Viva 'Vis-viva'

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Introduction

Long live the vis-viva equation. There is sometimes more than one way of telling a story with the same ending, and this is particularly true in the field of applied mathematics where there are often different ways of obtaining the same result to a given problem, so that what might be lost or not appreciated in one approach can be found and appreciated in another. It is the purpose here to illustrate this by presenting a well-known example drawn from the field of orbital dynamics, namely the development of what is called the Vis-viva equation. This equation is simply an expression relating the square of the velocity of an orbiting object, for example a planet orbiting a sun, to orbit parameters and scientific constants. It is a standard workhorse equation that is used extensively today by orbit control specialists wishing to determine and affect velocities of spacecraft orbiting significantly larger masses, and it was developed from work carried out centuries ago by Gottfried Leibniz (1646 – 1716). The first story will simply acknowledge the equation and how it arose. The second story will present an alternative approach based on calculus, trigonometry, algebra and computations set against the backdrop of an ellipse geometry. This second story leads to the vis-viva equation in disguise, so to speak, and examples of the speeds of two planets, Earth and Mercury, orbiting the Sun will be discussed. Where relevant, the nomenclature of orbit dynamics will be acknowledged. The discussion will involve primarily a planet's orbit velocity of translation and detailed considerations will not be given to effects due to its speed of rotation.

The first story

The vis-viva equation arose from considerations of energy conservation and Kepler's laws of planetary motion. Googling the words 'vis-viva equation' will produce a raft of connections to the subject in addition to YouTube videos of lectures on the topic. It is

$$V^2 = \frac{GM}{a}\left(\frac{2}{\rho_s} - 1\right), \tag{1}$$

where V denotes the orbit velocity in metres per second (m/s). The parameter G ('big G') is a universal gravitational constant and M is the mass of the Sun. Values for them can be found on internet web sites such as Wikipedia. The parameter a (in metres) is the length of the semi-major axis of the elliptical orbit and ρ_s is the normalised length of the radius vector from the Sun to the orbit point of interest; it is normalised with respect to the length a. If the length is stated in kilometres, its magnitude in (1) must be multiplied by 1000. With respect to values for the parameters M and G, there is a consensus about the former which indicates that

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$M = 1.9885 \times 10^{30}$ kg. However, the value recommended by NIST (USA: National Institute of Standards and Technology) for the latter seems to vary in time, for example, from between about 6.672×10^{-11} Nm^2/kg^2 (1973) and 6.6743×10^{-11} Nm^2/kg^2 (2018) [1]. Whatever the improving status about its true value, due perhaps to an increase in sophistication of measurement set-ups to determine it, experimentalists will doubtless continue with efforts to further refine its status. A value for the parameter a in (1) can be found from perihelion and aphelion data shown in a Table later. It is 1.495975×10^{11} m. For a circular orbit, $\rho_s = 1$ in (1) and V then reduces to the form $\sqrt{GM/a}$. With the above higher value for the universal gravitational constant and with values for the mass of the Sun and the said radius, it follows that $V = 29.79$ km/s. If instead, the lower value for G is assumed, a value for V of 29.78 km/s is obtained. This is the value for the mean orbital velocity cited in [1]. The issue of determining an appropriate value for big G will be raised again later. The product GM is sometimes denoted by the single parameter μ, known as the 'standard gravitational parameter', not to be confused with its later notational use here.

If eccentricity effects are to be considered, so that the orbit is not circular, the parameter ρ_s in (1) will vary with orbit position. Such matters will be considered hereunder to determine a representative description of a planet's speed throughout its orbit, which brings us to the next story.

The second story

To start this story, it is pertinent first to determine, by another very simple appreciation, the velocity of the Earth about the Sun were the orbit to be circular. By Kepler's law, the radius vector from the Sun, then at the centre of a circle whose radius is given above, will sweep out equal areas in equal times and thus the orbital speed is everywhere the same, i.e. a simple constant velocity figure of magnitude ($2\pi a/365.256$ km/ day), i.e. 29.78 km/ s, which agrees with the above figure obtained from the Vis-viva equation with an appropriate value for G. It should be appreciated that a sidereal year of length 365.256 Gregorian days has been assumed. The figure of 365.256 days is the time taken for planet Earth to orbit the Sun just once when its position is referenced to distant fixed stars. It is different to the figure of 365 days that is assumed commonly in the Gregorian calendar. If the latter figure is used in the calculations, a slightly different result will be obtained.

