GROUPS GENERATED BY TWO PARABOLIC LINEAR FRACTIONAL TRANSFORMATIONS

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0. Introduction and summary. We are interested in the structure of a group $\mathbf{G}$ of linear fractional transformations of the extended complex plane that is generated by two parabolic elements A and B, and, particularly, in the question of when such a group $\mathbf{G}$ is free. We shall, as usual, represent elements of $\mathbf{G}$ by matrices with determinant 1, which are determined up to change of sign. Two such groups $\mathbf{G}$ will be conjugate in the full linear fractional group, and hence isomorphic, provided they have, up to a change of sign, the same value of the invariant $\tau = \mathrm{Trace}(AB) - 2$. We put aside the trivial case that $\tau = 0$, where $\mathbf{G}$ is abelian. In the study of these groups, two normalizations have proved convenient. Sanov (**17**) and Brenner (**3**) took the generators in the form

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix},$$

while Chang, Jennings, and Ree (**4**) took them in the form

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

These parameters are connected by the relations $\tau = \mu^2 = 2\lambda$. We shall reserve the letters τ, μ, and λ for this meaning.

Sanov showed that $\mathbf{G}$ is free for $\mu = 2$, and characterized explicitly the matrices representing elements of $\mathbf{G}$. From the fact that $\mathbf{G}$ is free for any μ at all, it follows immediately that $\mathbf{G}$ is free for all transcendental μ. Brenner showed that $\mathbf{G}$ is free provided $|\mu| \geqq 2$. From this it follows immediately that $\mathbf{G}$ is free provided μ is an algebraic number with an algebraic conjugate μ^* such that $|\mu^*| \geqq 2$. In particular, algebraic numbers μ such that $\mathbf{G}$ is free are dense in the complex plane. Ree (**15**) has shown that the μ for which $\mathbf{G}$ is not free are dense in the circle $|\mu| < 1$, and in a domain in the plane containing the open intervals joining -2 to 2 and $-i\sqrt{2}$ to $i\sqrt{2}$. Hirsch (**7**) raised the question of which algebraic numbers μ, for example with $-2 < \mu < 2$, yield free groups $\mathbf{G}$. We do not yet know of any rational value of μ in this interval for which $\mathbf{G}$ is free. Brenner's sufficient condition for $\mathbf{G}$ to be free, that $|\mu| \geqq 2$, is equivalent to the condition $|\lambda| \geqq 2$. Chang, Jennings, and Ree improved this to the weaker condition that all of $|\lambda|$, $|\lambda - 1|$, and $|\lambda + 1|$ are at least 1.

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Let F be the set of values of λ for which $\mathbf{G}$ is free, and R the set of values for which $\mathbf{G}$ is not free. In Corollary 3 we show that $\lambda \in R$ implies that R is dense in some neighbourhood of λ. It then follows that there is a largest region, say F^*, contained in F. A region is the closure of an open set, and in this case F^* is the closure of the interior points of F. The complement of F^*, say R^*, is the smallest open region containing R. An open region is the interior of a region, and in this case, R^* is the interior of the closure of R.

In § 1 we further improve results aimed at determining F^*, in particular we improve on the result of Chang, Jennings, and Ree, weakening their condition in two independent directions.

In § 2 we examine values of λ which are not free, making several additions to known results. It should be remarked that the largest known region in F^* and the largest known open region in R^* do not exhaust the complex plane: a substantial part of the annulus $\frac{1}{2} \leqq |\lambda| < 2$ remains in doubt. The known results are summarized in a diagram which follows the statement of Theorem 4. Section 3 contains a few tentative observations about the structure of $\mathbf{G}$ for real λ.

1. Groups that are free. In proving their result, Chang, Jennings, and Ree used an instance of a classical argument, which Ford (**5**) called "the method of combination" and Fricke and Klein (**6**) called "the method of composition". Macbeath (**12**) formulated a general statement of this argument. After stating Macbeath's result in a slightly different formulation, which we have proved and used elsewhere (**11**), we restate and reprove the result of Chang, Jennings, and Ree.

THEOREM 1 (Macbeath). *Let $\mathbf{G}$ be a group of permutations of an infinite set Ω. Let $\mathbf{G}$ be generated by two of its subgroups $\mathbf{A}$ and $\mathbf{B}$, at least one of which has order greater than 2. Let Γ and Δ be disjoint non-empty subsets of Ω. Suppose now that $1 \neq A \in \mathbf{A}$ implies $A\Gamma \subseteq \Delta$ and $1 \neq B \in \mathbf{B}$ implies $B\Delta \subseteq \Gamma$. Then $\mathbf{G}$ is the free product of its subgroups $\mathbf{A}$ and $\mathbf{B}$.*

A proof is given in (**11**).

If S is any set in the extended complex plane, we write S^N for the interior of the complement of S. Note that if S is an open region (interior of its own closure), then S^N also is an open region, and $(S^N)^N = S$.

COROLLARY 1. *Let λ be any complex number, and let $\mathbf{G}$ be the group of linear fractional transformations of the complex plane Ω generated by the two elements*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad T = JBJ.$$

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Let Γ be an open region in Ω, neither empty nor all of Ω, and let $\Sigma = J\Gamma^N$. Now **G** *is a free group, freely generated by A and B, provided that*

(1) $A^k\Gamma \cap \Gamma = \emptyset$ *for all* $k \neq 0$,

(2) $T^k\Sigma \cap \Sigma = \emptyset$ *for all* $k \neq 0$.

To prove this, let $\Delta = \Gamma^N$. Then (2) is equivalent to (2'): $B^k\Delta \cap \Delta = \emptyset$ for all $k \neq 0$. Now (1) implies that, for $k \neq 0$, the open set $A^k\Gamma$ is contained in the complement $\tilde{\Gamma}$ of Γ, hence in the interior Δ of $\tilde{\Gamma}$. Similarly, (2') implies that $B^k\Delta \subset \Gamma$ for $k \neq 0$.

THEOREM 2 (Chang, Jennings, and Ree). *Let λ be a complex number lying in none of the open discs of radius* 1 *with centres* -1, 0, $+1$. *Then the group* **G** *generated by*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

is a free group, freely generated by A and B.

