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NOTE ON THE THREE-DIMENSIONAL LOG CANONICAL ABUNDANCE IN CHARACTERISTIC > 3

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Abstract. In this paper, we prove the nonvanishing and some special cases of the abundance for log canonical threefold pairs over an algebraically closed field k of characteristic $p > 3$. More precisely, we prove that if (X,B) be a projective log canonical threefold pair over k and $K_X + B$ is pseudo-effective, then $\kappa(K_X + B) \geq 0$, and if $K_X + B$ is nef and $\kappa(K_X + B) \geq 1$, then $K_X + B$ is semi-ample.

As applications, we show that the log canonical rings of projective log canonical threefold pairs over k are finitely generated and the abundance holds when the nef dimension $n(K_X + B) \leq 2$ or when the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is nontrivial. Moreover, we prove that the abundance for klt threefold pairs over k implies the abundance for log canonical threefold pairs over k.

§1. Introduction

Over the last decade, the minimal model program (MMP) for threefolds over a field of characteristic > 3 has been largely established. First, Hacon and Xu proved the existence of minimal models for terminal threefolds over an algebraically closed field k of characteristic > 5 (see [13]). Then Cascini, Tanaka, and Xu proved that arbitrary terminal threefold over k is birational to either a minimal model or a Mori fiber space (see [6]). Base on it, Birkar and Waldron established the MMP for klt threefolds over k (see [4], [5]). Moreover, there are some generalizations of it in various directions. For example, see [15], [26] for its generalization to log canonical (lc) pairs, [10]–[12] for its generalization to low characteristics, [9] for its generalization to imperfect base fields, and [3] for its analog in mixed characteristics.

Now we can run MMPs for lc threefold pairs over a perfect field of characteristic > 3 (see Theorem 2.12). Hence, a central problem remaining is the following conjecture.

1.1 Abundance conjecture

Let (X,B) be a projective lc threefold pair over a perfect field k of characteristic > 3. If $K_X + B$ is nef, then it is semi-ample.

Remark 1.1. The abundance conjecture for lc surface pairs over any field of positive characteristic is proved in [24], and for slc surface pairs over any field of positive characteristic, it is proved in [22].

Remark 1.2 (From a perfect field to its algebraic closure). Many properties of singularities and positivity, for example, klt, lc, semi-ampleness, and Iitaka dimensions, are preserved under the base change from a perfect field to its algebraic closure (see, for example, [10, Rem. 2.7]). In this paper, we sometimes do such base changes and assume that we work over algebraically closed fields. However, some conditions need that the base

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field is algebraically closed, for example, conditions about nef dimensions (see Section 2.8 for definition) and Albanese maps.

When K_X+B is big, Birkar and Waldron proved it in characteristic >5 (see [5], [26]), then Hacon and Witaszek proved it in characteristic 5 (see [12]). When (X,B) is klt and the characteristic of k is greater than 5, Waldron proved it in the case of $\kappa(X,K_X+B)=2$ (see [25]), Das, Waldron, and Zhang proved it in the case of $\kappa(X,K_X+B)=1$ (see [8], [29]), Witaszek proved it in the case when the nef dimension $n(X,K_X+B)\leq 2$ (see [27]), and Zhang proved it in the case when the Albanese map $a_X: X\to \mathrm{Alb}(X)$ is nontrivial (see [30]). In conclusion, the abundance holds when (X,B) is klt, the characteristic of k is greater than 5 and one of the following conditions holds:

(1) $\kappa(X,K_X+B)\geq 1$,
(2) the nef dimension $n(X,K_X+B)\leq 2$,
(3) the Albanese map $a_X: X\to \mathrm{Alb}(X)$ is nontrivial.

The above works on the abundance for klt pairs in characteristic >5 can be generalized to the case when the characteristic is greater than 3 by some careful modifications (see Section 3). Then it is natural to ask the following question.

Question 1.3. How can we generalize a result on the abundance for klt threefold pairs to lc threefold pairs?

In characteristic 0, this is done in [18]. However, the approach there needs vanishing theorems and the termination of flips for threefolds. The vanishing theorems may fail in positive characteristic and the termination of flips for threefolds is unknown in positive characteristic for lack of a good understanding of terminal threefold singularities in positive characteristic. In this paper, we propose a new method to solve Question 1.1 and generalize most of results on the abundance for klt pairs in characteristic >5 to lc pairs in characteristic >3. We first prove the nonvanishing theorem for lc threefold pairs over a perfect field k of characteristic >3.

Theorem 1.4 (Theorem 4.4). *Let (X,B) be a projective lc threefold pair over a perfect field k of characteristic >3. If K_X+B is pseudo-effective, then $\kappa(X,K_X+B)\geq 0$.*

As a corollary, we have the following result on termination of flips.

Theorem 1.5 (Theorem 4.5). *Let (X,B) be a projective lc threefold pair defined over a perfect field k of characteristic $p>3$ such that K_X+B is pseudo-effective. Then every sequence of (K_X+B)-flips terminates. In particular, any (K_X+B)-MMP terminates with a minimal model.*

Secondly, we prove the following result which is the main technical result of this paper.

Theorem 1.6 (Theorem 5.1). *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is nef and $\kappa(X,K_X+B)\geq 1$, then K_X+B is semi-ample.*

Combined with the results on klt pairs, we deduce the following statements.

THEOREM 1.7 (Theorem 6.1). *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. Then the log canonical ring*

$$R(K_X+B)=\oplus_{m=0}^{\infty}H^0(\lfloor m(K_X+B)\rfloor)$$

is finitely generated.

THEOREM 1.8 (Theorem 6.2). *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is nef and the nef dimension $n(X,K_X+B)\leq 2$, then K_X+B is semi-ample.*

THEOREM 1.9 (Theorem 6.3). *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is nef and* dim Alb$(X)\neq 0$, *then K_X+B is semi-ample.*

It turns out that the following result follows from Theorems 4.4 and 1.6.

THEOREM 1.10 (Theorem 6.4). *Let k be an algebraically closed field of characteristic >3. Assume we have:*

(1) *abundance for terminal threefolds over k holds, and*
(2) *any effective nef divisor D on any klt Calabi–Yau threefold pair (Y,Δ) ((Y,Δ) is klt and $K_Y+\Delta\sim_{\mathbb{Q}}0$) over k is semi-ample.*

Then the abundance conjecture for threefold pairs over k holds. In particular, the abundance conjecture for klt threefold pairs over k implies the abundance conjecture for lc threefold pairs over k.

1.2 Outline of the proof of Theorem 1.6

For simplicity, we assume that k is an uncountable algebraically closed field of characteristic >3 (the uncountability is used for defining the nef reduction map). We first prove the nonvanishing theorem for projective lc threefold pairs over k (see Theorem 4.4) as follows. By Theorem 2.20, after replacing, we can assume that (X,B) is $\mathbb{Q}$-factorial and dlt, and moreover, X is terminal. Then we run a K_X-MMP which is (K_X+B)-trivial by Definition 2.16. It terminates by Lemma 2.19. If we get a minimal model, then we can use the nonvanishing for klt pairs (see Theorem 3.10) to prove the assertion. Otherwise, we get a Mori fiber space. It implies that the nef dimension $n(K_X+B)\leq 2$. We can use Witaszek's weak canonical bundle formula to handle the case of $n(K_X+B)=2$. The case of $n(K_X+B)=1$ is trivial by descending K_X+B along the nef reduction map of K_X+B. Finally, we need to handle the case of $n(K_X+B)=0$. In this case, K_X+B is numerically trivial. Then the semi-ampleness of K_X+B preserves under any step of MMPs. By Theorem 2.12, we can run a $(K_X+B-\lfloor B\rfloor)$-MMP which terminates. It terminates with a Mori fiber space and then we can descend K_X+B along the Mori fiber space to prove its semi-ampleness. In conclusion, the nonvanishing holds. As a corollary, we have the termination of flips for pseudo-effective lc threefold pairs over k (see Theorem 4.5).

Now, let (X,B) be a projective lc threefold pair over k such that K_X+B is nef. We assume $\kappa(K_X+B)=2$, which is the most difficult case. Then K_X+B is endowed with a map $h:X\to Z$ to a normal proper algebraic space of dimension 2 by Lemma 5.3.

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We replace (X,B) by a $\mathbb{Q}$-factorial dlt modification by Theorem 2.20. Then one of the following cases holds:

Case I: $K_X+B-\varepsilon\lfloor B\rfloor$ is not pseudo-effective for any rational $\varepsilon>0$,

Case II: $K_X+B-\varepsilon\lfloor B\rfloor$ is pseudo-effective for any sufficiently small rational $\varepsilon>0$.

In Case I, we first prove that $\lfloor B\rfloor$ must dominate Z (see Proposition 5.5). Then we deduce the semi-ampleness of K_X+B by adjunction (see Proposition 5.6).

In Case II, we first modify the pair (X,B) by running several MMP which are (K_X+B)-trivial (see Definition 2.16) so that all h-exceptional prime divisors are connected components of $\lfloor B\rfloor$. Then after further modification we can construct an equidimensional fibration $h_\varepsilon : X\to Z_\varepsilon$ to a normal projective surface. Finally, we descend K_X+B to Z_ε and prove its semi-ampleness (see Proposition 5.10).

1.3 Notation and conventions

• We say that X is a variety if it is an integral and separated scheme which is of finite type over a field k.

• We say that a morphism $f : X\to Y$ is a contraction if X and Y are normal algebraic spaces (we refer to [1] for definition and basic properties of algebraic spaces), $f_*\mathcal{O}_X=\mathcal{O}_Y$, and f is proper.

• We say that a morphism $f : X\to Y$ of algebraic spaces is equidimensional if all fibers X_y of f are of the same dimension for $y\in Y$.

• Let $f : X\to Y$ be a surjective morphism of integral algebraic spaces. We say that a $\mathbb{Q}$-divisor D on X is f-exceptional if $\dim(f(\mathrm{Supp}\ D))<\dim Y-1$.

• We call a divisor $D\subseteq X$ vertical with respect to a contraction f if $f|_D$ is not dominant.

• We call (X,B) a pair if X is a normal variety and B is an effective $\mathbb{Q}$-divisor on X such that K_X+B is $\mathbb{Q}$-Cartier. For more notions in the theory of MMP such as klt (dlt, lc) pairs, flips, divisorial contractions and so on, we refer to [20].

• Let X be a normal projective variety over a field k, and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. If $|mD|=\emptyset$ for all $m>0$, we define the Kodaira dimension $\kappa(X,D)=-\infty$. Otherwise, let $\Phi : X\dashrightarrow Z$ be the Iitaka map (we refer to [21, 2.1.C]) of D and we define the Kodaira dimension $\kappa(X,D)$ to be the dimension of the image of Φ. Sometimes we write $\kappa(D)$ for $\kappa(X,D)$. We denote $\kappa(X,K_X)$ by $\kappa(X)$. And for a projective variety Y over a field k admitting a smooth model $\tilde{Y}$, we define $\kappa(Y):=\kappa(\tilde{Y})$.

• Let X be a normal projective variety of dimension n over a field k, and let D be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then we can define

$$\nu(D):=\max\{k\in\mathbb{N}|D^k\cdot A^{n-k}>0 \text{ for an ample divisor } A \text{ on } X\}.$$

§2. Preliminaries

In this section, we recall some basic results.

2.1 Keel's results on semi-ampleness

In this subsection, we survey Keel's work on basepoint free theorem for nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors in positive characteristic (see [17]). It is proved that to show the semi-ampleness of a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on a projective variety X, it suffices to show the semi-ampleness of D on $\mathbb{E}(L)$, which is a closed subset of X defined below.

DEFINITION 2.1. Let L be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a projective scheme X over a field. An irreducible subvariety $Z \subset X$ is called exceptional for L if $L|_Z$ is not big, that is, if $L^{\dim Z} \cdot Z = 0$. The exceptional locus of L, denoted by $\mathbb{E}(L)$, is the closure of the union of all exceptional subvarieties.

REMARK 2.2. $\mathbb{E}(L)$ is actually the union of finitely many exceptional subvarieties by [17, 1.2].

DEFINITION 2.3. A nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor L on a proper scheme X over a field is endowed with a map (EWM) $f : X \to Z$ if f is a proper map to a proper algebraic space Z such that it contracts a closed subvariety Y, that is, $\dim(f(Y)) < \dim(Y)$, if and only if $L|_Y$ is not big. We may always assume that such a map has geometrically connected fibers.

REMARK 2.4. By definition, if L is endowed with a map $f : X \to Z$, then a curve $C \subseteq X$ is contracted by f if and only if $L \cdot C = 0$. Moreover, if $f' : X \to Z'$ is a contraction which only contracts L-numerically trivial curves, then by the rigidity lemma (see [19, II.5.3]) f factors through f'.

LEMMA 2.5. *Let $p : Y \to X$ be a proper surjective morphism between reduced algebraic spaces of finite type over a field of positive characteristic. Let L be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that p^*L is semi-ample. If X is normal, then L is semi-ample.*

Proof. This lemma follows from [17, Lem. 2.10]. ∎

The following theorem is the main result of [17].

THEOREM 2.6 [17, Th. 0.2]. *Let L be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on a scheme X, projective over a field of positive characteristic. Then L is semi-ample (resp. EWM) if and only if $L|_{\mathbb{E}(L)}$ is semi-ample (resp. EWM).*

2.2 Nef reduction map

In this subsection, we recall the notion of nef reduction map.

