $N' = N/B$ is ν-semiprime: in particular, for each ideal I of N, the near-ring $N/P_\nu(I)$ is ν-semiprime. By definition, a ν-semiprime near-ring does not contain any ideal (respectively left ideal or N-subgroup) X_ν such that $X_\nu^2 = (0)$; moreover,

PROPOSITION 2.1. *If N is ν-semiprime ($\nu = 0, 1, 2$) and if X_ν is a two-sided ideal, a left ideal or a an N-subgroup of N such that there is a positive integer n for which $X_\nu^n = (0)$, then X is zero.*

PROOF. For the sake of brevity, write X instead of X_ν. The statement is true by assumption if n is 2. To obtain a contradiction, suppose now that $X^n = (0)$ with $n > 2$ and $X^{n-1} \neq (0)$. Then there exist $(n-1)$ elements $x_1, x_2, \dots, x_{n-1}$ of X such that the product $y = x_1 \cdot \ldots \cdot x_{n-1}$ is different from zero.

If $\nu = 0, 1$ consider the (two-sided or left) ideal I generated by y. Since I is contained in the (respectively, two-sided or left) ideal (X^{n-1}) generated

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by X^{n-1}, it follows that

$$I^2 \subseteq (X^{n-1}) \cdot I \subseteq (X^{n-1}) \cdot X .$$

As right distributivity implies $(X^{n-1}) \cdot X \subseteq (X^n)$ and X^n is zero by assumption, it follows that $I^2 = (0)$, that is, $I = (0)$, since N is ν-semiprime. So y is zero, which is a contradiction.

If $\nu = 2$, consider the N-subgroup $I = X \cdot x_2 \cdot x_3 \cdot \ldots \cdot x_{n-1}$. From $I \subseteq X^{n-1}$ it follows $I^2 \subseteq X^{2n-2} \subseteq X^n = (0)$, that is, $I = (0)$, since N is 2-semiprime. Consequently $x_1 \cdot \ldots \cdot x_{n-1} = y$ is zero which contradicts the choice of y.

Therefore, if $X^n = (0)$, also $X^{n-1} = (0)$ and, from the inductive assumption, X is zero.

Thus N is ν-semiprime $(\nu = 0, 1, 2)$ if and only if N has no nilpotent two-sided ideal, left ideal, N-subgroup (respectively).

The 0-semiprime near-rings were studied by many authors (for instance, see [3], [5]), while the 2-semiprime ones with descending chain condition on N-subgroups were studied by Blackett in [1]. Here some new properties are pointed out in the case where N is 1-semiprime. In this case N has no nilpotent left ideal; nevertheless we will suppose that N has at least one non-zero nil left ideal. Among other things, this fact implies that in N the descending chain condition on left ideals (and a fortiori on N-subgroups) does not hold; so this study is complementary to Blackett's one. Besides, observe that the nil ideals of N cannot be minimal as N-subgroups, because, for every near-ring N, the following result holds.

PROPOSITION 2.2. *If H is a minimal N-subgroup of N, then H is either nilpotent of index 2 or idempotent.*

3. 1-semiprime near-rings

From now on N will be a 1-semiprime near-ring with at least one non-zero nil left ideal. By Zorn's lemma, this assumption forces N to have at least one left ideal which is maximal in the family of nil left ideals: call each of them a *maximal nil left ideal*. Now, for every left ideal L, denote by $(0 : L)$ the annihilator of L; $(0 : L)$ is a two-sided ideal of N and we prove

PROPOSITION 3.1. *Let L be a maximal nil left ideal and L' be a nil left ideal of N. Then $(0 : L) \subseteq (0 : L')$.*

PROOF. Call S the left ideal generated by $(0 : L) \cdot L'$; S is nil since it is contained in L': actually it will be proved to be zero and hence the result will hold.

First of all observe that the set $(0 : L) \cdot L'$ is contained in $(0 : L)$, so $SL = (0)$. As a consequence, the left ideal $L + S$ is nil: in fact, for every $l \in L$ and every $s \in S$, let h and k be the least positive integers such that $l^h = 0 = s^k$. It is a routine calculation to verify that, if $n = \max(h, k)$, the element $(l + s)^n$ belongs to S and then $l + s$ is nilpotent. For instance, if $n = 2$,

$$(l + s)^2 = l(l + s) + s(l + s) = (l(l + s) - l^2) + l^2 + (s(l + s) - sl) + sl$$

and, since by assumption $l^2 = 0$ and sl belongs to SL, which is zero, $(l+s)^2$ is the sum of two elements of S. Now, since L is a maximal nil left ideal, the nil left ideal $L + S$ must coincide with L, and therefore S must be contained in L. As it is also contained in $(0 : L)$, S^2 is zero, and so S is zero, for N is 1-semiprime.

Consequently in a 1-semiprime near-ring N all the maximal nil left ideals have the same annihilator: it will be called the *nil-annihilator* of N and will be denoted by $\alpha(N)$.

