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ON AN INEQUALITY OF KOLMOGOROV AND STEIN

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A.N. Kolmogorov showed that, if $f, f', \dots, f^{(n)}$ are bounded continuous functions on $\mathbb{R}$, then $\|f^{(k)}\|_\infty \leqslant C_{k,n} \|f\|_\infty^{1-k/n} \|f^{(n)}\|_\infty^{k/n}$ when $0 < k < n$. This result was extended by E.M. Stein to Lebesgue L^p-spaces and by H.H. Bang to Orlicz spaces. In this paper, the inequality is extended to more general function spaces.

1. INTRODUCTION

Kolmogorov [8] showed that, if $f, f', \dots, f^{(n)}$ are bounded continuous functions on $\mathbb{R}$, then

$$\|f^{(k)}\|_\infty \leqslant C_{k,n} \|f\|_\infty^{1-k/n} \|f^{(n)}\|_\infty^{k/n}$$

when $0 < k < n$, where $C_{k,n} = K_{n-k} / K_n^{1-k/n}$, and

$$K_i = \frac{4}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^{j(i+1)}}{(2j+1)^{i+1}}.$$

This is the best constant. Kolmogorov's result was extended to Lebesgue L^p-spaces by Stein [10] and to Orlicz spaces by Bang [1].

In this paper, the methods of these authors are modified to prove the analogous result for other function spaces on $\mathbb{R}$. For variants and applications of such results, see, for example, [4, 9, 11]. In particular, our results apply to amalgams of L^p and ℓ^q, as defined and studied in, for example, [2, 3, 5, 6, 7].

To formulate our result, we need several definitions. First, if f is a function on $\mathbb{R}$, we denote by $\tau(t)f$ its translate: $\tau(t)f(s) = f(s+t)$. Next, let $D(\mathbb{R})$ be a space of test functions, such as $C_c^\infty(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$. We require that translations act continuously in $D(\mathbb{R})$.

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Let $X(\mathbb{R})$ be a Banach space of functions on $\mathbb{R}$. We say that $X(\mathbb{R})$ is *τ-stable* provided that $\tau(t)f$ is in $X(\mathbb{R})$ whenever t is in $\mathbb{R}$ and f is in $X(\mathbb{R})$ and further there is a constant C_X such that

$$\|\tau(t)f\|_X \leqslant C_X \|f\|_X \qquad \forall f \in X(\mathbb{R}) \quad \forall t \in \mathbb{R}. \tag{1}$$

Examples of τ-stable spaces include Lebesgue spaces, Lorentz spaces, Orlicz spaces, and, more generally, rearrangement invariant function spaces, and spaces involving derivatives, such as the Sobolev spaces $W^{p,k}(\mathbb{R})$. For all these spaces, $C_X = 1$. Other examples of τ-stable spaces include amalgams and weighted Lebesgue spaces $L^p(\mathbb{R}, w)$ of functions f such that

$$\left(\int_{\mathbb{R}} |f(t)|^p \, w(t)\, dt\right)^{1/p} < \infty,$$

where the weight w is positive, bounded, and bounded away from 0. For these spaces, $C_X > 1$ in general.

Let $\|f\|_X$ denote the norm of f in $X(\mathbb{R})$, and for g in $D(\mathbb{R})$, let $\|g\|_{X^*}$ denote the norm of duality with $X(\mathbb{R})$, that is,

$$\|g\|_{X^*} = \sup\left\{\left|\int_{\mathbb{R}} g(t)\, f(t)\, dt\right| : f \in X(\mathbb{R}),\ \|f\|_X \leqslant 1\right\}.$$