We proceed by making choices for the sets Γ, Δ, and Σ in Corollary 1. We choose for Γ the set $|\mathrm{Re}\, z| < 1$. The set Γ^N then consists of the set Δ_1, $\mathrm{Re}\, z > 1$, and Δ_2, $\mathrm{Re}\, z < -1$, so that $\Delta = \Delta_1 \cup \Delta_2$. The set $\Sigma = J\Delta = J\Delta_1 \cup J\Delta_2 = \Sigma_1 \cup \Sigma_2$, where Σ_1 is the disc $|z - \frac{1}{2}| < \frac{1}{2}$, and Σ_2 is the disc $|z + \frac{1}{2}| < \frac{1}{2}$. It is immediate that (1) is satisfied for this choice of Γ, and (3) will be satisfied if the discs Σ_1 and Σ_2 are disjoint from the translated discs $T^k\Sigma_1$ and $T^k\Sigma_2$ for all $k \neq 0$. Now all of these discs have radius $\frac{1}{2}$ and centres of the form $k\lambda \pm \frac{1}{2}$. The condition that all these centres are at least a distance one apart is precisely the hypothesis of the theorem.

THEOREM 3. *Let K be the convex hull of the set consisting of the circle $|z| = 1$ together with the two points $z = \pm 2$. If the complex number λ is not in the interior of K, then* **G**, *as above, is freely generated by A and B.*

Let Σ be, as before, the region bounded by the two circles C_1 and C_2 of radius $\frac{1}{2}$ with centres $\pm\frac{1}{2}$, and let λ satisfy the hypothesis of Theorem 2. Then we know that $T^k\Sigma \cap \Sigma = \emptyset$ for all $k \neq 0$. For arbitrary $u \neq 0$, let

$$U = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

and let

$$\Sigma^* = U\Sigma \quad \text{and} \quad T^* = UTU^{-1} = \begin{pmatrix} 1 & u\lambda \\ 0 & 1 \end{pmatrix}.$$

It follows that $T^{*k}\Sigma^* \cap \Sigma^* = \emptyset$ for all $k \neq 0$. Let $\Delta^* = J\Sigma^*$ and $\Gamma^* = (\tilde{\Delta}^*)_i$. The group G^* generated by A and

$$JT^*J = \begin{pmatrix} 1 & 0 \\ u\lambda & 1 \end{pmatrix}$$

will be free provided $A^k\Gamma^* \cap \Gamma^* = \emptyset$ for all $k \neq 0$.

Now Γ^* is an oblique strip bounded by the two parallel lines $L_1 = JUC_1$ and $L_2 = JUC_2$, symmetric in the origin. The desired condition will hold provided L_1 passes through the point $+1$ and hence L_2 passes through -1. If L is the line $\operatorname{Re} z = 1$, then $C_1 = JL$, and $L_1 = JUJL = VL$ for

$$V = JUJ = \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Now $+1$ on VL is equivalent to $V^{-1}(+1) = u$ lying on L, that is, to $\operatorname{Re} u = 1$.

This shows that if λ satisfies the hypothesis of Theorem 2, then $\mathbf{G}$ is free for all $\lambda_1 = u\lambda$, where u ranges over the line $\operatorname{Re} u = 1$; equivalently, $\mathbf{G}$ is free for all λ_1 lying on the line through λ that is perpendicular to the line from λ through the origin. With this, the conclusion of Theorem 3 follows immediately from that of Theorem 2.

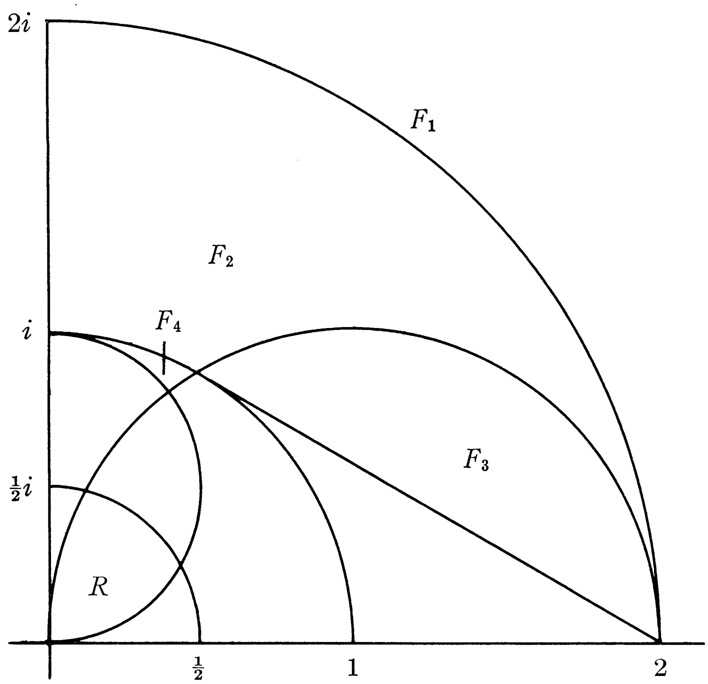
THEOREM 4. *Let λ be a complex number such that $|\lambda \pm \frac{1}{2}i| \geqq \frac{1}{2}$ and $|\lambda \pm 1| \geqq 1$. Then $\mathbf{G}$, defined as above, is freely generated by A and B.*

Before beginning the proof, we offer a comment and an illustrative figure. First, we remark that the set of λ for which $\mathbf{G}$ is free is symmetric with respect to reflection in both the real and the imaginary axes. It is immediate that if $\mathbf{G}$ is free for a complex number λ it is also free for the complex conjugate $\bar{\lambda}$ of λ. Furthermore, replacing λ by $-\lambda$ has the effect of replacing the generators A and B for $\mathbf{G}$ by the generators A and B^{-1}, and thus does not affect the question of whether $\mathbf{G}$ is free on the given generators. In view of this symmetry, we may confine our figure to the first quadrant.