DEFINITION 2.7. Let X be a normal projective variety defined over an uncountable field, and let L be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We call a rational map $\phi : X \dashrightarrow Z$ a nef reduction map of L if Z is a normal projective variety and there exist open dense subsets $U \subseteq X$, $V \subseteq Z$ such that:

(1) $\phi|_U : U \to Z$ is proper, its image is V and $\phi_*\mathcal{O}_U = \mathcal{O}_V$,

(2) $L|_F \equiv 0$ for all fibers F of ϕ over V, and

(3) if $x \in X$ is a very general point and C is a curve passing through it, then $C \cdot L = 0$ if and only if C is contracted by ϕ.

It is proved that a nef reduction map exists over an uncountable algebraically closed field.

THEOREM 2.8 [2, Th. 2.1]. *A nef reduction map exists for normal projective varieties defined over an uncountable algebraically closed field. Furthermore, it is unique up to birational equivalence.*

For a nef reduction map $\phi : X \dashrightarrow Z$ of L, the nef dimension of L is defined to be $\dim Z$ and denoted by $n(X, L)$. When the base field is countable and algebraically closed, we can define

$$n(X, L) := n(X_K, L_K)$$

by [27, Prop. 2.16], where K is an uncountable algebraically closed field that contains k, and X_K, L_K are the base changes of X, L to K. It satisfies $\kappa(X,L) \leq n(X,L)$. Sometimes we write $n(L)$ for $n(X,L)$.

Lemma 2.9 [5, Lem. 7.2]. *Let X be a normal projective variety of dimension ≤ 3 over an uncountable algebraically closed field of characteristic $p > 0$. Suppose L is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X with $\kappa(L) = n(L) \leq 2$. Then L is EWM to a proper algebraic space Z of dimension equal to $\kappa(L)$.*

The following lemma is very useful for descending a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor along a fibration.

Lemma 2.10. *Let $f : X \to Z$ be a projective contraction between normal quasi-projective varieties over a field of characteristic $p > 0$, and let L be a f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $L|_F \sim_{\mathbb{Q}} 0$, where F is the generic fiber of f. Assume* $\dim Z \leq 3$. *Then there exists a diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Z' & \xrightarrow{\psi} & Z \end{array}$$

with ϕ, ψ projective birational, and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Z' such that $\phi^* L \sim_{\mathbb{Q}} f'^* D$. Moreover, if Z is $\mathbb{Q}$-factorial and f is equidimensional, then we can take $X' = X$ and $Z' = Z$.

Proof. It is an adaptation of a result of Kawamata [16, Prop. 2.1]. See [25, Lem. 3.2] for a proof in this setting. ∎

2.3 Abundance theorem for surfaces

Abundance for slc surfaces over an arbitrary field of characteristic > 0 is known.

Theorem 2.11 [22, Th. 1]. *Let (X, Δ) be a projective slc surface pair over a field of characteristic > 0. If $K_X + \Delta$ is nef, then it is semi-ample.*

2.4 MMP for threefolds in positive characteristic

In this subsection, we recall the theory of MMP for projective lc threefold pairs over a perfect field of characteristic $p > 3$. Moreover, we define a partial MMP over an algebraically closed field of characteristic $p > 3$ (see Definition 2.16). We will use this construction to study the abundance in Section 5.

Theorem 2.12 [15, Th. 1.1] and [12]. *Let (X, B) be a lc threefold pair over a perfect field k of characteristic > 3 and $f : X \to Y$ a projective surjective morphism to a quasi-projective variety. If $K_X + B$ is pseudo-effective (resp. not pseudo-effective) over Y, then we can run a $(K_X + B)$-MMP over Y to get a log minimal model (resp. Mori fiber space) over Y.*

We recall the notion of MMP with scaling. Let (X, B) be a projective lc threefold pair over a perfect field k of characteristic > 3 and $A > 0$ an $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Suppose that there is $t_0 > 0$ such that $(X, B + t_0 A)$ is lc and $K_X + B + t_0 A$ is nef. We describe how to run a $(K_X + B)$-MMP with scaling of A as follows.

Let $\lambda_0 = \inf\{t|\ K_X + B + tA \text{ is nef}\}$. Suppose we can find a $(K_X + B)$-negative extremal ray R_0 which satisfies $(K_X + B + \lambda_0 A) \cdot R_0 = 0$ (In general, it is possible that there is no such extremal ray). This is the first ray we contract in our MMP. If the contraction is a Mori fiber contraction, we stop. Otherwise, let X_1 be the result of the divisorial contraction or flip. Then $K_{X_1}+B_{X_1}+\lambda_0 A_{X_1}$ is also nef, where B_{X_1} and A_{X_1} denote the birational transforms on X_1 of B and A, respectively. We define $\lambda_1 = \inf\{t \mid K_{X_1}+B_{X_1}+tA_{X_1} \text{ is nef}\}$. The next step in our MMP is chosen to be a $(K_{X_1}+B_{X_1})$-negative extremal ray R_1 which is $(K_{X_1}+B_{X_1}+\lambda_1 A_{X_1})$ -trivial. So long as we can find the appropriate extremal rays, contractions and flips, we can continue this process.

PROPOSITION 2.13. *Let (X,B) be a $\mathbb{Q}$-factorial projective lc threefold pair over an algebraically closed field k of characteristic >3, and W let be an effective $\mathbb{Q}$-divisor such that K_X+B+W is nef. Then either:*
(1) there is a (K_X+B)-negative extremal ray which is (K_X+B+W)-trivial, or
(2) $K_X+B+(1-\varepsilon)W$ is nef for any sufficiently small rational $\varepsilon>0$.

Proof. It is an adaptation of [18, Lem. 5.1]. Note that the proof there only uses the fact that for any (K_X+B)-negative extremal ray R there is a rational curve C such that C generates R and $-(K_X+B)\cdot C \le 6$, which holds in our setting by [15, Th. 1.3] and [12]. ∎

REMARK 2.14. The assumption that k is algebraically closed is used for the fact that for any (K_X+B)-negative extremal ray R there is a rational curve C such that C generates R and $-(K_X+B)\cdot C \le 6$.

COROLLARY 2.15. *Let (X,B) be a $\mathbb{Q}$-factorial projective lc threefold pair over an algebraically closed field k of characteristic >3, and let A be an effective $\mathbb{Q}$-divisor such that $(X,B+A)$ is lc and K_X+B+A is nef. If K_X+B is not nef, then we can run a (K_X+B)-MMP with scaling of A.*

Proof. Let $\lambda := \inf\{t \mid K_X+B+tA \text{ is nef}\}$ be the nef threshold. Then the only assertion is that we can find a (K_X+B)-negative extremal ray R such that $(K_X+B+\lambda A)\cdot R=0$. We apply Proposition 2.13 by letting $W:=\lambda A$. ∎

In this paper, we will use the following construction.

DEFINITION 2.16. Let (X,B) be a $\mathbb{Q}$-factorial projective lc threefold pair over an algebraically closed field k of characteristic >3, and let A be an effective $\mathbb{Q}$-divisor such that $(X,B+A)$ is lc and K_X+B+A is nef. We can run a partial (K_X+B)-MMP with scaling of A as follows.

Let $\lambda_0 = \inf\{t \mid K_X+B+tA \text{ is nef}\}$. If $\lambda_0<1$, then we stop. Otherwise, by Proposition 2.13, there exists a (K_X+B)-negative extremal ray R_0 which satisfies $(K_X+B+A)\cdot R_0=0$. We contract this extremal ray. If the contraction is a Mori fiber contraction, we stop. Otherwise, let $\mu_0 : X \dashrightarrow X_1$ be the divisorial contraction or flip. Repeat this process for $(X_1,\mu_{0*}B),\mu_{0*}A$ and so on.

We call this construction a (K_X+B)-MMP which is (K_X+B+A)-trivial.

The following lemma tells us what the output of this construction is if it terminates.

LEMMA 2.17. *Let (X,B) be a $\mathbb{Q}$-factorial projective lc threefold pair over an algebraically closed field k of characteristic >3, and let A be an effective $\mathbb{Q}$-divisor such that $(X,B+A)$ is lc and K_X+B+A is nef.*

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If a (K_X+B)-MMP which is (K_X+B+A)-trivial terminates, then its output is a $\mathbb{Q}$-factorial projective lc pair $(X',B'+A')$, and either:

(1) X' has the structure of a Mori fiber space $X'\to Y$, $K_{X'}+B'+A'$ is the pullback of a $\mathbb{Q}$-divisor from Y, and Supp *A' dominates Y, or*

(2) $K_{X'}+B'+(1-\varepsilon)A'$ is nef for any sufficiently small rational $\varepsilon>0$.

Moreover, $K_{X'}+B'+A'$ is semi-ample if and only if K_X+B+A is semi-ample.

Proof. We only need to prove that, if a (K_X+B)-MMP which is (K_X+B+A)-trivial terminates with a Mori fiber space $f:(X',B'+A')\to Y$, then Supp A' dominates Y. It is clear since f only contracts curves which have positive intersections with A'. ∎

We will use the following results on termination of flips.

Theorem 2.18 [26, Th. 1.6] and [12]. *Let (X,B) be a projective lc threefold pair over a perfect field k of characteristic $p>3$. If M is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then any sequence of (K_X+B)-flips which are also M-flips terminates.*

Lemma 2.19. *Let (X,B) be a $\mathbb{Q}$-factorial projective lc threefold pair over an algebraically closed field k of characteristic >3 such that K_X+B+A is nef. If X is terminal, then any K_X-MMP which is (K_X+B)-trivial terminates.*

Proof. Since every step of a K_X-MMP which is (K_X+B)-trivial is a step of a K_X-MMP, the assertion follows from [20, Th. 6.17]. ∎

2.5 Dlt modifications and adjunction

The following result helps us to reduce some problems for lc pairs to $\mathbb{Q}$-factorial dlt pairs.

Theorem 2.20. *Let (X,B) be a lc threefold pair over a perfect field k of characteristic >3. Then (X,B) has a crepant $\mathbb{Q}$-factorial dlt model. Moreover, we can modify X so that it is terminal.*

Proof. For the first assertion, see [4, Th. 1.6] and [12]. Let us prove that we can make X terminal. We take a crepant $\mathbb{Q}$-factorial dlt model $g:(X',B')\to(X,B)$ by the first assertion. Hence, by replacing (X,B) by (X',B'), we may assume that (X,B) is $\mathbb{Q}$-factorial and dlt. Let $U\subseteq X$ be the largest open set such that $(U,B|_U)$ is a snc pair. Then $\operatorname{codim}_X(X\backslash U)\geq 2$. Let $f:(X',\Theta')\to(X,0)$ be a terminal model of $(X,0)$ as in [4, Th. 1.7] such that $K_{X'}+\Theta'=f^*K_X$. Then f is an isomorphism over the smooth locus of X; in particular, f is an isomorphism over U. Let $Z=X\backslash U$. Define $B':=\Theta'+f^*B$ on X' so that

$$K_{X'}+B'=f^*(K_X+B),$$

and (X',B') is lc.

It remains to show that (X',B') is a dlt pair. Let $U'=f^{-1}(U)$ and $Z'=X'\backslash U'$. Then $(U',B'|_{U'})$ is a snc pair. If E is an exceptional divisor with center in Z', then its center in X is contained in Z. Hence $a(E,X',B')=a(E,X,B)>-1$. This completes the proof. ∎

For $\mathbb{Q}$-factorial dlt threefold pairs, we have the following result on adjunction.

Theorem 2.21. *Let (X,B) be a $\mathbb{Q}$-factorial projective dlt threefold pair over a perfect field k of characteristic >0. If $(K_X+B)|_{\lfloor B\rfloor}$ is nef, then $(K_X+B)|_{\lfloor B\rfloor}$ is semi-ample.*

Proof. By [11, Rem. 3.9], we know that all lc centres of $\mathbb{Q}$-factorial three-dimensional dlt pairs are normal up to a universal homeomorphism. Hence, we can argue as in [26, §5] to prove that the S_2-fication (see, for example, [26, 2.2]) of $\lfloor B \rfloor$ is a universal homeomorphism and $(K_X+B)|_{\lfloor B \rfloor}$ is semi-ample. ∎

2.6 Some known results on the abundance

The following theorem collects the recent results toward the abundance conjecture in positive characteristics.

THEOREM 2.22. *Let (X,B) be a projective klt threefold pair over an algebraically closed field k of characteristic >5 such that K_X+B is nef. Assume that one of the following conditions holds:*
(1) $\kappa(X,K_X+B)\geq 1$,
(2) the nef dimension $n(X,K_X+B)\leq 2$,
(3) the Albanese map $a_X : X \to \mathrm{Alb}(X)$ is nontrivial.
Then K_X+B is semi-ample.

Proof. For (1), the case of $\kappa(X,K_X+B)=3$ is proved in [5, Th. 1.2], the case of $\kappa(X,K_X+B)=2$ is proved in [25, Th. 1.3] and the case of $\kappa(X,K_X+B)=1$ is proved in [29, Th. 3.1] and [8, Th. A]. For (2), it is proved in [27, Th. 5]. For (3), see [30, Th. 1.1] and [27, Cor. 4.13]. ∎

Moreover, the nonvanishing theorem for terminal threefolds has been proved in [28].

THEOREM 2.23 [28, Th. 1.1]. *Let X be a projective terminal threefold over an algebraically closed field k of characteristic >5. If K_X is pseudo-effective, then $\kappa(X,K_X)\geq 0$.*

Based on it, the nonvanishing theorem for klt threefold pairs is proved in [27].

THEOREM 2.24 [27, Th. 3]. *Let (X,B) be a projective klt threefold pair over a perfect field k of characteristic >5. If K_X+B is pseudo-effective, then $\kappa(K_X+B)\geq 0$.*

§3. Klt threefold pairs in characteristic >3

In this section, we generalize the results in Section 2.6 to the case when the characteristic is greater than 3. Note that in Section 2.6, we always assume that the characteristic of the base field is greater than 5. Actually, the assumption of characteristic >5 is used for the following assertions. Let k be an algebraically closed field of characteristic >5. Then we have the following propositions hold:

P 1: (MMP) We can run MMP for lc threefold pairs over k (see, for example, [15]).