We say that $X(\mathbb{R})$ is *$D(\mathbb{R})$-full* if the map from $D(\mathbb{R})$ to $X(\mathbb{R})^*$ is continuous, so $\|g\|_{X^*} < \infty$ for all g in $D(\mathbb{R})$, and, if f is in $D(\mathbb{R})^*$ (the dual space of $D(\mathbb{R})$) and

$$\sup\left\{\left|\int_{\mathbb{R}} g(t) f(t)\, dt\right| : g \in D(\mathbb{R}),\ \|g\|_{X^*} \leqslant 1\right\} < \infty,$$

then f is in $X(\mathbb{R})$ and $\|f\|_X$ is equal to the left hand side of the inequality above. For example, take $D(\mathbb{R})$ to be $C_c^\infty(\mathbb{R})$. If $X(\mathbb{R})$ is an amalgam of L^p and ℓ^p where $1 < p, q \leqslant \infty$ (in particular if $X(\mathbb{R}) = L^p(\mathbb{R})$ for such p) then $X(\mathbb{R})$ is $D(\mathbb{R})$-full; however, if $X(\mathbb{R}) = L^1(\mathbb{R})$, then $X(\mathbb{R})$ is not $D(\mathbb{R})$-full: the problem is the measures.

Theorem 1. *Suppose that $X(\mathbb{R})$ is a τ-stable $D(\mathbb{R})$-full Banach space of functions on $\mathbb{R}$. [The word "functions" here is intended to include generalised functions such as distributions.] If f and its generalised derivative $f^{(n)}$ are in $X(\mathbb{R})$, then $f^{(k)}$ is in $X(\mathbb{R})$ when $0 < k < n$ and*

$$\|f^{(k)}\|_X \leqslant C_X\, C_{k,n} \|f\|_X^{1-k/n} \|f^{(n)}\|_X^{k/n}.$$

Proof: Take a function h in $D(\mathbb{R})$, and define $F : \mathbb{R} \to \mathbb{C}$ by the formula

$$F(t) = \int_{\mathbb{R}} f(s+t)\, h(s)\, ds = \int_{\mathbb{R}} f(s)\, h(s-t)\, ds.$$

Now

$$|F(t) - F(t')| = \left| \int_{\mathbb{R}} f(s) \left[h(s-t) - h(s-t') \right] ds \right|$$
$$\leqslant \|f\|_X \, \|\tau(-t)h - \tau(-t')h\|_{X^*} \longrightarrow 0$$

as $t \to t'$ in $\mathbb{R}$, since $\tau(-t)h \to \tau(-t')h$ in $D(\mathbb{R})$ and hence in $X^*(\mathbb{R})$. Furthermore, F is bounded, since

$$|F(t)| \leqslant \|\tau(t)f\|_X \, \|h\|_{X^*} \leqslant C_X \, \|f\|_X \, \|h\|_{X^*}.$$

Moreover,

$$F^{(n)}(t) = \int_{\mathbb{R}} f^{(n)}(t+s)\, h(s)\, ds,$$

and so, similarly, $F^{(n)}$ is continuous and bounded, and

$$\|F^{(n)}\|_\infty \leqslant C_X \, \|f^{(n)}\|_X \, \|h\|_{X^*}.$$

Finally, since $h^{(k)}$ is in $D(\mathbb{R})$ and

$$F^{(k)}(t) = (-1)^k \int_{\mathbb{R}} f(s)\, h^{(k)}(s-t)\, ds,$$

$F^{(k)}$ is also bounded and continuous.

By Kolmogorov's inequality applied to F,

$$\|F^{(k)}\|_\infty \leqslant C_{k,n} \|F\|_\infty^{1-k/n} \|F^{(n)}\|_\infty^{k/n}$$
$$\leqslant C_X \, C_{k,n} \, \|f\|_X^{1-k/n} \, \|f^{(n)}\|_X^{k/n} \|h\|_{X^*}.$$

Since $X(\mathbb{R})$ is $D(\mathbb{R})$-full, by hypothesis, and

$$\left| \int_{\mathbb{R}} h(s)\, f^{(k)}(s)\, ds \right| \leqslant C_X \, C_{k,n} \|f\|_X^{1-k/n} \, \|f^{(n)}\|_X^{k/n} \, \|h\|_{X^*}$$

for all h in $D(\mathbb{R})$, it follows that $f^{(k)}$ is in $X(\mathbb{R})$, and

$$\|f^{(k)}\|_X \leqslant C_X \, C_{k,n} \, \|f\|_X^{1-k/n} \, \|f^{(n)}\|_X^{k/n},$$

as required. ∎

This theorem implies, for instance, Stein's theorem, except for $L^1(\mathbb{R})$. Similarly, it does not give a result for amalgams involving L^1. However, we have several extensions of this result which take care of these examples.