In the figure, the region F_1 consists of those λ with $|\lambda| \geqq 2$, for which Brenner showed that $\mathbf{G}$ is free. The result of Chang, Jennings, and Ree, shows that $\mathbf{G}$ is free also for λ in the additional region F_2. Theorems 3 and 4 show that, further, $\mathbf{G}$ is free for λ in the additional regions F_3 and F_4. By way of contrast, the quarter disc R is an open set in which values of λ for which $\mathbf{G}$ is not free are dense. Among the radial spines emanating from R (which will be described in Theorem 5) along which values of λ for which $\mathbf{G}$ is not free are dense, are the segment of the real axis from 0 to $+2$ and the segment of the imaginary axis from 0 to $+i$.

The plan of the proof is as follows. In view of earlier remarks, we may suppose that λ lies inside the open curvilinear triangular region F_4. We start as in the proof of Theorem 2 by taking Γ to be the strip $|\operatorname{Re} z| < 1$ and Σ to be $J\Gamma^N$. For these choices, (1) holds but (2) fails, with Σ overlapping $T^{-1}\Sigma$ and $T\Sigma$. To restore (2) we replace Σ by a smaller set Σ^1. Now, however, $\Gamma^1 = J\Sigma^{1N}$ is larger than Γ, and (1) fails, with Γ^1 overlapping $A^{-1}\Gamma^1$ and $A\Gamma^1$. To restore (1) we replace Γ^1 by a smaller set Γ^2. Now $\Sigma^2 = J\Gamma^{2N}$ is larger than Σ^1, but we find that, nonetheless, (2) remains valid. The conclusion now follows from Corollary 1 applied to Γ^2 and Σ^2.

In the proof we assume that λ is in F_4. We also use the fact that both $|\lambda|$ and $\operatorname{Im} \lambda$ attain their minima on the closure $\bar{F}_4$ of F_4 at the lowest vertex, $(2 + 4i)/5$, whence we have $|\lambda| > 2/\sqrt{5}$ and $\operatorname{Im} \lambda > 4/5$.



To begin the proof we repeat that Γ, defined by $|\mathrm{Re}\, z| < 1$, satisfies condition (1). As in the proof of Theorem 2, $\Sigma = J\Gamma^N$ is the union of two discs Σ_1 and Σ_2 of radius $\frac{1}{2}$ and with centres at $-\frac{1}{2}$ and $+\frac{1}{2}$. Now $T^k\Sigma_1$ and Σ_2 have centres $-\frac{1}{2} + k\lambda$ and $+\frac{1}{2}$, and will be disjoint provided their centres are at a distance $|k\lambda - 1| \geqq 1$. But it is clear geometrically that the hypotheses $|\lambda - 1| > 1$ and $\mathrm{Re}\, \lambda > 0$ imply that $|k\lambda - 1| > 1$ for all $k \neq 0$. Similarly, $T^k\Sigma_i$ and Σ_i will be disjoint provided $|k\lambda| \geqq 1$. Since $|\lambda| > 2/\sqrt{5} > \frac{1}{2}$, this will be the case provided $|k| \geqq 2$. It follows that, if we form Σ^1 from Σ by deleting the closures of $\Sigma_1 \cap T^{-1}\Sigma_1$ and $\Sigma_2 \cap T\Sigma_2$, then Σ^1 will satisfy condition (2).

From the definition of Σ^1, it follows that $\Gamma^1 = J\Sigma^{1N}$ is the union of $\Gamma_1{}^1 = \Gamma_1 \cup JT^{-1}\Sigma_1$ and $\Gamma_1{}^2 = \Gamma_2 \cup JT\Sigma_2$. Examination shows that $JT\Sigma_2$ lies in the fourth quadrant, while $JT^{-1}\Sigma_1$ lies in the second quadrant. When we have shown, in the next paragraph, that the common diameter of these two discs is less than two, it will follow that $A^k\Gamma^1 \cap \Gamma^1 = \emptyset$ for $|k| \geqq 2$, and also that

$$\Gamma^1 \cap A\Gamma^1 = \Gamma^1 \cap AJT^{-1}\Sigma_1 = P_2 \text{ and } \Gamma^1 \cap A^{-1}\Gamma^1 = \Gamma^1 \cap A^{-1}JT\Sigma_2 = P_1.$$

The boundary of $T\Sigma_2$ is the circle C: $|z - v| = \frac{1}{2}$, where $v = \lambda + \frac{1}{2}$. Thus $JT\Sigma_1$ has boundary JC with equation $|1/z - v| = \frac{1}{2}$, or $|z| = 2|v| \cdot |z - 1/v|$. Generally, for $h > 0$, the circle $|z - a| = h|z - b|$ has diameter

$$d = \left| \frac{2h(a-b)}{h^2 - 1} \right| .$$

Thus JC has diameter

$$d = \left| \frac{4}{4|v^2| - 1} \right| .$$

Since $v = \lambda + \frac{1}{2}$ with $\operatorname{Re} \lambda > 0$, we have $|v| > \frac{1}{2}$, and the denominator is positive. Thus $d < 2$ is equivalent to $|v^2| > \frac{3}{4}$. Since $|v^2| = |\lambda|^2 + \operatorname{Re} \lambda + \frac{1}{4}$, it suffices that $|\lambda|^2 > \frac{1}{2}$, which follows from $|\lambda| > 2/\sqrt{5}$.

It is immediate from the foregoing that $\Gamma^2 = \Gamma^1 - (\bar{P}_1 \cup \bar{P}_2)$ will satisfy (1). To complete the proof, it remains to show that $\Sigma^2 = J\Gamma^{2N}$ satisfies condition (2). From the definition of Γ^2 it follows that $\Sigma^2 = \Sigma^1 \cup D_1 \cup D_2$, where $D_1 = JA^{-1}JT\Sigma_2$ and $D_2 = JAJT^{-1}\Sigma_1$. We know already that $T^k\Sigma^1 \cap \Sigma^1 = \emptyset$ for $k \neq 0$, whence it suffices to show that D_1 and D_2 are disjoint from $T^k\Sigma^2$ for $k \neq 0$. By symmetry, it is enough to show that D_1 is disjoint from $T^k\Sigma^2$ for $k \neq 0$. If $k < 0$, then $T^k\Sigma^2$ lies in the lower half plane, and is disjoint from D_1, which lies in the upper half plane. It remains to show D_1 disjoint from $T^k\Sigma^2$ for $k > 0$.