P 2: (Elliptic fibration) Let $g : X \to Z$ be a fibration of normal varieties of relative dimension one over k. Assume that the generic fiber X_η of g is a curve with arithmetic genus $p_a(X_\eta)=1$. Then the geometric generic fiber $X_{\overline{\eta}}$ of g is a smooth elliptic curve over $\overline{K(Z)}$ (see [30, Prop. 2.11]).

P 3: (Dlt adjunction) Let (X,B) be a $\mathbb{Q}$-factorial projective dlt threefold pair over k. Then every irreducible component of $\lfloor B \rfloor$ is normal. If, moreover, $(K_X+B)|_{\lfloor B \rfloor}$ is nef, then it is semi-ample (see [7, §2] and [26, Th. 1.3]).

P 4: (Classification of surface F-singularity) Klt surface singularities over k are strongly F-regular (see [14]).

REMARK 3.1. These proposition are not independent. For example, the proof of **P 1** uses **P 4**.

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Now we assume that the characteristic of k is just greater than 3. Then **P 1** and **P 2** hold by [12] and [30, Prop. 2.11]. Although **P 3** may not hold, it is not far from being true. More precisely, if (X,B) is a $\mathbb{Q}$-factorial dlt threefold pair over k, then every irreducible component of $\lfloor B\rfloor$ is normal up to a universal homeomorphism by [11, Rem. 3.9]. If, moreover, $(K_X+B)|_{\lfloor B\rfloor}$ is nef, then it is semi-ample by Theorem 2.21. Finally, **P 4** may not hold.

First, we generalize the results on subadditivity of Kodaira dimensions in [30] to the case when the characteristic is greater than 3 (see Theorem 3.4). To do this, we need the following lemmas.

Lemma 3.2 (cf. [30, Lem. 4.10]). *Let $(\hat{X},\hat{B})$ be a $\mathbb{Q}$-factorial projective dlt threefold pair over an algebraically closed field k of characteristic >3, and let $\hat{f}:\hat{X}\to Y$ be a fibration to a normal variety. Assume that $K_{\hat{X}}+\hat{B}$ is nef and $\hat{B}=G_1+G_2+\cdots+G_n$ is a sum of prime Weil divisors. Denote the normalization of G_j by G_j^ν for every $j=1,2,\dots,n$. Then for every $j=1,2,\dots,n$, $(K_{\hat{X}}+\hat{B})|_{G_j}$ is semi-ample. Moreover, a general fiber F_j of the Iitaka fibration induced by $(K_{\hat{X}}+\hat{B})|_{G_j^\nu}$ is integral. We denote the image of F_j along the normalization $G_j^\nu\to G_j$ by $\hat{F}_j$.*

Assume, in addition, that:

(a) there exist $N>0$ and two different effective Cartier divisors $\hat{D}_i$, $i=1,2$ such that

$$\hat{D}_i\sim N(K_{\hat{X}}+\hat{B})+\hat{f}^*L_i$$

for some $L_i\in\operatorname{Pic}^0(Y)$ and that $\operatorname{Supp}\,\hat{D}_i\subseteq\operatorname{Supp}\,\hat{B}$,

(b) there exist effective divisors $\hat{G}_1,\hat{G}_2,\hat{G}'_1,\hat{G}'_2$ such that

$$\hat{D}_1=a_{11}\hat{G}_1+a_{12}\hat{G}_2+\hat{G}'_1,\hat{D}_2=a_{21}\hat{G}_1+a_{22}\hat{G}_2+\hat{G}'_2,$$

where $a_{11}>a_{21}\geq 0$ and $a_{22}>a_{12}\geq 0$, and

(c) G_1,G_2 are two irreducible components of $\hat{G}_1,\hat{G}_2$ respectively, such that for $i,j\in\{1,2\}$ and $i\neq j$, $\hat{F}_j$ dominates Y and

$$\hat{F}_j\cap\operatorname{Supp}(\hat{G}''_j:=\hat{G}_i+\hat{G}'_1+\hat{G}'_2)=\emptyset.$$

Then both L_1 and L_2 are torsion line bundles.

Furthermore, condition (c) holds, if for $j=1,2$, G_j is not a component of $\hat{G}''_j$ and $\kappa(F_j)\geq 0$.

Proof. By Theorem 2.21, we have $(K_{\hat{X}}+\hat{B})|_{\hat{B}}=(K_{\hat{X}}+\hat{B})|_{\lfloor\hat{B}\rfloor}$ is semi-ample. In particular, $(K_{\hat{X}}+\hat{B})|_{G_j}$, and hence $(K_{\hat{X}}+\hat{B})|_{G_j^\nu}$ are semi-ample for every $j=1,2,\dots,n$. Moreover, a general fiber F_j of the Iitaka fibration induced by $(K_{\hat{X}}+\hat{B})|_{G_j^\nu}$ is integral by [30, Prop. 2.1]. Hence, the first assertion holds.

Now we assume (a), (b), and (c). Note that

$$(K_{\hat{X}}+\hat{B})|_{F_1}=\big((K_{\hat{X}}+\hat{B})|_{G_1^\nu}\big)|_{F_1}\sim_{\mathbb{Q}}0$$

since $(K_{\hat{X}}+\hat{B})|_{G_1^\nu}$ is semi-ample and F_1 is a general fiber of the Iitaka fibration of $(K_{\hat{X}}+\hat{B})|_{G_1^\nu}$. We have

$$\begin{aligned} a_{21}\hat{f}^*L_1|_{F_1} &\sim_{\mathbb{Q}} a_{21}(N(K_{\hat{X}}+\hat{B})+\hat{f}^*L_1)|_{F_1} \\ &\sim_{\mathbb{Q}} a_{21}\hat{D}_1|_{F_1} && \text{(by (a))} \\ &\sim_{\mathbb{Q}} a_{21}(a_{11}\hat{G}_1+a_{12}\hat{G}_2+\hat{G}_1')|_{F_1} && \text{(by (b))} \\ &\sim_{\mathbb{Q}} a_{11}a_{21}\hat{G}_1|_{F_1} && \text{(by (c))}. \end{aligned}$$

Similarly, $a_{11}\hat{f}^*L_2|_{F_1} \sim_{\mathbb{Q}} a_{11}a_{21}\hat{G}_1|_{F_1}$. Hence, we have $a_{21}\hat{f}^*L_1|_{F_1} \sim_{\mathbb{Q}} a_{11}\hat{f}^*L_2|_{F_1}$. It follows that $a_{21}L_1 \sim_{\mathbb{Q}} a_{11}L_2$ by [30, Lem. 2.4]. Similarly, we have $a_{22}L_1 \sim_{\mathbb{Q}} a_{12}L_2$. We then deduce that $L_1 \sim_{\mathbb{Q}} L_2 \sim_{\mathbb{Q}} 0$ since $a_{11} > a_{21} \geq 0$ and $a_{22} > a_{12} \geq 0$. Hence the second assertion holds.

It remains to prove the third assertion. As $\kappa(F_j) \geq 0$, we have the canonical divisor $K_{F_j^\nu} \geq 0$, where F_j^ν is the normalization of F_j. Applying the adjunction formula, we get

$$\begin{aligned} 0 \sim_{\mathbb{Q}} (K_{\hat{X}}+\hat{B})|_{F_j^\nu} &\sim_{\mathbb{Q}} ((K_{\hat{X}}+\hat{B})|_{G_j^\nu})|_{F_j^\nu} \\ &\sim_{\mathbb{Q}} ((K_{\hat{X}}+G_j)|_{G_j^\nu}+(\hat{B}-G_j)|_{G_j^\nu})|_{F_j^\nu} \\ &\sim_{\mathbb{Q}} (K_{G_j^\nu}+C_j)|_{F_j^\nu}+(\hat{B}-G_j)|_{F_j^\nu} \\ &\sim_{\mathbb{Q}} K_{F_j^\nu}+C_j|_{F_j^\nu}+(\hat{B}-G_j)|_{F_j^\nu}, \end{aligned}$$

where $C_j \geq 0$ on G_j^ν. It implies that $(\hat{B}-G_j)|_{F_j^\nu} \leq 0$. Since F_j is general, $\hat{F}_j$ is not contained in $\hat{B}-G_j$. Hence, $\hat{F}_j \cap \operatorname{Supp}(\hat{B}-G_j) = \emptyset$. By our assumption, G_j is not a component of $\hat{G}_j''$. Thus, $\operatorname{Supp}(\hat{G}_j'') \subseteq \operatorname{Supp}(\hat{B}-G_j)$. It follows that $\hat{F}_j \cap \operatorname{Supp}(\hat{G}_j'') = \emptyset$. ∎

LEMMA 3.3. *Let (X,B) be a projective klt threefold pair over an algebraically closed field k of characteristic >3. Assume that K_X+B is nef and $\kappa(X,K_X+B) \geq 1$. Then K_X+B is semi-ample.*

Proof. The case of $\kappa(X,K_X+B)=3$ follows from [12, Th. 1.3]. In the cases of $\kappa(X,K_X+B)=1$ or 2, the assertion is proved when the characteristic of k is greater than 5 in [25, Th. 1.3], [29, Th. 3.1], and [8, Th. A]. And it uses the assumption of characteristic >5 for **P 1**. When the characteristic of k is greater than 3, by Theorem 2.12, **P 1** also holds. Hence, we can argue as in the proofs of [25, Th. 1.3], [29, Th. 3.1], and [8, Th. A] to prove the assertion. ∎

Now we can deduce the following result on subadditivity of Kodaira dimensions in characteristic >3.

THEOREM 3.4. *Let $f: X \to Y$ be a fibration from a $\mathbb{Q}$-factorial projective threefold to a smooth projective variety of dimension 1 or 2, over an algebraically closed field k of characteristic $p>3$. Assume that there is an effective $\mathbb{Q}$-divisor B on X such that (X,B) is klt. Assume that Y is of maximal Albanese dimension. Moreover, we assume that if $\kappa(X_\eta, K_{X_\eta}+B_\eta) = \dim X - \dim Y - 1$, where X_η is the generic fiber of f and $K_{X_\eta}+B_\eta := (K_X+B)|_{X_\eta}$, then B does not intersect the generic fiber X_ξ of the relative Iitaka fibration $I: X \dashrightarrow Z$ induced by K_X+B on X over Y.*

Then

$$\kappa(X,K_X+B) \geq \kappa(X_\eta, K_{X_\eta}+B_\eta)+\kappa(Y).$$

Proof. The case when the characteristic is greater than 5 is proved in [30, Th. 1.4]. Using Theorem 2.12 and Lemma 3.3, we can argue as in the proof of [30, Th. 1.4] except in the cases when:

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(1) Y is an elliptic curve or a simple abelian surface, and $K_X + B$ is f-big, or

(2) Y is an elliptic curve, $\kappa(X_\eta, K_{X_\eta} + B_\eta) = 1$ and B does not intersect the generic fiber X_ξ of the relative Iitaka fibration $I : X \dashrightarrow Z$ induced by $K_X + B$ on X over Y.

Now we assume that we are in one of these cases. We first make some reductions as follows. In the case (1), if the characteristic of k is greater than 5, then the proof of [30, Th. 4.2] reduces the assertion to the case when:

• the denominators of coefficients of B are not divisible by p,

• $K_X + B$ is a nef and f-ample,

• $\nu(K_X + B) \leq 2$,

• there exist a sufficiently divisible positive integer l and a coherent sheaf $\mathcal{F}$ such that $\mathcal{F}$ is a subsheaf of $f_*\mathcal{O}_X(l(K_X + B))$,

• there exists an isogeny $\tau : Y_1 \to Y$ between abelian varieties, some $P_i \in \mathrm{Pic}^0(Y_1)$ and a generically surjective homomorphism

$$\tau^*\mathcal{F} \cong \oplus_{i=1}^{r_1} P_i.$$

In the case (2), if the characteristic of k is greater than 5, the proof of [30, Th. 4.3] reduces the assertion to the case when:

• $K_X + B$ is nef,

• there exists a commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\sigma} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y \end{array}$$

where σ is a log resolution, h is a fibration to a smooth projective surface which is birational to the relative Iitaka fibration induced by $\sigma^*(K_X + B)$ on W over Y,

• there exist a nef and g-big divisor C on Z such that $\sigma^*(K_X + B) \sim_{\mathbb{Q}} h^*C$,

• the geometric generic fiber of g is either a smooth elliptic curve or a rational curve,

• $\nu(Z, C) = 1$,

• there exist a sufficiently divisible positive integer l and a nef sub-vector bundle V of $f_*\mathcal{O}_X(l(K_X + B))$ of rank $r \geq 2$,

• there exists a flat base change $\pi : Y_2 \to Y$ between elliptic curves such that

$$\pi^*V \cong \oplus_{i=1}^{r_2} L'_i,$$

where $L'_i \in \mathrm{Pic}^0(Y_2)$.

When the characteristic of k is greater than 3, using Theorem 2.12 and Lemma 3.3, we can also argue as in the proofs of [30, Ths. 4.2 and 4.3] to make such reductions.