Let $X_u(\mathbb{R})$ denote the closed subspace of $X(\mathbb{R})$ of all functions f such that

$$\|\tau(t)f - f\|_X \to 0 \;\text{ as }\; t \to 0.$$

Corollary 2. *Suppose that $X(\mathbb{R})$ is a τ-stable $D(\mathbb{R})$-full Banach space of functions on $\mathbb{R}$. If f is in $X_u(\mathbb{R})$ and its generalised derivative $f^{(n)}$ is in $X(\mathbb{R})$, then $f^{(k)}$ is in $X_u(\mathbb{R})$ when $0 < k < n$.*

Proof: By the theorem,

$$\begin{aligned}\left\|\tau(t)f^{(k)} - f^{(k)}\right\|_X &\leqslant C_X\, C_{k,n}\left\|\tau(t)f - f\right\|_X^{1-k/n}\left\|\tau(t)f^{(n)} - f^{(n)}\right\|_X^{k/n}\\ &\leqslant C_X\, C_{k,n}\left\|\tau(t)f - f\right\|_X^{1-k/n}\left[C_X\left\|f^{(n)}\right\|_X + \left\|f^{(n)}\right\|_X\right]^{k/n}\\ &\longrightarrow 0,\end{aligned}$$

as $t \to 0$, so $f^{(k)}$ is in $X_u(\mathbb{R})$, as required. ∎

The second variant of the result involves another subspace of $X(\mathbb{R})$. Given a nonnegative integer m, we say that a Banach space $X(\mathbb{R})$ of functions on $\mathbb{R}$ is *stable under multiplication by* $C^m(\mathbb{R})$ if, whenever f is in $X(\mathbb{R})$ and φ, φ', ..., $\varphi^{(m)}$ are bounded and continuous on $\mathbb{R}$, the pointwise product φf is in $X(\mathbb{R})$ and

$$\left\|\varphi f\right\|_X \leqslant C_{X,m}\left\|f\right\|_X\left\|\varphi\right\|_{C^m} = C_{X,m}\left\|f\right\|_X\sum_{j=0}^{m}\left\|\varphi^{(j)}\right\|_\infty.$$

If $X(\mathbb{R})$ is stable under multiplication by $C^m(\mathbb{R})$ for some m, then we denote by $X_0(\mathbb{R})$ the closed subspace of $X(\mathbb{R})$ of all functions f for which

$$\lim_{\varepsilon\to 0+}\left\|\varphi_\varepsilon f - f\right\|_X = 0,$$

where $\varphi_\varepsilon(x) = e^{-\varepsilon x^2}$.

Corollary 3. *Suppose that $X(\mathbb{R})$ is a τ-stable $D(\mathbb{R})$-full Banach space of functions on $\mathbb{R}$, stable under multiplication by $C^m(\mathbb{R})$ for some nonnegative integer m. If f is in $X_0(\mathbb{R})$ and its generalised derivative $f^{(n)}$ is in $X(\mathbb{R})$, then $f^{(k)}$ is in $X_0(\mathbb{R})$ when $0 < k < n$.*

Proof: By the theorem, $f^{(j)}$ is in $X(\mathbb{R})$, when $0 \leqslant j \leqslant n$. By Leibniz's rule for the derivative of a product,

$$\begin{aligned}\left\|\varphi_\varepsilon f^{(k)} - f^{(k)}\right\|_X &= \left\|(\varphi_\varepsilon f)^{(k)} - f^{(k)} - \sum_{j=0}^{k-1}\binom{k}{j}\varphi_\varepsilon^{(k-j)}f^{(j)}\right\|_X\\ &\leqslant \left\|(\varphi_\varepsilon f - f)^{(k)}\right\|_X + \sum_{j=0}^{k-1}\binom{k}{j}\left\|\varphi_\varepsilon^{(k-j)}f^{(j)}\right\|_X.\end{aligned}$$