We have noted that D_1 lies in the upper half plane. Also, D_1 meets Σ_1, which lies entirely in the left half plane. Therefore, to show that D_1 lies entirely in the second quadrant, it suffices to show that the boundary of D_1 does not meet Y, the imaginary axis. If C is the boundary of $T\Sigma_2$, then D_1 has boundary $JA^{-1}JC$, and the condition $JA^{-1}JC \cap Y = \emptyset$ is equivalent to the condition $C \cap JAJY = \emptyset$. Now $JY = Y$, whence $AJY = AY$ is the line $\operatorname{Re} z = 2$, and $JAJY$ is the circle $|z - \frac{1}{4}| = \frac{1}{4}$. Since C is the circle with centre $\lambda - \frac{1}{2}$ and radius $\frac{1}{2}$, it suffices to show that the distance $|\lambda - \frac{1}{4}|$ between the centres exceeds $\frac{3}{4}$, the sum of the radii. For this it is easily verified that the minimum value of $|\lambda - \frac{1}{4}|$ on $\bar{F}_4$ is attained at the vertex $(2 + 4i)/5$, where it exceeds $\frac{3}{4}$.

We have shown that D_1 lies in the left half plane. By symmetry, D_2 lies in the right half plane. Since $\Sigma_2{}^1$, as a subset of Σ_2, lies in the right half plane, so does $\Sigma_2{}^2 = \Sigma_2{}^1 \cup D_2$, and with it $T^k\Sigma_2{}^2$ for all $k > 0$. Thus $D_1 \cap T^k\Sigma_2{}^2 = \emptyset$ for all $k > 0$.

We must show that $D_1 \cap T^k\Sigma_1{}^2 = \emptyset$ for all $k > 0$. Since $\Sigma_1{}^2 = \Sigma_1{}^1 \cup D_1$ and $\Sigma_1{}^1 \subset \Sigma_1$, this splits into showing that $D_1 \cap T^k\Sigma_1 = \emptyset$ and $D_1 \cap T^kD_1 = \emptyset$ for all $k > 0$. We begin by showing that $D_1 \cap T\Sigma_1 = \emptyset$. Let

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};$$

then $A = U^2$, $T\Sigma_2 = UT\Sigma_1$, and $D_1 = JU^{-2}JUT\Sigma_1$. Setting

$$W = JU^{-2}JU = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix},$$

we must show that $WT\Sigma_1 \cap T\Sigma_1 = \emptyset$. Now W is an involution with fixed points $p, p' = (-1 \pm i)/2$. Since $T\Sigma_1$ has centre $\lambda - \frac{1}{2}$ and radius $\frac{1}{2}$, the hypothesis that $|\lambda - \frac{1}{2}i| > \frac{1}{2}$ implies that p does not lie in $T\Sigma_1$. Let E be the disc with the same centre as $T\Sigma_1$ and with p on its boundary B; then it suffices to show that $WE \cap E = \emptyset$. Since W is a non-Euclidean half turn about p, B and WB are externally tangent at p, and it suffices to show that WE is the finite region bounded by WB, that is, that ∞ is not in WE. Since $\operatorname{Im} \lambda > \frac{1}{2}$, the centre $\lambda - \frac{1}{2}$ of E lies above the horizontal line through p, and therefore is nearer to p, on its boundary, than to $-\frac{1}{2}$. Thus $-\frac{1}{2}$ is not in E, and $W(-\frac{1}{2}) = \infty$ is not in WE.

To show that $D_1 \cap T^k\Sigma_1 = \emptyset$ for $k > 1$, it suffices to observe that $T\Sigma_1$ lies above the common tangent line H separating E from WE, and that, since $\operatorname{Im} \lambda > 0$, $T^k\Sigma_1$ will lie above H for all $k \geqq 1$, while $D_1 \subset WE$ will lie below H.

To complete the proof we must show that $D_1 \cap T^kD_1 = \emptyset$ for all $k > 0$. This amounts to showing that D_1 has diameter $d < |\lambda|$. From the fact that $D_1 = JA^{-1}JT\Sigma_2$ and our knowledge of $T\Sigma_2$, we conclude that the boundary of D_1 has an equation of the form $|z + \frac{1}{2}| = 2|\lambda| \cdot |z + \frac{1}{2} + \frac{1}{4}\lambda|$. A formula used earlier gives $d = 1/|4r^2 - 1|$, where $r = |\lambda|$. Since $r > \frac{1}{2}$, we have $d = 1/(4r^2 - 1)$, and the condition $d > r$ becomes $1 < 4r^3 - r$, or that $f(r) = 4r^3 - r - 1$ be positive. It is routine to check that this is true for all $r > 2/\sqrt{5}$. This completes the proof of Theorem 4.

2. Points where G is not free. We remark that, for $\mu \neq 0$, if **G** is not freely generated by

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix},$$

then **G** is not a free group. For it is a special case of a well-known result that if a free group is generated by two elements but not freely, then it has rank less than two, and hence is abelian. But, for $\mu \neq 0$,

$$AB = \begin{pmatrix} 1 + \mu^2 & \mu \\ \mu & 1 \end{pmatrix}$$

is not equal to

$$BA = \begin{pmatrix} 1 & \mu \\ \mu & 1 + \mu^2 \end{pmatrix}.$$

The following theorem is about groups **G** that are not free. This theorem and Corollary 3 are slight extensions of results of Ree (**15**).

THEOREM 5. *Let μ_0 be a complex number such that $\mu_0^{2^n} = -4$ for some positive integer n. Then values of μ for which the group* **G** *generated by*

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}$$

is not free are dense on the line segment joining μ_0 to the origin.