If the characteristic of k is greater than 5, then the argument in [30, Steps 2, 3 of the proof of Th. 4.2 and Steps 2, 3 of the proof of Th. 4.3] implies that there exist an integer m_1 and some divisors $D_i \in |m_1(K_X + B) + f^*L_i|$, $i = 1, 2, \ldots, r$ for some $L_i \in \mathrm{Pic}^0(Y)$. Moreover, we can construct a pair $(\hat{X}, \hat{B})$ and divisors $\hat{D}_1, \hat{D}_2$ satisfying all conditions of Lemma 3.2. When the characteristic of k is greater than 3, using Theorem 2.12, we can also argue as in the proofs of [30, Ths. 4.2 and 4.3] to prove these assertions. By Lemma 3.2, L_1 and L_2 are

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torsions. Hence, there exist a sufficiently divisible integer $N>0$ and two different divisors among D_i, say, $D_1 \neq D_2$ such that

$$ND_j \in |Nm_1(K_X+B)+NL_j| = |Nm_1(K_X+B)|$$

for $j=1,2$. Hence we have $\kappa(X,K_X+B)\geq 1$. In the case (2), it implies that

$$\kappa(X,K_X+B)\geq 1 = \kappa(X_\eta, K_{X_\eta}+B_\eta).$$

In the case (1), by Lemma 3.3, K_X+B is semi-ample. Thus, for a sufficiently divisible $M>0$, the linear system $|M(K_X+B)|$ has no base point. Since $K_{X_\eta}+B_\eta$ is big, the restriction $|M(K_X+B)||_{X_\eta}$ on the generic fiber X_η defines a generically finite morphism. It implies that

$$\kappa(X,K_X+B)\geq \dim X_\eta = \kappa(X_\eta, K_{X_\eta}+B_\eta).$$

In conclusion, the assertion holds. ∎

Using this result on subadditivity of Kodaira dimensions in characteristic >3, we deduce the following results on the abundance with nontrivial Albanese maps in characteristic >3.

Lemma 3.5. *Let (X,B) be a $\mathbb{Q}$-factorial projective klt threefold pair over an algebraically closed field k of characteristic >3. Assume that K_X+B is nef, X is non-uniruled and the Albanese map $a_X: X\to \mathrm{Alb}(X)$ is nontrivial. Then K_X+B is semi-ample.*

Proof. The case when the characteristic of k is greater than 5 is proved in [29, Th. 1.1]. When the characteristic of k is greater than 3, by the proof of [29, Th. 1.1], we only need to prove the following assertions.

(1) Let $f_1: X_1\to Y_1$ be a separable fibration from a smooth projective threefold to a smooth projective variety of dimension 1 or 2 over k. Denote by $\tilde{X}_{1,\overline{\eta}}$ a smooth projective birational model of $X_{1,\overline{\eta}}$, where $X_{1,\overline{\eta}}$ is the geometric generic fiber of f_1. Then

$$\kappa(X_1)\geq \kappa(\tilde{X}_{1,\overline{\eta}})+\kappa(Y_1).$$

(2) Let X_2 be a $\mathbb{Q}$-factorial projective klt threefold over k with $K_{X_2}\sim_{\mathbb{Q}} 0$, and let D be an effective and nef $\mathbb{Q}$-divisor on X_2. Assume that X_2 has a morphism $f_2: X_2\to Y_2$ to an elliptic curve and that $X_{2,\overline{\eta}}$ has at most canonical singularities, where $X_{2,\overline{\eta}}$ is the geometric generic fiber of f_2. Then either $D=0$ or $\kappa(X_2,D)\geq 1$.

(1) is proved when the characteristic of k is greater than 5 in [29, Cor. 2.9]. It uses the assumption of characteristic >5 for the fact that canonical singularities over k are F-pure. This fact holds in characteristic 5 by [14, Th. 1.2]. Hence, (1) follows from the proof of [29, Cor. 2.9]. For (2), it suffices to show that if $\kappa(X_2,D)=0$, then $D=0$. We assume that $\kappa(X_2,D)=0$. We denote the generic fiber of f_2 by $X_{2,\eta}$. Note that

$$D_\eta := D|_{X_{2,\eta}} \sim_{\mathbb{Q}} K_{X_{2,\eta}}+D_\eta$$

and $(X_{2,\eta},D_\eta)$ is lc after replacing D by a small multiple. By Theorem 2.11, D_η is semi-ample. Hence $\kappa(X_{2,\eta},D_\eta)\geq 0$. If $\kappa(X_{2,\eta},D_\eta)\neq 1$, then by Theorem 3.4, we have $\kappa(X_{2,\eta},D_\eta)=0$. Hence $D_\eta\sim_{\mathbb{Q}} 0$. Note that f_2 is equidimensional since Y_2 is a normal curve. By Lemma 2.10, D descends to an effective $\mathbb{Q}$-divisor on Y_2. Hence $D=0$. Otherwise, we have $\kappa(X_{2,\eta},D_\eta)=1$. Then we may apply the proof of [29, Cor. 2.10] to the case of the characteristic of k is greater than 3. Therefore, the assertion holds. ∎

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REMARK 3.6. The non-uniruled assumption is used in the proof of [29, Th. 1.1].

THEOREM 3.7. *Let (X,B) be a $\mathbb{Q}$-factorial projective klt threefold pair over an algebraically closed field k of characteristic >3. Assume that K_X+B is nef and the Albanese map $a_X: X \to \mathrm{Alb}(X)$ is nontrivial. Denote by $f: X \to Y$ the fibration arising from the Stein factorization of a_X and by X_η the generic fiber of f. Assume, moreover, that $B=0$ if:*
(1) $\dim Y = 2$ *and* $\kappa(X_\eta, (K_X+B)|_{X_\eta}) = 0$*, or*
(2) $\dim Y = 1$ *and* $\kappa(X_\eta, (K_X+B)|_{X_\eta}) = 1$*.*
Then K_X+B is semi-ample.

Proof. The case when the characteristic is greater than 5 is proved in [30, Th. 1.2]. By Lemma 3.5, we can assume that X is uniruled. Moreover, by Lemma 3.3, we can assume that $\kappa(X, K_X+B) \le 0$.

Since X is uniruled, we have $\dim Y = 1$ or 2. Note that $K_{X_\eta}+B_\eta$ is semi-ample by the abundance for surfaces (Theorem 2.11) and curves. In particular, $\kappa(X_\eta, K_{X_\eta}+B_\eta) \ge 0$. Therefore by Theorem 3.4, we have $\kappa(X, K_X+B) = 0$, and hence $\kappa(Y) = \kappa(X_\eta, K_{X_\eta}+B_\eta) = 0$. If $\dim Y = 1$, then the assertion is proved when the characteristic of k is greater than 5 in [30, Th. 4.4]. Using Theorem 3.4, we can argue as in the proof of [30, Th. 4.4] to prove that K_X+B is semi-ample. Otherwise, we have $\dim Y = 2$. Then $B=0$ by our assumption and f is an elliptic fibration by [30, Prop. 2.11]. Hence X is non-uniruled. We obtain a contradiction. Thus, K_X+B is semi-ample. ∎

COROLLARY 3.8. *Let X be a projective terminal threefold over an algebraically closed field k of characteristic >3. If K_X is pseudo-effective, then $\kappa(X, K_X) \ge 0$.*

Proof. The case when the characteristic of k is greater than 5 is proved in [28, Th. 1.1]. Using Theorem 3.7, we can argue as in the proof of [28, Th. 1.1] to prove the assertion. ∎

Now we can generalize Theorem 2.22 to the case when the characteristic is greater than 3.

THEOREM 3.9. *Let (X,B) be a projective klt threefold pair over an algebraically closed field k of characteristic >3 such that K_X+B is nef. Assume that one of the following conditions holds:*
(1) $\kappa(X, K_X+B) \ge 1$,
(2) the nef dimension $n(X, K_X+B) \le 2$,
(3) the Albanese map $a_X: X \to \mathrm{Alb}(X)$ *is nontrivial.*
Then K_X+B is semi-ample.

Proof. See Lemma 3.3 for (1). For (2), it is proved when the characteristic of k is greater than 5 in [27, Th. 5]. Using Theorem 3.7 in the case of $n(X, K_X+B) = 0$, we can argue as in the proof of [27, Th. 5] to prove the assertion. For (3), it is proved when the characteristic of k is greater than 5 in [27, Cor. 4.13]. Using Theorem 3.7 and (2), we can argue as in the proof of [27, Cor. 4.13] to prove the assertion. ∎

Moreover, we can deduce the nonvanishing theorem for klt threefold pairs in characteristic >3.

THEOREM 3.10. *Let (X,B) be a projective klt threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is pseudo-effective, then $\kappa(K_X+B) \ge 0$.*

Proof. It is proved when the characteristic of k is greater than 5 in [27, Th. 3]. Using Corollary 3.8 and (2) of Theorem 3.9, we can argue as in the proof of [27, Th. 3] to prove the assertion. ∎

§4. Nonvanishing theorem for lc threefold pairs

In this section, we show the nonvanishing theorem for projective lc threefold pairs. First, we recall a standard lemma on modifying a pair by some birational transform.

LEMMA 4.1. *Let (X,B) be a $\mathbb{Q}$-factorial dlt threefold pair over an algebraically closed field k of characteristic >3. Suppose that K_X+B is nef and there exists an effective $\mathbb{Q}$-divisor D such that $D\equiv K_X+B$. Then there exists a $\mathbb{Q}$-factorial dlt pair (Y,B_Y) such that:*
(1) K_Y+B_Y is nef,
(2) $n(K_Y+B_Y)=n(K_X+B)$,
(3) $\kappa(K_X+B)\le\kappa(K_Y+B_Y)\le\kappa(K_X+B+rD)$ for some $r>0$,
(4) $K_Y+B_Y\equiv\Delta$ for an effective $\mathbb{Q}$-divisor Δ with $\operatorname{Supp}\Delta\subseteq\lfloor B_Y\rfloor$,
(5) $(Y\backslash\operatorname{Supp}\Delta,B_Y)\cong(X\backslash\operatorname{Supp}D,B)$.
Moreover, if $D\sim_{\mathbb{Q}}K_X+B$, then $K_Y+B_Y\sim_{\mathbb{Q}}\Delta$ in (4).

Proof. It follows from Theorem 2.12 and the proof of [27, Lem. 4.6]. ∎

The following lemma is proved by Witaszek via his weak canonical bundle formula.

LEMMA 4.2 [27, Lem. 4.8]. *Let (X,B) be a projective $\mathbb{Q}$-factorial threefold pair over an algebraically closed field k of characteristic >3 such that the coefficients of B are at most one. Assume that $L:=K_X+B$ is nef and $n(L)=2$. Then the following hold:*
(1) there exists an effective $\mathbb{Q}$-divisor D such that $L\equiv D$,
(2) if $L|_{\operatorname{Supp}D}\sim_{\mathbb{Q}}0$ for some D as above, then $\kappa(L)\ge 0$,
(3) if $L|_{\operatorname{Supp}D}\not\equiv 0$ for some D as above, or $L|_{\operatorname{Supp}D}\sim_{\mathbb{Q}}0$ and $L\sim_{\mathbb{Q}}D$, then $\kappa(L)=2$.

Then we can deduce the following proposition.

PROPOSITION 4.3. *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is nef and $n(X,K_X+B)=2$, then $\kappa(K_X+B)=2$.*

Proof. The proof is similar to the proof of [27, Prop. 4.10]. By Theorem 2.20, replacing (X,B) by a $\mathbb{Q}$-factorial dlt model, we may assume that (X,B) is $\mathbb{Q}$-factorial and dlt. By Lemma 4.2, there exists an effective $\mathbb{Q}$-divisor D satisfying $K_X+B\equiv D$. Now by Lemma 4.1, we have a $\mathbb{Q}$-factorial dlt pair (Y,B_Y) such that for some $r>0$,
- K_Y+B_Y is nef,
- $n(K_Y+B_Y)=n(K_X+B)$ and $\kappa(K_Y+B_Y)\le\kappa(K_X+B+rD)$,
- $K_Y+B_Y\equiv E_Y$, where E_Y is an effective $\mathbb{Q}$-divisor such that $\operatorname{Supp}E_Y\subseteq\lfloor B_Y\rfloor$.

By Theorem 2.21, $(K_Y+B_Y)|_{\lfloor B_Y\rfloor}$, and hence $(K_Y+B_Y)|_{\operatorname{Supp}E_Y}$ are semi-ample. Applying Lemma 4.2 to (Y,B_Y) and E_Y, we have $\kappa(K_Y+B_Y)\ge 0$.

We claim that in fact $\kappa(K_Y+B_Y)\ge 2$. We apply Lemma 4.1 to (Y,B_Y) and an effective $\mathbb{Q}$-divisor which is $\mathbb{Q}$-linearly equivalent to K_Y+B_Y, then we obtain a $\mathbb{Q}$-factorial dlt pair (Z,B_Z) satisfying:
- K_Z+B_Z is nef,
- $n(K_Z+B_Z)=n(K_Y+B_Y)$ and $\kappa(K_Z+B_Z)=\kappa(K_Y+B_Y)$,
- $K_Z+B_Z\sim_{\mathbb{Q}}E_Z$, where E_Z is an effective $\mathbb{Q}$-divisor such that $\operatorname{Supp}E_Y\subseteq\lfloor B_Y\rfloor$.

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Similarly, we have $(K_Z+B_Z)|_{\mathrm{Supp}\ E_Z}$ is semi-ample. Therefore, by Lemma 4.2, we have $\kappa(K_Y+B_Y)=\kappa(K_Z+B_Z)=2$. It implies that $\kappa(K_X+B+rD)\geq 2$. Since $K_X+B\equiv D$, it is clear that $(K_X+B)|_D\not\equiv 0$. Finally, by Lemma 4.2, we have $\kappa(K_X+B)=2$. ∎

Now we can prove the nonvanishing theorem for projective lc threefold pairs.