Furthermore, the following statement holds

PROPOSITION 3.2. *The nil-annihilator of N coincides with the nil-annihilator of any sum of maximal nil left ideals of N.*

PROOF. Let L, L' be two maximal nil left ideals and let x be an element of $\alpha(N)$. For all $l \in L$, $l' \in L'$ we have

$$x(l + l') = x(l + l') - xl \in L'.$$

But $x(l + l')$ belongs also to $\alpha(N)$ and therefore

$$x(l + l') \in L' \cap (0 : L') = (0).$$

This proves that $(0 : L) = \alpha(N) \subsetneq (0 : (L + L'))$. Since the converse is obvious, one sees that $(0 : L) = (0: (L + L'))$.

By induction the result may be extended to any finite sum of maximal nil left ideals and also to those which are not finite, since every element of such a sum is a finite sum of elements of maximal nil left ideals.

4. Properties of the nil-annihilator of N

The nil-annihilator of N is a two-sided ideal different from N, because, if $\alpha(N)$ coincided with N, then for every maximal nil left ideal L this would imply $L^2 \subseteq NL = (0)$, contradicting the fact that N is 1-semiprime.

PROPOSITION 4.1. *The nil-annihilator of N is not nil and does not contain any non-zero nil left ideal.*

Indeed if L' is a nil left ideal contained in $\alpha(N)$ and L is a maximal nil left ideal containing L' it follows that

$$L' \subseteq L \cap \alpha(N) = L \cap (0 : L) = (0).$$

PROPOSITION 4.2. $\alpha(N)$ *is a* 0*-semiprime ideal.*

PROOF. Let B be a two-sided ideal such that B^n is contained in $\alpha(N)$. It must be proved that B is contained in $\alpha(N)$, that is, for every maximal nil left ideal L of N, the product BL is zero. Indeed the left ideal K generated by BL is contained in $B \cap L$ and therefore

$$K^n \subseteq B^n \cap L \subseteq \alpha(N) \cap L = (0)$$

which implies $K = (0)$ (because N is 1-semiprime) and consequently $BL = (0)$.

PROPOSITION 4.3. *The nil-annihilator of N is zero if and only if every two-sided ideal contains a non-zero nil left ideal.*

PROOF. Let $\alpha(N)$ be different from zero: then it is a two-sided ideal which contains no non-zero nil left ideal. On the contrary, if $\alpha(N) = (0)$, for every non-zero two-sided ideal B and for every maximal nil left ideal L, BL is different from zero.

Let x be a non-zero element of BL: the left ideal generated by x is the required ideal since it is non-zero, is contained in $B \cap L$ and so nil.

Consider now the factor near-ring $N' = N/\alpha(N)$ and the canonical epimorphism $\pi: N \to N'$. If L is a nil left ideal of N, then by Proposition 4.1, $\pi(L)$ is a non-zero nil left ideal of N', so N' too contains a non-zero nil left ideal. On the other hand, since $\alpha(N)$ is 0-semiprime, N' is 0-semiprime (see 4.2): if N' is also 1-semiprime, its nil-annihilator can be defined and one has

THEOREM 4.4. *If N' is* 1*-semiprime, then $\alpha(N')$ is zero.*

PROOF. In order to prove that $\alpha(N')$ is zero, it is sufficient to show that if B is a two-sided ideal of N with $\pi(B) = \alpha(N')$, then B is contained in $\alpha(N)$, or, equivalently, that for every maximal nil left ideal L of N the product BL is zero.

Now, the left ideal K generated by BL is nil (since it is contained in L); so also its image $\pi(K)$ is nil and contained in $\pi(B) = \alpha(N')$. But $\alpha(N')$ does not contain any non-zero nil left ideal: thus K must be contained in $\alpha(N)$ and, by the same argument, K and its generating set BL must be zero.

The assumption of Theorem 4.4 is satisfied when N is 2-semiprime and has a left identity. Moreover, we have

THEOREM 4.5. *If N is a 2-semiprime near-ring with a left identity and a non-zero nil left ideal, then N' is 2-semiprime too (and consequently contains a non-zero nil left ideal and $\alpha(N')$ is zero.)*

PROOF. In order to prove that N' has no non-zero nilpotent N'-subgroups, first of all we remark that if S' is a nilpotent N'-subgroup of N' and S is its preimage in N, then S is an N-subgroup of N.

Since S' is nilpotent, there is a positive integer n such that S^n is contained in $\alpha(N)$. So, for any maximal nil left ideal L, the product S^nL is zero and consequently

$$(SL)^n = (SL) \cdot \ldots \cdot (SL) \subseteq S \cdot S^{n-1} \cdot L = S^n L = (0).$$

Let now sl be any element of SL: Nsl is an N-subgroup, nilpotent of index at most n, for Nsl is contained in SL; therefore Nsl is zero, as N is 2-semiprime. Since N has a left identity, this implies $sl = 0$, for each $s \in S$ and $l \in L$. Thus SL is zero, so S is contained in the nil-annihilator of N and $S' = \pi(S)$ is zero.

The remaining properties are consequences of the Theorem 4.4 and the preceding remarks.

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