The proof consists in showing that, for a dense set of values of μ on the described segment, a certain commutator has finite order. Let T be any unimodular matrix and $T' = [A, T] = ATA^{-1}T^{-1}$. We write

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and calculate the entries of

$$T' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

in terms of those of T. Now,

$$AT = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + c\mu & b + d\mu \\ c & d \end{pmatrix}$$

and

$$A^{-1}T^{-1} = \begin{pmatrix} 1 & -\mu \\ 0 & 1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d + c\mu & -b - a\mu \\ -c & a \end{pmatrix},$$

whence, using $ad - bc = 1$,

$$T' = \begin{pmatrix} 1 + ac\mu + c^2\mu^2 & * \\ c^2\mu & 1 - ac\mu \end{pmatrix}.$$

We see, in particular, that $c' = c^2\mu$, whence $c'\mu = (c\mu)^2$, and that T' has trace $t' = 2 + c^2\mu^2 = 2 + c'$.

Define T_n recursively by taking

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_{n+1} = [A, T_n].$$

Note that, although T_0 need not belong to $\mathbf{G}$, we have $T_1 = AT_0A^{-1}T_0^{-1} = AB$, so that T_n is in $\mathbf{G}$ for all $n \geqq 1$. Let

$$T_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \quad \text{and} \quad t_n = \operatorname{Tr} T_n.$$

Now $c_0 = 1$ and $c_{n+1}\mu = (c_n\mu)^2$ implies that $c_n\mu = \mu^{2^n}$, whence $t_n = 2 + c_n\mu = 2 + \mu^{2^n}$.

Suppose that $\mu = r\mu_0$, where $\mu_0^{2^n} = -4$ and $0 \leqq r \leqq 1$. Then $t_n = 2 - 4r^{2^n}$. If $t_n = 2\cos\theta$, for θ a rational multiple of π, then T_n has finite order. Now numbers $t = 2\cos\theta$ for θ a rational multiple of π are dense in the interval $[-2, +2]$. Since the map carrying r into t_n is a homeomorphism from the interval $[0, 1]$ onto $[-2, 2]$, it follows that the values of r for which $\mathbf{G}$ is not free are dense in the interval $[0, 1]$. This completes the proof of the theorem.

We remark that Ree has shown that the two open segments joining -2 to 2 and $(1 + i)$ to $-(1 + i)$ arising in Theorem 5 are contained in a connected set that is open in the topology of the plane, and in which the values of μ for which $\mathbf{G}$ is not free are dense.

COROLLARY 2 (Ree). *Values of μ for which* **G** *is not free are dense in the unit disc.*

COROLLARY 3. *Let R' be the set of μ for which* **G** *is not free. If $\mu_0 \in R'$, then a neighbourhood of μ_0 lies in $\overline{R'}$.*

Proof. We treat μ as an indeterminate. If $\mu_0 \in R'$, then there is a matrix $\begin{pmatrix} * & * \\ p_{21}(\mu) & * \end{pmatrix}$ generated by A and B, such that $p_{21}(\mu)$ is not the zero polynomial, and that $p_{21}(\mu_0) = 0$. Using this matrix for T_0 in the proof of Theorem 5, we find that $t_n = 2 + (\mu p_{21}(\mu))^{2^n}$. The set where $|\mu p_{21}(\mu)| < 1$, say D, is an open set containing μ_0. If $\mu' \in D$, then in every neighbourhood of μ' there is some value, say μ'', and a value of n, such that $t_n = 2\cos\theta$ for θ a rational multiple of π. Thus T_n has finite order, and $\mu'' \in R'$. This completes the proof.

Remark. For the extreme values of μ, satisfying $\mu^{2^n} = -4$, **G** is free provided $n = 1$ or $n = 2$, but **G** need not be free for such μ if $n = 3$.

If $n = 1$, then $\mu^2 = -4$, whence $2\lambda = \mu^2 = -4$, and $\lambda = -2$. If $n = 2$, then $\mu^4 = -4$, $\mu^2 = \pm 2i$, $2\lambda = \mu^2 = \pm 2i$, and $\lambda = \pm i$. That **G** is free for such μ follows from Theorem 1.

Let $n = 3$. Four of the roots of $\mu^8 = -4$, including that with smallest angle, satisfy the equation $\mu^4 - 2\mu^2 + 2 = 0$. We show that **G** is not free for μ satisfying this equation. More explicitly, we note that

$$T_3 = [A, [A, B]] = A(ABA^{-1}B^{-1})A^{-1}(BAB^{-1}A^{-1})$$

is conjugate to $S = [A^{-1}, B][A, B]$. Write

$$A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 1+\alpha\beta & \alpha \\ \beta & 1 \end{pmatrix}, \qquad A^{-1}B^{-1} = \begin{pmatrix} 1+\alpha\beta & -\alpha \\ -\beta & 1 \end{pmatrix},$$

and

$$[A, B] = \begin{pmatrix} 1+\alpha\beta+\alpha^2\beta^2 & -\alpha^2\beta \\ \alpha\beta^2 & 1-\alpha\beta \end{pmatrix}.$$

Now

$$S = [A^{-1}, B][A, B] = \begin{pmatrix} 1-\mu^2+\mu^4 & -\mu^3 \\ * & * \end{pmatrix}\begin{pmatrix} * & -\mu^3 \\ * & 1-\mu^2 \end{pmatrix} = \begin{pmatrix} * & b \\ * & * \end{pmatrix},$$

where $b = -\mu^3(2 - 2\mu^2 + \mu^4) = 0$. That **G** is not free now follows from the fact that the affine group stabilizing a point in $\mathrm{PSL}_2(\mathbf{C})$ is metabelian. More explicitly, computation shows that

$$B' = SBS^{-1} = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}$$

for some ν, whence B' commutes with B.

We remark that the solution of $\mu^8 = -4$ with smallest angle yields $\lambda = \frac{1}{2}(1 + i)$, well away from the free regions discussed in § 1.