THEOREM 4.4. *Let (X,B) be a projective lc threefold pair over a perfect field k of characteristic >3. If K_X+B is pseudo-effective, then $\kappa(X,K_X+B)\geq 0$.*

Proof. We pass to an uncountable algebraically closed field. Replacing (X,B) by its log minimal model by Theorem 2.12, we can assume that K_X+B is nef. By Theorem 2.20, we can take a $\mathbb{Q}$-factorial dlt model (X',B') of (X,B) such that (X',B') is $\mathbb{Q}$-factorial and dlt, and moreover X' is terminal. We replace (X,B) by (X',B'). If $\lfloor B\rfloor=0$, then the proposition follows from Theorem 3.10. Hence we can assume that $\lfloor B\rfloor\neq 0$.

Now, by Definition 2.16, we run a K_X-MMP which is (K_X+B)-trivial. By Lemma 2.19, it terminates with a pair (X'',B''). Note that $(X,(1-\varepsilon)B)$ is klt and every step of a K_X-MMP which is (K_X+B)-trivial is a step of a $(K_X+(1-\varepsilon)B)$-MMP for any sufficiently small rational $\varepsilon>0$. Hence we have $(X'',(1-\varepsilon)B'')$ is klt for any sufficiently small rational $\varepsilon>0$. If $K_{X''}+(1-\varepsilon)B''$ is nef for any sufficiently small rational $\varepsilon>0$, then we have $\kappa(K_{X''}+(1-\varepsilon)B'')\geq 0$ by Theorem 3.10 since $(X'',(1-\varepsilon)B'')$ is klt. Hence we have

$$\kappa(K_X+B)=\kappa(K_{X''}+B'')\geq\kappa(K_{X''}+(1-\varepsilon)B'')\geq 0.$$

Otherwise, by Lemma 2.17, we get a Mori fiber space

$$\begin{tikzcd} X \arrow[r, dashed] & X'' \arrow[d, "f"] \\ & Z \end{tikzcd}$$

and $\mathbb{Q}$-divisors C on Z such that

$$K_{X''}+B''\sim_{\mathbb{Q}} f^*C.$$

Hence we have

$$n(K_X+B)\leq \dim Z\leq 2.$$

If $n(K_X+B)=2$, by Proposition 4.3, we have $\kappa(K_X+B)=2$. If $n(K_X+B)=1$, then by Theorem 2.8, we get a nef reduction map of K_X+B, $g:X\to Z'$. Then g is an equidimensional morphism since Z' is a normal curve and g is proper over the generic point of Z'. By Theorem 2.11, we have $(K_X+B)|_G\sim_{\mathbb{Q}} 0$, where G is the generic fiber of g. Hence by Lemma 2.10, K_X+B descends to an ample divisor on Z'. Therefore, K_X+B is semi-ample.

If $n(K_X+B)=0$, then K_X+B is numerically trivial. By Theorem 2.12, there exists a $(K_X+B-\lfloor B\rfloor)$-MMP which terminates. Since $\lfloor B\rfloor>0$, this MMP terminates with a Mori fiber space

$$\begin{tikzcd} X \arrow[r, dashed] & Y \arrow[d, "f'"] \\ & Z'' \end{tikzcd}$$

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There are $\mathbb{Q}$-divisors C' on Z'', B_Y on Y such that B_Y is the birational transform of B on Y and

$$K_Y + B_Y \sim_{\mathbb{Q}} f'^* C'.$$

Now, by Theorem 2.20, we can take a dlt modification

$$\mu : (Y', B_{Y'}) \to (Y, B_Y).$$

Note that $\lfloor B_{Y'} \rfloor$ dominates Z'' since f' only contract curves which have positive intersections with $\lfloor B_Y \rfloor$. Since $(K_{Y'} + B_{Y'})|_{\lfloor B_{Y'} \rfloor}$ is semi-ample by Theorem 2.21, we deduce that C', and hence $K_X + B$ are semi-ample by Lemma 2.5. ∎

As a corollary, we have the following result on termination of flips.

THEOREM 4.5. *Let (X,B) be a projective lc threefold pair defined over a perfect field k of characteristic $p > 3$ such that $K_X + B$ is pseudo-effective. Then every sequence of $(K_X + B)$-flips terminates. In particular, any $(K_X + B)$-MMP terminates with a minimal model.*

Proof. By Theorem 4.4, we have $\kappa(K_X + B) \geq 0$. Then the proposition follows from Theorem 2.18. ∎

§5. Abundance conjecture for lc threefold pairs

In this section, we show the abundance for lc threefold pairs whose Kodaira dimension ≥ 1. To be precise, we prove the following result.

THEOREM 5.1. *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic > 3. If $K_X + B$ is nef and $\kappa(X, K_X + B) \geq 1$, then $K_X + B$ is semi-ample.*

5.1 Preparation

Before proving Theorem 5.1, we make some preparations.

LEMMA 5.2. *Let X be a normal projective variety of dimension 3 over an algebraically closed field, and let D be a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\kappa(X,D) = 2$. Then $n(X,D) = 2$.*

Proof. We pass to an uncountable algebraically closed field. Consider the Iitaka map of D. After resolving the indeterminacies and replacing D by its pullback, we can assume that the Iitaka map of D is a morphism. Since D is nef and not big, it has to be numerically trivial on all fibers of the Iitaka map. Hence we have $n(X,D) \leq 2$. Then by the equality $\kappa(X,D) \leq n(X,D)$, we have $n(X,D) = 2$. ∎

LEMMA 5.3. *Let X be a normal projective variety of dimension 3 over an uncountable algebraically closed field of characteristic > 0. Assume D is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $\kappa(X,D) = 2$. Then D is endowed with a map $h : X \to Z$ to a normal proper algebraic space Z of dimension 2.*

If, moreover, $D|_G \sim_{\mathbb{Q}} 0$, where G is the generic fiber of h, then there exists a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[d, "h_1"'] & X \arrow[d, "h"] \\ Z_1 \arrow[r, "\psi"] & Z \end{tikzcd}$$

where Z_1 is a smooth projective surface, X_1 is a normal projective threefold, ϕ, ψ are birational morphisms, and $h_1 : X_1 \to Z_1$ is an equidimensional fibration. Moreover, there exists a nef and big $\mathbb{Q}$-divisor D_1 on Z_1 such that $\phi^* D \sim_{\mathbb{Q}} h_1^* D_1$.

Proof. By Lemma 5.2, we have $\kappa(X, D) = n(X, D) = 2$. Hence, by Lemma 2.9, D is endowed with a map $h : X \to Z$ to a normal proper algebraic space Z of dimension 2.

Assume, moreover, $D|_G \sim_{\mathbb{Q}} 0$, where G is the generic fiber of h. By Theorem 2.8, we get a nef reduction map $f : X \dashrightarrow Y$ of D. Resolving the indeterminacies of f and replacing D by its pullback, we can assume that $f : X \to Y$ is a morphism to a normal surface.

Now we apply Lemma 2.10 to f and D. Then we get a commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z' \arrow[r, "\psi'"] & Y \end{tikzcd}$$

with ϕ', ψ' projective birational, and an $\mathbb{Q}$-divisor C on Z' such that $\phi'^* D \sim_{\mathbb{Q}} f'^* C$. Moreover, we can apply the flattening trick [23, Th. 5.2.2] to f', and we get the following commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\phi''"] \arrow[d, "h_1"'] & X' \arrow[r, "\phi'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z_1 \arrow[r, "\psi''"] & Z' \arrow[r, "\psi'"] & Y \end{tikzcd}$$

where Z_1 is a normal projective surface, X_1 is a normal projective threefold, ϕ'', ψ'' are birational morphisms, and $h_1 : X_1 \to Z_1$ is a flat fibration. Replacing Z_1 by a smooth resolution and X_1 by the normalization of main component of the fiber product of h_1 and the resolution, we may assume that Z_1 is smooth.

Let $\phi := \phi' \circ \phi'', D_1 := \psi''^* C$. Then we have

$$\phi^* D \sim_{\mathbb{Q}} h_1^* D_1.$$

Since h_1 only contracts curves which are $\phi^* D$-numerically trivial, we know that the morphism $h \circ \phi : X_1 \to Z$ factors through h_1. In other words, there exists a natural map $\psi : Z_1 \to Z$ making the following diagram commutative:

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[d, "h_1"'] & X \arrow[d, "h"] \\ Z_1 \arrow[r, "\psi"] & Z \end{tikzcd}$$

This completes the proof of the lemma. ∎

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5.2 The case of $\kappa(K_X+B)=2$

In this subsection, we focus on the case of $\kappa(K_X+B)=2$, which is the most difficult case.

Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3 such that K_X+B is nef and $\kappa(K_X+B)=2$. We pass to an uncountable base field. After replacing (X,B), we can assume that (X,B) is $\mathbb{Q}$-factorial and dlt by Theorem 2.20. Then one of the following cases holds:

Case I: $K_X+B-\varepsilon\lfloor B\rfloor$ is not pseudo-effective for any rational $\varepsilon>0$.

Case II: $K_X+B-\varepsilon\lfloor B\rfloor$ is pseudo-effective for any sufficiently small rational $\varepsilon>0$.

Note that by Lemma 5.3, K_X+B is endowed with a map $h: X\to Z$ to a normal proper algebraic space Z of dimension 2. We will run several MMP which are (K_X+B)-trivial. It is clear that every step of such construction is still over Z.

5.2.1. Proof of Case I

In this part, we prove Case I (see Proposition 5.6). More precisely, we first prove that $\lfloor B\rfloor$ must dominate Z in this case. Then we deduce the semi-ampleness of K_X+B by adjunction.

Lemma 5.4. *Let $\phi: Z'\to Z$ be a birational morphism from a $\mathbb{Q}$-factorial projective normal surface to a normal proper algebraic space of dimension 2. Assume that S is an effective Weil divisor on Z'. Then we can take a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A such that $A\geq S$ and $A\cdot E=0$ for any curve E which is ϕ-exceptional.*

Proof. We will write $A=S+H+\sum_\alpha a_\alpha C_\alpha$, where H is a sufficiently ample effective divisor such that $S+H$ is ample, $C_\alpha,\alpha\in I=\{1,2,\dots,r\}$ are all ϕ-exceptional curves and a_α are some nonnegative rational numbers. It is clear that $A\geq S$. We only need to choose appropriate $a_\alpha\geq 0$ such that $A\cdot E=0$ for any curve E which is ϕ-exceptional.

Note that

$$\begin{aligned} & A\cdot C_\beta=0,\beta\in I\\ \iff\qquad & (\sum_\alpha a_\alpha C_\alpha)\cdot C_\beta=-(S+H)\cdot C_\beta,\beta\in I\\ \iff & [C_\beta\cdot C_\alpha]_{\alpha,\beta\in I}[a_\alpha]_{\alpha\in I}=[-(S+H)\cdot C_\beta]_{\beta\in I},\end{aligned}$$

where $[C_\beta\cdot C_\alpha]_{\alpha,\beta\in I}$ is a matrix with element $C_\beta\cdot C_\alpha$ at row β and column α, and $[a_\alpha]_{\alpha\in I},[-(S+H)\cdot C_\beta]_{\beta\in I}$ are column vectors with elements $a_\alpha,-(S+H)\cdot C_\beta$ at rows α,β, respectively. Since $-(S+H)\cdot C_\beta<0$ for $\beta\in I$, to get a solution of $[a_\alpha]_{\alpha\in I}$ with $a_\alpha>0$ we only need to prove that the symmetric matrix $[C_\beta\cdot C_\alpha]_{\alpha,\beta\in I}$ is negative definite.

Consider a resolution of singularities $\phi': Z''\to Z'$. We first prove that the proposition holds for the morphism $\phi\circ\phi': Z''\to Z$. Let $C'_\alpha,\alpha\in J$ be all $\phi\circ\phi'$-exceptional curves. Since $\phi\circ\phi'$ is a contraction, for any closed point $x\in Z$, $(\phi\circ\phi')^{-1}(x)$ is connected. Hence, different connected components of $\bigcup_{\alpha\in J}C'_\alpha$ maps to different closed points. We apply [1, Th. 4.5] to the morphism $\phi\circ\phi'$, then we know that the intersection matrix of any connected component of $\bigcup_{\alpha\in J}C'_\alpha$ is negative definite. Note that the intersection matrix of $\bigcup_{\alpha\in J}C'_\alpha$ is the direct sum of intersection matrices of all connected components of $\bigcup_{\alpha\in J}C'_\alpha$. Hence, the intersection matrix of $\bigcup_{\alpha\in J}C'_\alpha$ is negative definite.