By the theorem,

$$\left\|(\varphi_\varepsilon f - f)^{(k)}\right\|_X \leqslant C_X\, C_{k,n}\left\|\varphi_\varepsilon f - f\right\|_X^{1-k/n}\left\|(\varphi_\varepsilon f - f)^{(n)}\right\|_X^{k/n} \longrightarrow 0$$

as $\varepsilon \to 0$, since $\|\varphi_\varepsilon f - f\|_X \to 0$ and $\|(\varphi_\varepsilon f - f)^{(n)}\|_X$ is bounded, by another application of Leibniz's rule. Further, when $0 \leqslant j < k$,

$$\|\varphi_\varepsilon^{(k-j)} f^{(j)}\|_X \leqslant C_{X,m} \|\varphi_\varepsilon^{(k-j)}\|_{C^m} \|f^{(j)}\|_X,$$

and it is easy to check that $\|\varphi_\varepsilon^{(k-j)}\|_{C^m} \to 0$ as $\varepsilon \to 0^+$. ∎

A third variant of the result combines the themes of the two previous corollaries. Assume that $X(\mathbb{R})$ is stable under multiplication by $C^m(\mathbb{R})$, and let $X_1(\mathbb{R})$ denote the subspace of $X(\mathbb{R})$ of all functions f such that $\varphi_\varepsilon f$ is in $X_u(\mathbb{R})$ for all ε in $\mathbb{R}^+$.

Corollary 4. *Suppose that $X(\mathbb{R})$ is a τ-stable $D(\mathbb{R})$-full Banach space of functions on $\mathbb{R}$, stable under multiplication by $C^m(\mathbb{R})$ for some integer m. If f is in $X_1(\mathbb{R})$ and its generalised derivative $f^{(n)}$ is in $X(\mathbb{R})$, then $f^{(k)}$ is in $X_1(\mathbb{R})$ when $0 < k < n$.*

Proof: This proof combines the ingredients of the proofs of the last two corollaries. We need to show that

$$\|\tau(t)(\varphi_\varepsilon f^{(k)}) - (\varphi_\varepsilon f^{(k)})\|_X \longrightarrow 0$$

as $t \to 0$, which we do by induction. Suppose that

$$\|\tau(t)(\varphi_\varepsilon f^{(j)}) - (\varphi_\varepsilon f^{(j)})\|_X \longrightarrow 0,$$

when $0 \leqslant j < k$. Observe that

$$\begin{aligned}
&\tau(t)(\varphi_\varepsilon f^{(k)}) - (\varphi_\varepsilon f^{(k)}) \\
&\quad = \tau(t)(\varphi_\varepsilon f)^{(k)} - (\varphi_\varepsilon f)^{(k)} - \sum_{j=0}^{k-1} \binom{k}{j} \left[\tau(t)(\varphi_\varepsilon^{(k-j)} f^{(j)}) - (\varphi_\varepsilon^{(k-j)} f^{(j)})\right].
\end{aligned}$$

Now

$$\begin{aligned}
&\|\tau(t)(\varphi_\varepsilon f)^{(k)} - (\varphi_\varepsilon f)^{(k)}\|_X \\
&\quad = \left\|\left(\tau(t)(\varphi_\varepsilon f) - (\varphi_\varepsilon f)\right)^{(k)}\right\|_X \\
&\quad \leqslant C_X\, C_{k,n} \|\tau(t)(\varphi_\varepsilon f) - (\varphi_\varepsilon f)\|_X^{1-k/n} \left\|\left(\tau(t)(\varphi_\varepsilon f) - (\varphi_\varepsilon f)\right)^{(n)}\right\|_X^{k/n} \\
&\quad \longrightarrow 0
\end{aligned}$$

as $t \to 0$, since $\|\tau(t)(\varphi_\varepsilon f) - (\varphi_\varepsilon f)\|_X \to 0$ while $\|(\tau(t)(\varphi_\varepsilon f) - (\varphi_\varepsilon f))^{(n)}\|_X$ is bounded as $t \to 0$, by the arguments of the previous corollaries. Further,