The method used above can be given a more general formulation; this raises some interesting questions, but has not enabled us to gain much additional information. For this, we again view μ as an indeterminate. Then any word $W(\mu) = A^{a_1}B^{b_1} \ldots A^{a_n}B^{b_n}$ is given by a matrix whose entries are polynomials in μ with coefficients depending on $a_1, b_1, \ldots, a_n, b_n$. Write

$$W(\mu) = \begin{pmatrix} 1 + p_{11}(\mu) & p_{12}(\mu) \\ p_{21}(\mu) & 1 + p_{22}(\mu) \end{pmatrix}.$$

It is easy to see, for example writing $A = I + \mu E_{12}$ and $B = I + \mu E_{21}$, that $p_{12}(\mu) = c_1\mu + c_3\mu^3 + \ldots + c_{2n-1}\mu^{2n-1}$, an odd polynomial, where each coefficient c_{2k+1} is the sum of all products $a_{i_1}b_{j_1} \ldots a_{i_k}b_{j_k}a_{i_{k+1}}$ where the factors are, in order, a subsequence of the sequence $a_1, b_1, \ldots, a_n, b_n$. We note also that if W is a non-trivial reduced word, that is, if $n > 0$ and all $a_i, b_j \neq 0$, then p_{12} is a non-constant polynomial with leading coefficient

$$c_{2n-1} = a_1 b_1 \ldots b_{n-1} a_n.$$

The remaining p_{ij} are analogous.

Proposition 1. *A complex number μ determines a group* $\mathbf{G}$ *that is not free if and only if μ is a root of some polynomial $p(x)$ determined by a sequence $a_1, b_1, \ldots, a_n, b_n$ of non-zero integers in the following manner:*

$$p(x) = \sum c_{2k+1} x^{2k+1},$$

odd, where each $c_{2k+1} = \sum a_{i_1}b_{j_1} \ldots a_{i_k}b_{j_k}a_{i_{k+1}}$, sum over all

$$i_1 \leqq j_1 < i_2 \leqq j_2 < \ldots \leqq j_k < i_{k+1}.$$

If $\mathbf{G}$ is not free, then for some non-trivial reduced W we have $W(\mu) = I$, and hence $p_{12}(\mu) = 0$. Conversely, if $p_{12}(\mu) = 0$ for some non-trivial p_{12}, hence for some W that is not a power of B, we have

$$W(\mu) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

and, as before, $B' = WBW^{-1}$ commutes with B.

We next apply the method indicated above to find a few more points for which $\mathbf{G}$ is not free.

Proposition 2. $\mathbf{G}$ *is not free if $\mu = \sqrt{(2/n)}$ or $\mu = \sqrt[4]{(2/n)}$ for some positive integer n.*

To see this, note that if the trace $t(\mu)$ of $W = A^{a_1}B^{b_1} \ldots A^{a_n}B^{b_n}$ vanishes, then $W^2 = 1$ and (assuming always W reduced and non-trivial) $\mathbf{G}$ is not free. If $n = 1$, then $t(\mu) = 2 + a_1b_1\mu^2$ with roots $\mu = \pm \sqrt{(2/a_1b_1)}$. If $n = 2$, then $t(\mu) = 2 + k\mu^2 + c\mu^4$, where $c = a_1a_2b_1b_2$ and $k = (a_1 + a_2)(b_1 + b_2)$. To make $k = 0$ take $a_2 = -a_1$; then $t(\mu)$ has as roots the fourth roots of $2/a_1{}^2b_2b_3$.

The first half of the above proposition is contained in a more general result.

PROPOSITION 3. *If μ determines a group* **G** *that is not free, then so does $\mu' = \mu\sqrt{(2/n)}$ for every non-zero integer n.*

This is equivalent to the assertion that if λ determines a group **G** that is not free, then so does $\lambda' = \lambda/n$. But this follows from the fact that any relation between

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B' = B^n = \begin{pmatrix} 1 & 0 \\ n\lambda & 1 \end{pmatrix}$$

is *a fortiori* a relation between A and B.

Finally we explore the case, for $n = 3$, where W is periodic.

PROPOSITION 4. *If $\omega^6 = +1$, then points where $W = A^{-1}B^{-2}A^{-1}BA^2B$ has finite order are dense on the line segment from ω to the origin. If $\omega^6 = -1$, the same is true for $W = AB^{-2}ABA^{-2}B$.*

Let $W = A^{a_1}B^{b_1} \ldots B^{b_3}$. To make the coefficient of μ^2 in $t(\mu)$ vanish, we take $a_1 + a_2 + a_3 = 0$. Now calculation shows that $t(\mu) = 2 + k\mu^4 + c\mu^6$ with $k = -a_2{}^2b_1b_2 - a_1{}^2b_1b_3 - a_3{}^2b_2b_3$ and $c = \prod a_i \prod b_i$. To make $k = 0$, we choose $a_1, a_2, b_2, b_3 = 1$ and $a_3 = b_1 = -2$. Then $t(\mu) = 2 + 4\mu^6$, and $t(\mu)$ ranges from $+2$ to -2 as μ^6 ranges from 0 to -1. Changing the signs of the a_i gives similarly $t(\mu) = 2 - 4\mu^6$.

3. Structure of groups G for real μ. From (**3**) we know that **G** is free for $|\mu| \geqq 2$. As noted, it follows that **G** is free if μ is transcendental or is algebraic with an algebraic conjugate μ' such that $|\mu'| \geqq 2$. We know of no real μ not falling under one of the above heads for which **G** is free. It is well known (see **9**) that the only discontinuous **G** for $-2 < \mu < +2$ are essentially the Hecke groups, which, although not free, are free products of cyclic groups. (From this it follows that no **G**, for $-2 < \mu < +2$ can be shown free by Macbeath's method, using for Γ and Δ regions in the plane under its natural topology.) After examining the groups just mentioned, we find a few rational values of μ, with $|\mu| < 2$, for which we can show that **G** is not free; but we have not been able to settle even the case that $\mu = 7/4$.