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To prove that $[C_\beta \cdot C_\alpha]_{\alpha,\beta\in I}$ is negative definite, we only need to check $\phi'^*C_\alpha, \alpha \in I$ are linearly independent. This is clear since we have $\phi'^*C_\alpha = \tilde{C_\alpha} + E_\alpha$, where $\tilde{C_\alpha}$ are birational transforms of C_α and E_α are ϕ'-exceptional $\mathbb{Q}$-divisors. ∎

PROPOSITION 5.5. *Let (X, B) be a $\mathbb{Q}$-factorial projective dlt threefold pair over an algebraically closed field k of characteristic > 3 with $\kappa(K_X + B) = 2$. Assume $K_X + B$ is nef, and it is endowed with a map $h : X \to Z$. If $K_X + B - \varepsilon\lfloor B\rfloor$ is not pseudo-effective for any rational $\varepsilon > 0$, then $\lfloor B\rfloor$ dominates Z.*

Proof. We first prove the case when X is terminal. Since $K_X + B - \varepsilon\lfloor B\rfloor$ is not pseudo-effective for any rational $\varepsilon > 0$, $K_X + (1-\varepsilon)B$ is not pseudo-effective for any rational $\varepsilon > 0$. Then by Definition 2.16, we can run a K_X-MMP which is $(K_X + B)$-trivial. By Lemma 2.19, it terminates with a pair (X', B') since X is terminal. Moreover, since $\kappa(K_X + (1-\varepsilon)B) = \kappa(K_{X'} + (1-\varepsilon)B')$ and $K_X + (1-\varepsilon)B$ is not pseudo-effective for any small rational $\varepsilon > 0$, $K_{X'} + (1-\varepsilon)B'$ is not nef for any small rational $\varepsilon > 0$ by Theorem 4.4. Hence this MMP terminates with a Mori fiber space.

$$\begin{tikzcd} X \arrow[r, dashed, "f"] & Y \arrow[d, "h'"] \\ & Z' \end{tikzcd}$$

Denote the birational transform of B on Y by B_Y. Note that $K_Y + B_Y$ is endowed with a map $h_Y : Y \to Z$ and h_Y factors through h' since h' only contracts curves which are $(K_Y + B_Y)$-trivial. In other words, we have a commutative diagram:

$$\begin{tikzcd} & Y \arrow[dl, "h'"'] \arrow[dr, "h_Y"] & \\ Z' \arrow[rr, "\phi"] & & Z. \end{tikzcd}$$

Note that h' is equidimensional, Z' is $\mathbb{Q}$-factorial and ϕ is a birational map. Applying Lemma 5.4 to ϕ, we get a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor A on Z' such that $A \geq h'(\lfloor B_Y\rfloor)$ and $h'(F) \cdot A = 0$ for any h_Y-exceptional divisor F. Note that

$$\kappa(K_Y + B_Y - \lfloor B_Y\rfloor + ah'^*A) \geq \kappa(K_Y + B_Y) = \kappa(K_X + B) = 2$$

for some integer $a > 0$. Hence, there exists an effective $\mathbb{Q}$-divisor

$$M \sim_{\mathbb{Q}} K_Y + B_Y - \lfloor B_Y\rfloor + ah'^*A$$

such that $M \cdot C = (K_Y + B_Y - \lfloor B_Y\rfloor) \cdot C$ for any curve C in the fiber of h_Y. In other words, flips of a $(K_Y + B_Y - \lfloor B_Y\rfloor)$-MMP which is $(K_Y + B_Y)$-trivial are all M-flips. Therefore, by Theorem 2.18, a $(K_Y + B_Y - \lfloor B_Y\rfloor)$-MMP which is $(K_Y + B_Y)$-trivial terminates with a Mori fiber space

$$\begin{tikzcd} Y \arrow[r, dashed, "f'"] & Y' \arrow[d, "h''"] \\ & Z'' \end{tikzcd}$$

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such that $f'_*(\lfloor B_Y \rfloor)$ dominates Z'' by Lemma 2.17. Note that $K_{Y'}+B_{Y'}$ is endowed with a map $h_{Y'}: Y' \to Z$ and $h_{Y'}$ factors through h'' since h'' only contracts curves which are $(K_{Y'}+B_{Y'})$-trivial. Therefore $f'_*(\lfloor B_Y \rfloor)$, and hence $\lfloor B \rfloor$ dominate Z.

Now we turn to the general case. By Theorem 2.20, we can take a dlt modification $\mu: (X'',B'') \to (X,B)$ such that (X'',B'') is $\mathbb{Q}$-factorial and dlt, and X'' is terminal. If $K_X+B-\varepsilon\lfloor B \rfloor$ is not pseudo-effective for any rational $\varepsilon>0$, then $K_{X''}+B''-\varepsilon\lfloor B'' \rfloor$ is not pseudo-effective for any rational $\varepsilon>0$, since

$$\mu_*(K_{X''}+B''-\varepsilon\lfloor B''\rfloor) = K_X+B-\varepsilon\lfloor B \rfloor.$$

By the last paragraph, $\lfloor B'' \rfloor$ dominates Z. Note that $\lfloor B'' \rfloor$ dominates Z if and only if $\lfloor B \rfloor$ dominates Z since Z is of dimension 2 and μ is an isomorphism over a big open subset of X. Hence we have $\lfloor B \rfloor$ dominates Z. ∎

Now we can prove Case I.

Proposition 5.6. *Let (X,B) be a $\mathbb{Q}$-factorial projective dlt threefold pair over an algebraically closed field k of characteristic >3 such that K_X+B is nef and $\kappa(K_X+B)=2$. If $K_X+B-\varepsilon\lfloor B \rfloor$ is not pseudo-effective for any rational $\varepsilon>0$, then K_X+B is semi-ample.*

Proof. We pass to an uncountable base field. By Lemma 5.3, K_X+B is endowed with a map $h: X \to Z$ to a normal proper algebraic space Z of dimension 2. Now by Proposition 5.5, $\lfloor B \rfloor$ dominates Z.

Since $(K_X+B)|_G \equiv 0$, where G is the generic fiber of h and G is of dimension 1, we have $(K_X+B)|_G \sim_{\mathbb{Q}} 0$ by the abundance for curves. Then we can apply Lemma 5.3 to get a commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[d, "h_1"'] & X \arrow[d, "h"] \\ Z_1 \arrow[r, "\psi"] & Z \end{tikzcd}$$

where Z_1 is a smooth projective surface, X_1 is a normal projective threefold, ϕ,ψ are birational morphisms and $h_1: X_1 \to Z_1$ is a fibration. Moreover, there exists a nef and big $\mathbb{Q}$-divisor D_1 on Z_1 such that $\phi^*(K_X+B) \sim_{\mathbb{Q}} h_1^* D_1$. To show K_X+B is semi-ample, it suffices to show D_1 is semi-ample.

Let B_1 be the birational transform of B on X_1. Since $\lfloor B \rfloor$ dominates Z, we have $\lfloor B_1 \rfloor$ dominates Z_1. Moreover, we have $\phi^*(K_X+B)|_{\lfloor B_1 \rfloor}$ is semi-ample since $(K_X+B)|_{\lfloor B \rfloor}$ is semi-ample by Theorem 2.21. Hence by Lemma 2.5, D_1, and hence K_X+B are semi-ample. ∎

5.2.2. Proof of Case II

In this part, we prove Case II (see Proposition 5.10). First, we prove this case when K_X+B is endowed with an equidimensional map $h: X \to Z$. For the general case, we modify the pair (X,B) by running several MMP which are (K_X+B)-trivial so that all h-exceptional prime divisors are connected components of $\lfloor B \rfloor$. Then after further modification we can construct an equidimensional fibration $h_\varepsilon: X \to Z_\varepsilon$ to a normal projective surface. Finally, we descend K_X+B to Z_ε and prove its semi-ampleness.

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PROPOSITION 5.7. *Let D be a nef $\mathbb{Q}$-divisor on X with $\kappa(X,D)=2$, where X is a $\mathbb{Q}$-factorial normal projective threefold over an uncountable algebraically closed field k of characteristic >0. Suppose that D is endowed with an equidimensional map $h:X\to Z$ such that $D|_G\sim_{\mathbb{Q}}0$, where G is the generic fiber of h. Then Z is a projective variety and D is semi-ample.*

Proof. By Lemma 5.3, there is a commutative diagram as following:

$$\begin{array}{ccc} X_1 & \xrightarrow{\phi} & X \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h} \\ Z_1 & \xrightarrow{\psi} & Z \end{array}$$

where Z_1 is a smooth projective surface, X_1 is a normal projective threefold, ϕ,ψ are birational morphisms, and $h_1:X_1\to Z_1$ is an equidimensional fibration. Moreover, there exists a nef and big $\mathbb{Q}$-divisor D_1 on Z_1 such that $\phi^*D\sim_{\mathbb{Q}}h_1^*D_1$.

Since Z is a normal proper algebraic space of dimension 2, there exists an open set $U\subseteq Z$ such that U is a smooth quasi-projective variety and $T:=Z\backslash U$ consists of finitely many closed points on Z. By Lemma 2.10, we have $D|_{h^{-1}(U)}$ is $\mathbb{Q}$-linearly trivial over U since h is equidimensional and $D|_G\sim_{\mathbb{Q}}0$. Now we take a very ample divisor S on X, which does not contain any component of $h^{-1}(T)$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} S_1^{\nu} & \xrightarrow{\text{normalization}} & S_1=\phi^{-1}S & \longrightarrow & X_1 & \xrightarrow{h_1} & Z_1 \\ \downarrow{\scriptstyle \phi_{S^{\nu}}} & & \downarrow{\scriptstyle \phi_S} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ S^{\nu} & \xrightarrow{\text{normalization}} & S & \longrightarrow & X & \xrightarrow{h} & Z. \end{array}$$

The $\mathbb{Q}$-divisor $D|_{S^{\nu}}$ is nef and big. Consider the exceptional locus $\mathbb{E}(D|_{S^{\nu}})$. It is, the union of finitely many D-numerically trivial curves on S^{ν}. Note that $S\cap h^{-1}(T)$ contains no curve by our construction. Hence the image of $\mathbb{E}(D|_{S^{\nu}})$, via the natural map $S^{\nu}\to X$, is contained in finitely many fibers of h over some closed points in U. Therefore $(D|_{S^{\nu}})|_{\mathbb{E}(D|_{S^{\nu}})}$ is semi-ample, and by Theorem 2.6, $D|_{S^{\nu}}$ is semi-ample.

Denote the natural map $S_1^{\nu}\to Z_1$ by σ. Since $D|_{S^{\nu}}$ is semi-ample, we know that

$$\phi_{S^{\nu}}^*D|_{S^{\nu}}\sim_{\mathbb{Q}}\sigma^*D_1$$

is semi-ample. Then by Lemma 2.5, we have D_1 is semi-ample. Hence $\phi^*D\sim_{\mathbb{Q}}h_1^*D_1$ is semi-ample. Again by Lemma 2.5 it follows that D is semi-ample. Moreover, D induces the morphism $h:X\to Z$. Hence Z is projective. ∎

This proposition proves Case II when K_X+B is endowed with an equidimensional map $h:X\to Z$ by letting $D=K_X+B$. In general, this equidimensionality condition may fail. We need to modify the pair (X,B). To do this, we need the following lemmas.

LEMMA 5.8. *Let D be a nef $\mathbb{Q}$-divisor on X with $\kappa(X,D)=2$, where X is a $\mathbb{Q}$-factorial normal projective threefold over an uncountable algebraically closed field k of characteristic >0. Suppose that D is endowed with a map $h:X\to Z$ such that $D|_G\sim_{\mathbb{Q}}0$, where G is the generic fiber of h. Then any h-exceptional prime divisor F is not nef.*

Proof. By Lemma 5.3, we have the following commutative diagram:

$$\begin{tikzcd} X_1 \arrow[r, "\phi"] \arrow[d, "h_1"'] & X \arrow[d, "h"] \\ Z_1 \arrow[r, "\psi"] & Z \end{tikzcd}$$

where Z_1 is a smooth projective surface, X_1 is a normal projective threefold, ϕ,ψ are birational morphisms, and $h_1: X_1 \to Z_1$ is an equidimensional fibration such that, there exists a nef and big $\mathbb{Q}$-divisor D_1 on Z_1 such that $\phi^*D \sim_{\mathbb{Q}} h_1^*D_1$.

First, by the definition of EWM, we have D is numerically trivial on F. Let F_1 be the birational transform of F on X_1. Since D_1 is a nef and big $\mathbb{Q}$-divisor on Z_1, we can write $D_1 \sim_{\mathbb{Q}} A + E_1$ such that A is an ample effective $\mathbb{Q}$-divisor, and E_1 is an effective $\mathbb{Q}$-divisor. Moreover, we can choose A such that $\mathrm{Supp}(h_1^*A)$ doesn't contain any component of $\mathrm{Supp}(\phi^*F) \cup \mathrm{Exc}(\phi)$ since A is ample. We take a $\mathbb{Q}$-effective divisor Δ such that $D \sim_{\mathbb{Q}} \Delta$ and $\phi^*\Delta = h_1^*(A+E_1)$.

Now we take a very ample divisor H_1 on X_1. Since $h_1^*A \cdot F_1 \cdot H_1 > 0$, we have $\mathrm{Supp}(h_1^*A) \cap F_1 \neq \emptyset$. Let A_X be the birational transform of $\mathrm{Supp}(h_1^*A)$ on X. Then its intersection with F is of dimension one by our choice of A. If we take a very ample divisor H on X, it is clear that $A_X \cdot F \cdot H > 0$. Note that $\Delta \cdot F \cdot H = 0$ and $A_X \subseteq \mathrm{Supp}\,\Delta$. It implies that $F \subseteq \mathrm{Supp}\,\Delta$ and $F \cdot F \cdot H < 0$. ∎

LEMMA 5.9. *Let (X,B) be a $\mathbb{Q}$-factorial projective lc threefold pair over an algebraically closed field k of characteristic >3, and let D be an effective $\mathbb{Q}$-divisor such that* $\mathrm{Supp}\,D \subseteq \mathrm{Supp}\,B$. *Assume that K_X+B is nef and $K_X+B-\varepsilon D$ is pseudo-effective for any sufficiently small rational $\varepsilon > 0$. Then we have:*

(1) $\kappa(K_X+B-\varepsilon D) = \kappa(K_X+B)$ for any sufficiently small rational $\varepsilon > 0$,

(2) if $D \subseteq \lfloor B \rfloor$ is a reduced divisor, then any (K_X+B-D)-MMP which is (K_X+B)-trivial terminates with a pair (X',B') such that $K_{X'}+B'-\varepsilon D'$ is nef for any sufficiently small rational $\varepsilon > 0$, where D' is the birational transform of D on X',

(3) if $D \subseteq \lfloor B \rfloor$ is a prime divisor, then D is not contracted by any (K_X+B-D)-MMP which is (K_X+B)-trivial.