$$\begin{aligned}
&\|\tau(t)(\varphi_\varepsilon^{(k-j)} f^{(j)}) - (\varphi_\varepsilon^{(k-j)} f^{(j)})\|_X \\
&\quad \leqslant \left\|\left(\tau(t)\varphi_\varepsilon^{(k-j)} - \varphi_\varepsilon^{(k-j)}\right)\tau(t) f^{(j)}\right\|_X + \left\|\varphi_\varepsilon^{(k-j)}\left(\tau(t) f^{(j)} - f^{(j)}\right)\right\|_X \\
&\quad \leqslant C_{X,m} \|\tau(t)\varphi_\varepsilon^{(k-j)} - \varphi_\varepsilon^{(k-j)}\|_{C^m} \|\tau(t) f^{(j)}\|_X \\
&\qquad + C_{X,m} \|\varphi_\varepsilon^{(k-j)}(\varphi_{\varepsilon/2})^{-1}\|_{C^m} \|\varphi_{\varepsilon/2}(\tau(t) f^{(j)} - f^{(j)})\|_X \\
&\quad \longrightarrow 0
\end{aligned}$$

as $t \to 0$, by straightforward estimates of φ_ε and its derivatives, and the inductive hypothesis. $\square$

EXAMPLES. The Lorentz spaces $L^{p,q}(\mathbb{R})$ (where $1 < p < \infty$ and $1 \leqslant q \leqslant \infty$) are dual spaces except when $q = 1$, and are covered by the theorem. When $q = 1$, they are covered by Corollary 4.

The amalgams $\ell^q(L^p)$ are covered by the theorem if $p > 1$. If $q < \infty$ and $X = \ell^q(M)$, where M denotes the space of bounded complex measures, then $X_0 = \ell^q(L^1)$. If $X = \ell^\infty(M)$, then $X_1 = \ell^\infty(L^1)$.

REMARKS. It should be noted that the constant obtained in the theorem is, in general, not best possible. For example, it is easy to show that the best constant when $X = L^2$ is 1, by using the Fourier transform and Hölder's inequality. The point of the theorem is that there is a constant which works for all $D(\mathbb{R})$-full, τ-stable, Banach spaces for which the translation constant C_X of formula (1) is in a given range.

The hypothesis of Corollary 2 can be varied a little without changing the conclusion: more precisely, we may assume that f is in $X(\mathbb{R})$ and its generalised derivative $f^{(n)}$ is in $X_u(\mathbb{R})$. Similarly, the hypothesis of Corollary 3 can also be varied.

If we are interested in proving additive inequalities, that is, those of the form

$$\|f^{(k)}\|_X \leqslant C\left(\|f\|_X + \|f^{(n)}\|_X\right), \tag{2}$$

then more can be said. Indeed, the condition that X be τ-stable can be replaced by the condition that translations act continuously on X (that is, the map $(t, f) \mapsto \tau(t)f$ from $\mathbb{R} \times X$ to X is continuous, which implies that $\|\tau(t)f\|_X \leqslant \Omega(t)\ \|f\|_X$ for all f in X and t in $\mathbb{R}$, where $\Omega(t)$ grows at most exponentially as $|t|$ grows). By writing a function f as $\psi * f + (f - \psi * f)$, where ψ is a suitable test function, one sees that

$$\|f^{(k)}\|_X \leqslant \|\psi^{(k)} * f\|_X + \|(f - \psi * f)^{(k)}\|_X.$$

The first term on the right hand side can be controlled by a weighted L^1-norm of $\psi^{(k)}$ multiplied by $\|f\|_X$, and the second, after some integrations by parts, by a weighted L^1 norm of an $(n-k)$-fold integral of $\delta - \psi$ multiplied by $\|f^{(n)}\|_X$, where δ denotes the Dirac delta distribution. The conclusion at which one arrives is that the constant in the inequality (2) can be taken to depend only on k and n and the growth rate $\Omega(t)$. This result applies to spaces such as $L^p(\mathbb{R}, w)$, where the weight w does not vanish or grow too fast; the weights $w(x) = (1 + |x|)^\alpha$, where α is real, are examples of admissible weights.

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