For certain values of μ, the group **G** generated by

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix},$$

although not free, is a free product of cyclic groups. It is well known that, for $\mu = 1$, the two transvections A and B generate the unimodular group, $\mathbf{G} = \mathrm{PSL}_2(\mathbf{Z})$, and that this group is the free product of the cyclic groups generated by the two elements

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

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of order 2 and

$$S = AJ = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

of order 3. For other values of μ, the element J need not belong to $\mathbf{G}$, and it is profitable to study the larger group $\mathbf{H}$ obtained by adjoining J to $\mathbf{G}$. Since $B = JA^{-1}J$, the group $\mathbf{H}$ is generated by

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Among these groups, for μ real, the only groups acting discontinuously (on the upper half plane) are known to be the Hecke groups, where $\mu = 2\cos\theta$, θ a rational multiple of π, together with those for $|\mu| \geqq 2$.

A standard proof that $\mathbf{H}$ is discontinuous also reveals the structure of $\mathbf{H}$ as a free product; for example, one can recognize $\mathbf{H}$ as the conformal subgroup of the Coxeter group generated by reflections in the sides of the hyperbolic triangle with vertices ∞, i, and $e^{2\pi i/q}$. We indicate here a proof based on Macbeath's lemma.

THEOREM 6. *Let* $\mathbf{H}$ *be the group of linear fractional transformations generated by*

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\mu = 2\cos(p/q)\pi$ *for* p *and* q *relatively prime integers,* $q > 2$. *Then* $\mathbf{H}$ *is the free product of the two cyclic groups generated by* J, *of order* 2, *and the other by*

$$S = AJ = \begin{pmatrix} \mu & -1 \\ 1 & 0 \end{pmatrix},$$

of order q.

First, it is clear that J and S generate $\mathbf{H}$. Now $\mu = \zeta + \zeta^{-1}$, where ζ is a primitive qth root of unity. Since all primitive qth roots of unity are algebraic conjugates, $\mathbf{H}$ will be isomorphic to the group obtained by replacing ζ by the special choice $\zeta = e^{\pi i/q}$. Now S, as well as J, satisfies the special hypothesis (of having minimal angle of rotation) of the theorem in (**11**, § 5). Moreover, $SJ = A$ has the real fixed point ∞, whence it follows by that theorem that $\mathbf{H}$ is the free product of the groups generated by J and by S.

THEOREM 7. *Let* $\mathbf{G}$ *be the group generated by*

$$A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix},$$

for μ *as above. If* q *is odd, then* $\mathbf{G}$ *coincides with* $\mathbf{H}$, *as above. If* $q = 2n$, *even, then* $\mathbf{G}$ *is a proper subgroup of* $\mathbf{H}$, *and is the free product of two cyclic groups generated by* A, *of infinite order, and by* $AB^{-1} = S^2$, *of order* n.

We verify that $A = SJ$, $B = S^{-1}J$, and $AB^{-1} = S^2$. If S has odd order, it is contained in the group generated by S^2, whence S is in $\mathbf{G}$ and also $J = SB$ is in $\mathbf{G}$, and it follows that $\mathbf{G} = \mathbf{H}$. Suppose now that $q = 2n$. Then $(AB^{-1})^n = 1$, and it remains to show that A and B satisfy no relation that is not a consequence of this one.

For this it will suffice to show that if $W = A^{a_1}B^{b_1}\ldots B^{b_t}$ is a non-trivial reduced word ($t > 0$ and all $a_i, b_i \neq 0$) that cannot be shortened by use of the relation $(AB^{-1})^n = 1$, then $W \neq 1$. We suppose then that $W = 1$, and yet that W does not contain any part $(AB^{-1})^n$, $(B^{-1}A)^n$, $(BA^{-1})^n$, or $(A^{-1}B)^n$; from this assumption we shall draw a contradiction. Form W_1 from W by making the literal substitutions: $A \to SJ$, $A^{-1} \to JS^{-1}$, $B \to S^{-1}J$, $B^{-1} \to JS$, and without making any cancellations in the resulting word. It is immediate that W_1 will contain no part JJJ, and, from the fact that W was reduced, that W_1 will contain no part SS^{-1}, $S^{-1}S$, $SJJS^{-1}$ or $S^{-1}JJS$. Let W_2 be obtained from W_1 by deleting all parts JJ. Then W_2 will contain no part JJ, SS^{-1}, or $S^{-1}S$. If W_2 contains no part S^q or S^{-q}, then W_2 is the normal form for a non-trivial element in the free product $\mathbf{H}$, contrary to our assumption that $W = 1$. We may suppose then that W_2 contains such a part, and, by symmetry, that W_2 contains a part S^q. Inspection shows that a sequence of q consecutive etters S in W_2 can arise only from a part of W_1 of the form

$$(JS)(SJ)(JS)\ldots(JS)(SJ),$$

and hence from a part $(B^{-1}A)^n$ of W. But this contradicts our assumption on W, completing the proof of the theorem.

Let $\mathbf{G}(\mu)$ be generated by A and B, for $\mu = 2\cos(p/q)\pi$, $(p, q) = 1$, $q \geqq 3$. From our knowledge of $\mathbf{G}(\mu)$, as a free product, it is easy to see that, if $hk > 1$ (except $hk = 2$ and $q = 3$), A^h and B^k generate a free subgroup $\mathbf{G}'$ of $\mathbf{G}(\mu)$, conjugate to $\mathbf{G}(\mu')$, where $\mu' = \sqrt{(hk)}\,\mu$. However, this gives us no new information, since μ' has a conjugate $\mu_0' = \sqrt{(hk)}\,2\cos(\pi/q)$ with $|\mu_0'| \geqq 2$.

In considering the groups $\mathbf{G}$ defined by μ as above, it suffices to study those with $\mu = 2\cos(\pi/q)$, $q > 0$. We have excluded the case $q = 1$, which gives Sanov's group, $\mu = 2$. We have excluded also the case $q = 2$, where $\mu = 0$ and $\mathbf{G}$ is trivial. For $q = 3$ we have $\mu = 1$, and $\mathbf{G}$ is the unimodular group. For $q = 4$, we have $\mu = \sqrt{2}$; it is easy to obtain an explicit description of this group.

Proposition 5. *The group* $\mathbf{G}$ *generated by*

$$A = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}$$

consists of all

$$T = \begin{pmatrix} a & b\sqrt{2} \\ c\sqrt{2} & d \end{pmatrix},$$

for a, b, c, *and* d *integers satisfying* $ad - 2bc = 1$.