Proof. Since $K_X+B-\varepsilon D$ is pseudo-effective for any sufficiently small rational $\varepsilon > 0$, by Theorem 4.4, we have $K_X+B-\varepsilon D$ is effective for any sufficiently small rational $\varepsilon > 0$. Hence there exists an effective $\mathbb{Q}$-divisor $\Delta_\varepsilon \sim_{\mathbb{Q}} K_X+B-2\varepsilon D$ for a sufficiently small rational $\varepsilon > 0$. Then we have

$$K_X+B \sim_{\mathbb{Q}} \Delta_\varepsilon + 2\varepsilon D, K_X+B-\varepsilon D \sim_{\mathbb{Q}} \Delta_\varepsilon + \varepsilon D.$$

This proves (1) since effective divisors with the same support have the same Kodaira dimension.

Assume that $D \subseteq \lfloor B \rfloor$ is a reduced divisor. Note that for any sufficiently small rational $\varepsilon > 0$, $K_X+B-\varepsilon D$ is pseudo-effective and every step of a $(K_X+B-\varepsilon D)$-MMP which is (K_X+B)-trivial is a step of a $(K_X+B-\varepsilon D)$-MMP. We choose a sufficiently small rational $\varepsilon_0 > 0$. By Theorem 4.5, we have a $(K_X+B-\varepsilon_0 D)$-MMP which is (K_X+B)-trivial terminates with a pair (X',B') such that $K_{X'}+B'-\varepsilon D'$ is nef for any sufficiently small rational $\varepsilon > 0$, where D' is the birational transform of D on X'. Since any (K_X+B-D)-

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MMP which is (K_X+B)-trivial is a $(K_X+B-\varepsilon_0 D)$-MMP which is (K_X+B)-trivial, we have (2) holds.

Assume, moreover, that D is a prime divisor. By (2), a (K_X+B-D)-MMP which is (K_X+B)-trivial terminates with a pair (X',B') such that $K_{X'}+B'-\varepsilon D'$ is nef for any sufficiently small rational $\varepsilon>0$, where D' is the birational transform of D on X'. We take a common resolution of X and X'

$$\begin{tikzcd} & W \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ X \arrow[rr, dashed, "f"] & & X'. \end{tikzcd}$$

Note that since every step of a (K_X+B-D)-MMP which is (K_X+B)-trivial is a step of a (K_X+B-D)-MMP, we have

$$\phi_1^*(K_X+B-D) \sim_{\mathbb{Q}} \phi_2^*(K_{X'}+B'-D')+E,$$

where E is an effective ϕ_2-exceptional $\mathbb{Q}$-divisor. It implies that

$$-\phi_1^*D-E \sim_{\mathbb{Q}} \phi_2^*(K_{X'}+B'-D')-\phi_1^*(K_X+B).$$

Applying the negativity lemma (see [20, Lem. 3.39]) to ϕ_2, we know that

$$-\phi_{2*}\phi_1^*D \neq 0.$$

Hence D is not contracted by f, that is, (3) holds. □

Now we can prove Case II.

Proposition 5.10. *Let (X,B) be a $\mathbb{Q}$-factorial projective dlt threefold pair over an algebraically closed field k of characteristic >3 such that K_X+B is nef and $\kappa(K_X+B)=2$. If $K_X+B-\varepsilon\lfloor B\rfloor$ is pseudo-effective for any sufficiently small rational $\varepsilon>0$, then K_X+B is semi-ample.*

Proof. We pass to an uncountable base field. By Proposition 5.6, K_X+B is endowed with a map $h: X\to Z$ to an algebraic space Z of dimension 2.

Step 1. We contract all h-exceptional prime divisors which have empty intersection with $\lfloor B\rfloor$.

Let F be a h-exceptional prime divisor such that $F\cap\lfloor B\rfloor=\emptyset$, then we can choose a sufficiently small rational ε such that $(X,B+\varepsilon F)$ is still dlt. Note that by Lemma 5.8, we have $K_X+B+\varepsilon F$ is not nef since K_X+B is numerically trivial on F. We run a $(K_X+B+\varepsilon F)$-MMP as follows.

For the first step, the extremal ray is (K_X+B)-numerically trivial since any curve which is $(K_X+B+\varepsilon F)$-negative must be contained in F. If it is a divisorial contraction, then F is contracted and the process terminates. Otherwise, we get a flip

$$\mu:(X,B+\varepsilon F) \dashrightarrow (X^+,B^++\varepsilon F^+)$$

such that $F^+\neq 0$. Note that $K_{X^+}+B^++F^+$ is still not nef. By Theorem 4.5, the process must terminate, hence F is contracted after finitely many steps. Since at every step, we only contract (K_X+B)-trivial curves, we can replace (X,B) by the output of this process.

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Moreover, since the number of h-exceptional prime divisors is finite, we can repeat this process until every h-exceptional divisor intersects $\lfloor B \rfloor$.

From now on, we can assume that every h-exceptional divisor intersects $\lfloor B \rfloor$.

Step 2. We reduce the proposition to the case when all h-exceptional prime divisors are connected components of $\lfloor B \rfloor$.

To this end, let $S \subseteq \lfloor B \rfloor$ be a prime divisor such that there exists a h-exceptional divisor F whose intersection with S is of dimension one. By Definition 2.16, we run a (K_X+B-S)-MMP which is (K_X+B)-trivial. By Lemma 5.9, it terminates with a pair (X_1,B_1) such that $K_{X_1}+B_1-\varepsilon S_1$ is nef for any sufficiently small rational $\varepsilon>0$, where S_1 is the birational transform of S on X_1. Moreover, $S_1 \neq 0$.

After replacing $(X,B),S$ by $(X_1,B_1),S_1$ ((X,B) may no longer be dlt), we can assume that $K_X+B-\varepsilon S$ is nef for any sufficiently small rational $\varepsilon>0$. Let F be a h-exceptional prime divisor such that it has non-empty intersection with S. Since K_X+B is numerically-trivial on F, we have $-S$ is nef on F, which implies that $S=F$. It is to say that after this process, there is no h-exceptional divisor F whose intersection with S is of dimension one.

Since the number of h-exceptional prime divisors is finite and it decreases strictly under the above process, we can repeat this process until there is no prime divisor $S \subseteq \lfloor B \rfloor$ such that there exists a h-exceptional divisor F whose intersection with S is of dimension one.

From now on, we can assume that all h-exceptional prime divisors are connected components of $\lfloor B \rfloor$.

Step 3. We further modify (X,B) and construct an equidimensional fibration $h_\varepsilon : X \to Z_\varepsilon$.

First, let F_h be the reduced h-exceptional divisor and run a (K_X+B-F_h)-MMP which is (K_X+B)-trivial by Definition 2.16. After replacing (X,B) by the output of this process, we can assume that $K_X+B-\varepsilon F_h$ is nef for any sufficiently small rational $\varepsilon>0$ as at Step 2.

We choose a sufficiently small rational $\varepsilon>0$. Note that by Lemma 5.9, we have $\kappa(K_X+B-\varepsilon F_h)=\kappa(K_X+B)=2$. Hence by Lemma 5.3, $K_X+B-\varepsilon F_h$ is endowed with a map $h_\varepsilon : X \to Z_\varepsilon$. We claim that there exists a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "h_\varepsilon"'] & & Z \\ & Z_\varepsilon \arrow[ur, "\psi_\varepsilon"'] & \end{tikzcd}$$

We only need to prove that any curve contracted by h_ε is contracted by h. Let C_1 be a curve contracted by h_ε, that is, $(K_X+B-\varepsilon F_h)\cdot C_1=0$.

If $C_1 \cap F_h=\emptyset$, then $(K_X+B-\varepsilon F_h)\cdot C_1=0$ implies $(K_X+B)\cdot C_1=0$. Hence C_1 is contracted by h. If $C_1\cap F_h \neq \emptyset$ and $C_1 \not\subseteq F_h$, then we have $C_1\cdot F_h>0$. But $K_X+B-2\varepsilon F_h$ is nef as well, that is,

$$(K_X+B-2\varepsilon F_h)\cdot C_1 = -\varepsilon F_h \cdot C_1 \geq 0.$$

We obtain a contradiction. Finally, if $C_1 \subseteq F_h$, then C_1 is always contracted by h.

We prove that h_ε is actually equidimensional. By the above diagram, we know that exceptional divisors of h_ε have to be exceptional divisors of h. Hence, all h_ε-exceptional divisors are supported in F_h. If F is a prime h_ε-exceptional divisor, we have both K_X+B

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and $K_X+B-\varepsilon F_h$ are numerically trivial on F, and hence F_h is numerically trivial on F, which is impossible since F is not nef by Lemma 5.8 and F is a connected component of F_h.

Step 4. Descend K_X+B to Z_ε and prove its semi-ampleness.

By Proposition 5.7, we have $K_X+B-\varepsilon F_h$ is semi-ample and Z_ε is a projective variety. Moreover, by Lemma 2.10, K_X+B descends to a nef and big divisor D_ε on Z_ε since h_ε is equidimensional and Z_ε is $\mathbb{Q}$-factorial by [25, Prop. 3.3].

By the projection formula for any curve $\Gamma \subseteq \mathbb{E}(D_\varepsilon)$, we have K_X+B is numerically trivial on $h_\varepsilon^{-1}(\Gamma)$. However, by our assumption, $h_\varepsilon^{-1}(\Gamma)$ has to be contained in F_h. Hence it is clear that

$$\mathbb{E}(D_\varepsilon) \subseteq h_\varepsilon(F_h).$$

Since h_ε is equidimensional, we have $h_\varepsilon^{-1}(h_\varepsilon(F_h))$ is the union of finitely many prime divisors. All these prime divisors are exceptional divisors of h since $\psi_\varepsilon \circ h_\varepsilon(F_h)$ is of dimension 0. Hence, we have

$$h_\varepsilon^{-1}(\mathbb{E}(D_\varepsilon)) \subseteq h_\varepsilon^{-1}(h_\varepsilon(F_h)) = F_h.$$

We take a dlt modification $g:(X',B') \to (X,B)$ such that g only extracts prime divisors E with discrepancies $a(E,X,B)=-1$ by [4, Lem. 7.7] and [12]. Then we have

$$(h_\varepsilon \circ g)^{-1}(\mathbb{E}(D_\varepsilon)) \subseteq g^{-1}(F_h) \subseteq g^{-1}(\lfloor B \rfloor) \subseteq \lfloor B' \rfloor.$$

Since $(K_{X'}+B')|_{\lfloor B' \rfloor}$ is semi-ample by Theorem 2.21, we have

$$(K_{X'}+B')|_{(h_\varepsilon \circ g)^{-1}(\mathbb{E}(D_\varepsilon))}$$

is semi-ample. Then by [5, Lem. 7.1], we have D_ε, and hence K_X+B are semi-ample. ∎

5.3 Proof of Theorem 5.1

Proof. Case of $\kappa(X,K_X+B)=3$: In this case, K_X+B is nef and big, hence the proposition holds by [26, Th. 1.1] and [12].

Case of $\kappa(X,K_X+B)=2$: After replacing (X,B) by its dlt modification, we can assume that (X,B) is a $\mathbb{Q}$-factorial dlt pair by Theorem 2.20. Then the proposition follows from Propositions 5.6 and 5.10.

Case of $\kappa(K_X+B)=1$: The proof is similar to the case of $\kappa(K_X+B)=2$ but easier.

After replacing (X,B) by its dlt modification, we can assume that (X,B) is a $\mathbb{Q}$-factorial dlt pair and X is terminal by Theorem 2.20. Then we have either:

(1): $K_X+B-\varepsilon\lfloor B \rfloor$ is not pseudo-effective for any rational $\varepsilon>0$, or

(2): $K_X+B-\varepsilon\lfloor B \rfloor$ is pseudo-effective for any sufficiently small rational $\varepsilon>0$.

In the case of (1), since $K_X+B-\varepsilon\lfloor B \rfloor$ is not pseudo-effective for any rational $\varepsilon>0$, $K_X+(1-\varepsilon)B$ is not pseudo-effective for any rational $\varepsilon>0$. Then by Definition 2.16, we can run a K_X-MMP which is (K_X+B)-trivial. By Lemma 2.19, it terminates with a pair (X',B') since X is terminal. Moreover, since

$$\kappa(K_X+(1-\varepsilon)B) = \kappa(K_{X'}+(1-\varepsilon)B')$$

and $K_X+(1-\varepsilon)B$ is not pseudo-effective for any small rational $\varepsilon>0$, $K_{X'}+(1-\varepsilon)B'$ is not nef for any small rational $\varepsilon>0$ by Theorem 4.4. Hence this K_X-MMP which is

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(K_X+B)-trivial terminates with a Mori fiber space. Then we have $n(K_X+B)\leq 2$. By Proposition 4.3, $n(K_X+B)=1$ since $\kappa(K_X+B)=1$. Then a nef reduction map of K_X+B, which exists by Theorem 2.8, is an equidimensional fibration to a normal curve. Hence we can descend K_X+B to an ample divisor on the curve by Lemma 2.10.

In the case of (2), by Definition 2.16, we run a $(K_X+B-\lfloor B\rfloor)$-MMP which is (K_X+B)-trivial which terminates by Lemma 5.9, and replace (X,B) by the output. (X,B) may no longer be dlt and X may no longer be terminal. However, we can assume that $K_X+B-\varepsilon\lfloor B\rfloor$ is nef and $(X,B-\varepsilon\lfloor B\rfloor)$ is klt for any sufficiently small rational $\varepsilon>0$. By Lemma 5.9, we have

$$\kappa(K_X+B-\varepsilon\lfloor B\rfloor)=\kappa(K_X+B)=1$$

for any sufficiently small rational $\varepsilon>0$. We choose a sufficiently small rational $\varepsilon>0$ such that $K_X+B-2\varepsilon\lfloor B\rfloor$ is nef and $\kappa(K_X+B-\varepsilon\lfloor B\rfloor)=1$. Then by Theorem 3.9, $|m(K_X+B-\varepsilon\lfloor B\rfloor)|$ induces a fibration $h':X\to Z'$ for a sufficiently divisible positive integer m since $(X,B-\varepsilon\lfloor B\rfloor)$ is klt. Denote the generic fiber of h' by G. By Theorem 2.11, $(K_X+B-2\varepsilon\lfloor B\rfloor)|_G$ is semi-ample. Note that $(K_X+B-\varepsilon\lfloor B\rfloor)|_G\sim_{\mathbb{Q}}0$. It implies that $(K_X+B-2\varepsilon\lfloor B\rfloor)|_G\sim_{\mathbb{Q}}0$, and hence $(K_X+B)|_G\sim_{\mathbb{Q}}0$. Then by Lemma 2.10, K_X+B descends to an ample divisor on Z'. Hence K_X+B is semi-ample. ∎

§6. Applications

In this section, we complete the proofs of the statements in the introduction.

THEOREM 6.1. *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. Then the log canonical ring*

$$R(K_X+B)=\oplus_{m=0}^{\infty}H^0(\lfloor m(K_X+B)\rfloor)$$

is finitely generated.

Proof. If $\kappa(K_X+B)=0$ or $-\infty$, the assertion is trivial. Otherwise, we have $\kappa(K_X+B)\geq 1$. After replacing (X,B) by its log minimal model by Theorem 2.12, we can assume that K_X+B is nef. Then the assertion follows from Theorem 5.1. ∎

THEOREM 6.2. *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is nef and $n(X,K_X+B)\leq 2$, then K_X+B is semi-ample.*

Proof. Case of $n(K_X+B)=0$: By Theorem 4.4, we have $\kappa(K_X+B)\geq 0$. Hence we have

$$\kappa(K_X+B)=n(K_X+B)=0.$$

Therefore $K_X+B\sim_{\mathbb{Q}}0$.

Case of $n(K_X+B)=1$: Let $\phi:X\dashrightarrow Z$ be a nef reduction map, which exists by Theorem 2.8. Since Z is a normal curve, X is normal and ϕ is proper over the generic point μ of Z, we have ϕ is indeed a morphism. Note that $(K_X+B)|_G\sim_{\mathbb{Q}}0$ by Theorem 2.11, where G is the generic fiber of ϕ. Since ϕ is equidimensional, we have $K_X+B\sim_{\mathbb{Q}}f^*A$ for an ample divisor on Z by Lemma 2.10. Hence K_X+B is semi-ample.

Case of $n(K_X+B)=2$: By Proposition 4.3, we have $\kappa(K_X+B)=2$. Then the proposition follows from Theorem 5.1. ∎

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THEOREM 6.3. *Let (X,B) be a projective lc threefold pair over an algebraically closed field k of characteristic >3. If K_X+B is nef and* $\dim \mathrm{Alb}(X)\neq 0$*, then K_X+B is semi-ample.*

Proof. After replacing (X,B) by its dlt modification, we can assume that (X,B) is a $\mathbb{Q}$-factorial dlt pair and X is terminal by Theorem 2.20. Moreover, by Theorems 4.4 and 5.1, we can assume that $\kappa(K_X+B)=0$. By Definition 2.16, we run a K_X-MMP which is (K_X+B)-trivial, which terminates by Lemma 2.19 since X is terminal.

If it terminates with a Mori fiber space, then we have $n(K_X+B)\leq 2$. Then the semi-ampleness of K_X+B follows from Theorem 6.2.

Otherwise, by Lemma 2.17, this K_X-MMP which is (K_X+B)-trivial terminates with a pair (X',B') such that $K_{X'}+(1-\varepsilon)B'$ is nef for any sufficiently small rational $\varepsilon>0$. Note that for any sufficiently small rational $\varepsilon>0$ we have $(X',(1-\varepsilon)B')$ is klt since $(X,(1-\varepsilon)B)$ is klt, and

$$\kappa(K_{X'}+(1-\varepsilon)B')=\kappa(K_{X'}+B')=\kappa(K_X+B)=0$$

by Lemma 5.9. Moreover, $\dim \mathrm{Alb}(X')\neq 0$ since $\dim \mathrm{Alb}(X)\neq 0$. Hence, by Theorem 3.9, $K_{X'}+(1-\varepsilon)B'$ is $\mathbb{Q}$-linearly trivial for any sufficiently small rational $\varepsilon>0$. Then $K_{X'}+B'$, and hence K_X+B are $\mathbb{Q}$-linearly trivial. ∎

THEOREM 6.4. *Let k be an algebraically closed field of characteristic >3. Assume we have:*

(1) abundance for terminal threefolds over k holds, and

(2) any effective nef divisor D on any klt Calabi–Yau threefold pair (Y,Δ) ((Y,Δ) is klt and $K_Y+\Delta\sim_{\mathbb{Q}}0$) over k is semi-ample.

Then the abundance conjecture for threefold pairs over k holds. In particular, the abundance conjecture for klt threefold pairs over k implies the abundance conjecture for lc threefold pairs over k.

Proof. Let (X,B) be a projective lc threefold pair over k such that K_X+B is nef. After replacing (X,B) by its dlt modification, we can assume that (X,B) is a $\mathbb{Q}$-factorial dlt pair and X is terminal by Theorem 2.20. Moreover, by Theorems 4.4 and 5.1, we can assume that $\kappa(K_X+B)=0$.

By Corollary 2.15, we run a K_X-MMP with scaling of B. It terminates by Lemma 2.19 since X is terminal. Hence we have a following sequence:

$$(X_0,B_0):=(X,B)\overset{\mu_1}{\dashrightarrow}(X_1,B_1)\overset{\mu_2}{\dashrightarrow}\cdots\overset{\mu_r}{\dashrightarrow}(X_r,B_r)$$

such that μ_i are $K_{X_{i-1}}$-MMP which are $(K_{X_{i-1}}+\lambda_{i-1}B_{i-1})$-trivial, where λ_i are the smallest numbers such that $K_{X_i}+\lambda_iB_i$ are nef and $\lambda_0>\lambda_1>\cdots>\lambda_r$. Moreover, (X_r,B_r) is the output of the K_X-MMP with scaling of B.

If (X_r,B_r) is a minimal model, then K_{X_r} is nef. By (1), K_{X_r} is semi-ample. Note that

$$\kappa(X_r,K_{X_r})\leq\kappa(X_r,K_{X_r}+B_r)=\kappa(X,K_X+B)=0.$$

Hence $K_{X_r}\sim_{\mathbb{Q}}0$. Since $K_{X_r}+\lambda B_r$ is nef for any $\lambda_{r-1}>\lambda>\lambda_r=0$, we have B_r is nef on X_r. By (2), we have B_r is semi-ample, and hence $B_r=0$ since

$$\kappa(B_r)=\kappa(K_{X_r}+B_r)=\kappa(K_X+B)=0.$$

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It implies that $B=0$ by a standard argument using the negativity lemma (see the proof of Lemma 5.9 for example). Hence $K_X+B=K_X\sim_{\mathbb{Q}}0$ by (1).

Otherwise, (X_r,B_r) is a Mori fiber space. Then we have $n(X_r,K_{X_r}+\lambda_rB_r)\le 2$, where $\lambda_r>0$. Hence $K_{X_r}+\lambda_rB_r$ is semi-ample by Theorem 6.2. Moreover, $K_{X_r}+\lambda_rB_r\sim_{\mathbb{Q}}0$ since

$$\kappa(X_r,K_{X_r}+\lambda_rB_r)\le\kappa(X_r,K_{X_r}+B_r)=\kappa(X,K_X+B)=0.$$

If $\lambda_r=1$, then $\lambda_0=\lambda_r=1$. It is to say that $K_X+B=K_{X_r}+\lambda_rB_r\sim_{\mathbb{Q}}0$. Therefore, we can assume that $\lambda_r<1$. Then we have $K_{X_r}+\lambda B_r$ is nef for any $\lambda_{r-1}>\lambda>\lambda_r$, and hence B_r is nef on X_r. By (2), we have B_r is semi-ample, and hence $B_r=0$ since

$$\kappa(B_r)=\kappa(K_{X_r}+B_r)=\kappa(K_X+B)=0.$$

It is impossible since $\lambda_r>0$. In conclusion, we have K_X+B is semi-ample. ∎

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References

[1] M. Artin, *Algebraic spaces*, Uspekhi Mat. Nauk **26** (1971), no. 1, 181–205.
[2] T. Bauer, F. Campana, T. Eckl, S. Kebekus, T. Peternell, S. Rams, T. Szemberg, and L. Wotzlaw, "A reduction map for nef line bundles" in *Complex geometry*, Springer, Berlin, 2002, pp. 27–36.
[3] B. Bhatt, L. Ma, Z. Patakfalvi, K. Schwede, K. Tucker, J. Waldron, and J. Witaszek, *Globally-regular varieties and the minimal model program for threefolds in mixed characteristic*, Publ. Math. Inst. Hautes Etudes Sci. **138** (2023), 1–159.
[4] C. Birkar, "Existence of flips and minimal models for 3-folds in char p" in *Annales Scientifiques de l'École Normale supérieure*, Vol. 49, Societe Mathematique de France, 2016, pp. 169–212.
[5] C. Birkar and J. Waldron, *Existence of Mori fibre spaces for 3-folds in char p*, Adv. Math. **313** (2017), 62–101.
[6] P. Cascini, H. Tanaka, and X. Chenyang, *On base point freeness in positive characteristic*, Ann. Sci. Ec. Norm. Sup´er. **48** (2015), no. 5, 1239–1272.
[7] O. Das and C. D. Hacon, *On the adjunction formula for 3-folds in characteristic* $p>5$, *Math. Z.* **284** (2016), no. 1, 255–269.
[8] O. Das and J. Waldron, *On the abundance problem for 3-folds in characteristic* $p>5$, *Math. Z.* **292** (2019), no. 3, 937–946.
[9] O. Das and J. Waldron, *On the log minimal model program for threefolds over imperfect fields of characteristic* $p>5$, *J. Lond. Math. Soc.* **106** (2022), no. 4, 3895–3937.
[10] Y. Gongyo, Y. Nakamura, and H. Tanaka, *Rational points on log Fano threefolds over a finite field*, J. Eur. Math. Soc. **21** (2019), no. 12, 3759–3795.
[11] C. Hacon and J. Witaszek, *On the relative minimal model program for threefolds in low characteristics*, Peking Math. J. **5** (2022), no. 2, 365–382.
[12] C. Hacon and J. Witaszek, *The minimal model program for threefolds in characteristic 5*, Duke Math. J. **171** (2022), no. 11, 2193–2231.
[13] C. Hacon and X. Chenyang, *On the three dimensional minimal model program in positive characteristic*, J. Am. Math. Soc. **28** (2015), no. 3, 711–744.
[14] N. Hara, *Classification of two-dimensional f-regular and f-pure singularities*, Adv. Math. **133** (1998), no. 1, 33–53.
[15] K. Hashizume, Y. Nakamura, and H. Tanaka, *Minimal model program for log canonical threefolds in positive characteristic*, Math. Res. Lett. **27** (2017), 1003–1024.
[16] Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. **79** (1985), no. 3, 567–588.
[17] S. Keel, *Basepoint freeness for nef and big line bundles in positive characteristic*, Ann. Math. **149** (1999), 253–286.

[18] S. Keel, K. Matsuki, and J. McKernan, *Log abundance theorem for threefolds*, Duke Math. J. **75** (1994), no. 1, 99–119.
[19] J. Kollár, *Rational curves on algebraic varieties*, Vol. 32, Springer Science & Business Media, Berlin, 2013.
[20] J. Kollár, J. Kollár, S. Mori, C. H. Clemens, and A. Corti, *Birational geometry of algebraic varieties*, Vol. 134, Cambridge University Press, Cambridge, 1998.
[21] R. K. Lazarsfeld, *Positivity in algebraic geometry I: Classical setting: Line bundles and linear series*, Vol. 48, Springer, Berlin, 2017.
[22] Q. Posva, *Abundance for slc surfaces over arbitrary fields*, Épijournal Géom. Algébrique **7** (2023).
[23] M. Raynaud and L. Gruson, *Criteres de platitude et de projectivité*, Invent. Math. **13** (1971), no. 1, 1–89.
[24] H. Tanaka, *Abundance theorem for surfaces over imperfect fields*, Math. Z. **295** (2020), no. 1, 595–622.
[25] J. Waldron, *Finite generation of the log canonical ring for 3-folds in char p*, Math. Res. Lett. **24** (2017), no. 3, 933–946.
[26] J. Waldron, *The lmmp for log canonical 3-folds in characteristic*, Nagoya Math. J. **230** (2018), 48–71.
[27] J. Witaszek, *On the canonical bundle formula and log abundance in positive characteristic*, Math. Ann. **381** (2021), no. 3, 1309–1344.
[28] C. Xu and L. Zhang, *Nonvanishing for* 3*-folds in characteristic* $p > 5$. *Duke Math. J.* **168** (2019), no. 7, 1269–1301.
[29] L. Zhang, *Abundance for non-uniruled 3-folds with non-trivial Albanese maps in positive characteristics*, J. Lond. Math. Soc. **99** (2019), no. 2, 332–348.
[30] L. Zhang, *Abundance for 3-folds with non-trivial Albanese maps in positive characteristic*, J. Eur. Math. Soc. **22** (2020), no. 9, 2777–2820.

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