That every element T of $\mathbf{G}$ has this form follows from the fact that the diagonal elements in $W(\mu)$ are even polynomials in μ, while the off-diagonal elements are odd polynomials. To show that every such T belongs to $\mathbf{G}$, we first verify that, if $a, c \neq 0$, then, by a modified division algorithm, $|a| + |c|$ will be diminished in passing to one of $A^{\pm 1}T$ or $B^{\pm 1}T$. It remains to consider the case that $a = 0$ or $c = 0$. However, $a = 0$ contradicts $ad - 2bc = 1$. If $c = 0$, then $ad = 1$; we may suppose that $a = d = 1$, and

$$T = \begin{pmatrix} 1 & b\sqrt{2} \\ 0 & 1 \end{pmatrix} = A^b,$$

in $\mathbf{G}$.

The next case, $q = 5$, has been studied from a somewhat different point of view by Rosen (**16**) and Leutbecher (**10**).

We conclude with some fragmentary remarks about the structure of $\mathbf{G}$ for rational μ. These groups, for $\mu = 1/p$, p a prime, have been studied by Ihara (**8**), and by Mennicke (**13**), who showed that $\mathbf{G}$ is the full unimodular group over the ring R_p of rationals with denominator a power of p. There is some reason to suppose that, for general μ, the polynomials $p_{12}{}^W(\mu)$ associated with words W will not tend to have common roots without good reason, for example unless W_1 and W_2 are in the normal closure of a third word W_3. One might expect then that the groups under consideration if not free might be free products of groups of a simple nature, perhaps with amalgamation. Since the present paper was submitted, the paper of Behr (**1**) has come to our attention, in which it is proved, *inter alia*, that $\mathbf{G}$, for $\mu = 1/p$, p a prime, is finitely presented, as well as the paper of Behr and Mennicke (**2**) in which a finite presentation is obtained. However, we have not been able to derive from their presentation any insight into the structure of $\mathbf{G}$.

The group $\mathbf{G}$ for $\mu = 1/p$ of course contains the unimodular group, and therefore satisfies the two defining relations for the unimodular group when expressed in terms of the generators A and B of $\mathbf{G}$. In the case $\mu = \frac{1}{2}$, we have shown that $\mathbf{G}$ satisfies other relations that are not consequences of these two. The argument consists in first observing that if $\mathbf{G}$ were defined by these two relations alone, it would be the result of adjoining a square root A (in the sense of Neumann (**14**)) to a certain element of the unimodular group $\mathbf{U}$, and, in that case, all involutions of $\mathbf{G}$ would be conjugates of those in $\mathbf{U}$. However, by calculation, we find an involution of the form AX, with X in $\mathbf{U}$, which cannot possibly be a conjugate of any element of $\mathbf{U}$.

In this connection we remark that we have shown (with H. Stark (oral communication) supplying the proof of an elementary lemma from number theory) that for $\mu = 1/p$, and p prime, the group $\mathbf{G}$ has exactly two conjugacy classes of involutions if p is of the form $p = 4k + 3$, and exactly one conjugacy class otherwise.

Any relation among the generators of $\mathbf{G}$ for a given value of μ carries with it a relation for μ/n, n an arbitrary integer. Thus, in seeking such relations, we are led to examine values $1 < \mu < 2$. Let $\mu = 3/2$, and define $W_0 = I$, $W_1 = A^{a_1}W_0$, $W_2 = B^{b_1}W_1$, $W_3 = A^{a_2}W_2$, and so forth, for non-zero integers $a_1, b_1, a_2, \ldots$. We seek to choose these integers in such a way that

$$W_n = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$$

for some $n > 0$. It will then follow as before that $B' = W_n^{-1}BW_n$ commutes with B, and hence that $\mathbf{G}$ is not free.

Examination of the second columns of the W_i shows that this reduces to the following problem. Given μ, to find integers $h_1, h_2, \ldots$ such that the recursive definition

$$x_0 = 0, \quad x_1 = 1, \quad x_{n+2} = x_n + h_{n+1}\mu x_{n+1}$$

leads to some $x_n = 0$, $n > 0$. The method of choosing the h_i always to minimize $|x_{n+2}|$ works for $\mu = 3/2, 4/3, 5/3, 5/4$, but does not seem to work for $\mu = 7/4$. We summarize our results for these cases.

PROPOSITION 6. *If $\mu = 3/2, 4/3, 5/3, 5/4$, then $\mathbf{G}$ is not free. Indeed, if we let $B' = W^{-1}BW$, then for each value of μ above there is an appropriate choice of W such that the corresponding B' will commute with B, thus yielding a relationship. For $\mu = 3/2$ the choice of $W = A^2B^{-2}AB^{-1}A$ yields $B' = B^{2^6}$; for $\mu = 3/4$, the choice of $W = A^{-18}B^2AB^{-1}A$ yields $B' = B^{3^8}$; for $\mu = 5/3$ the choice of $W = A^{18}B^{-2}ABA^2B^{11}A^2B^{-1}AB^{-1}A$ yields $B' = B^{2^{20}}$; for $\mu = 5/4$ the choice of $W = A^{-8}B^2A^5B^{-3}A^2B^{-1}A$ yields $B' = B^{4^{10}}$.*

We note also that, for $\mu = 3/2$, $\mathbf{G}$ is contained in the principal congruence group $\mathbf{U}(3)$ over R_2, and contains $\begin{pmatrix} 8 & 0 \\ 0 & 1/8 \end{pmatrix}$; but it does not appear to coincide with $\mathbf{U}(3)$. We repeat that $\mu = 7/4$ is the simplest case of rational μ where we cannot decide whether $\mathbf{G}$ is free.

We state a final result concerning rational values of μ for which $\mathbf{G}$ is not free.

PROPOSITION 7. *If $\mu = p/(p^2 + 1)$, p a positive integer, then $A^{h_3}B^{h_2}A = W$, with $h_2 = -(p^2 + 2)$, $h_3 = -(p^2 + 1)^2$, has the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Thus $\mathbf{G}$ is not free for any of these values of μ.*

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