If orbit eccentricity $e\,(= \sin\alpha)$ is considered, the simple approach must be eschewed. To this end, it is appropriate to consider first the typical schematic for a general elliptic orbit, such as that of Figure 1, where lengths have been normalised to that of the semi-major axis. In this Figure, the point P represents a planet on an elliptic path at the end of a radius vector from the focus, where the Sun is positioned. The usual rectangular Cartesian coordinate system (x-horizontal, y-vertical) is assumed.

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At any instant, the radius vector (RV) from the Sun subtends an angle ϕ with respect to the ellipse semi-major axis and the normalised length of this vector is, as before, denoted by ρ_s. In this notation, the usual parametric representation for the point P on the ellipse will be, in normalised form, $(\cos\theta, \cos\alpha \sin\theta)$. In the nomenclature of orbital mechanics, the parameter angle θ is known as the eccentric anomaly and the angle ϕ is known as the true anomaly.

Coincidentally, a normal (N) through the point P is shown, and this subtends similarly an angle ψ. An appreciation of this normal is required because the radius of curvature (of length R say) of the ellipse is centred on this line, and the orbit speed at P will then be $R\frac{d\psi}{dt}$, which it is the purpose here to determine in the light of Kepler's second law. In this respect the treatment here differs from that normally associated with the development of the vis-viva equation, where velocity considerations are not usually addressed via a radius of curvature approach. This facilitates the development of an alternative expression for the velocity of P, one that involves the sidereal period.

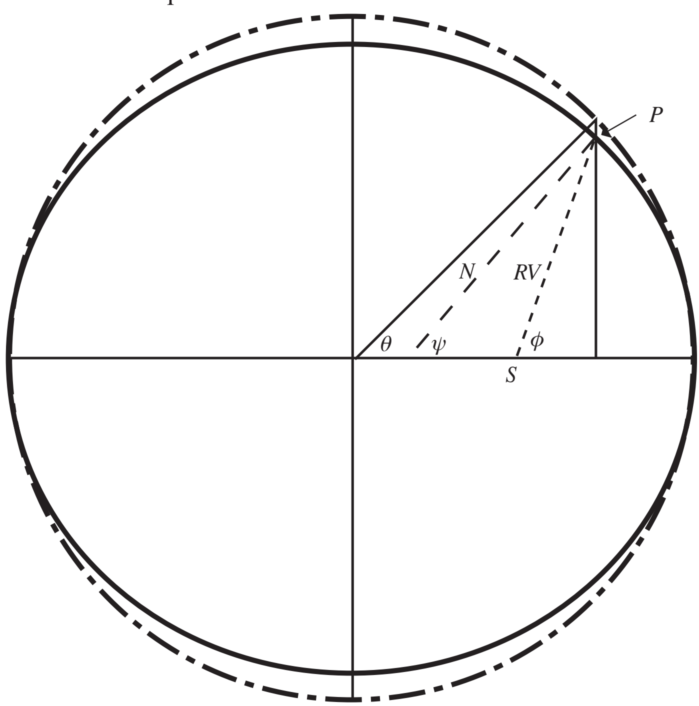

FIGURE 1: Normalised ellipse plus auxiliary circle showing a planet at point P on the ellipse with a radius vector (RV) to the Sun at a focus S, an inward normal (N) and associated angles of interest

Before exploiting Kepler's 2nd law, it will be necessary to highlight some relationships between the ellipse parameters that will be required later.

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Ellipse parameters relationships and preliminary results

Various equations relating the ellipse quantities evidenced in Figure 1 will be required for use later in expressions involving the planet's velocity. They can be established using standard trigonometric principles and are

$$\rho_S = 1 - \sin\alpha\cos\theta,$$
$$\tan\phi = \frac{\cos\alpha\sin\theta}{\cos\theta - \sin\alpha}. \tag{2}$$

With reference to Figure 1, the first equation in (2) can be derived by applying Pythagoras's theorem to the right-angled triangle formed by the hypotenuse line (RV), the perpendicular line from P to the semi-major axis and the relevant portion of that axis enclosed by these two lines. The second equation arises from trigonometric relationships in the aforesaid triangle. From the second equation in (2), it is possible to express the ellipse parameter θ explicitly in terms of α and ϕ by invoking half-angle formulae to obtain a quadratic equation in $\tan\frac{1}{2}\theta$ which can be solved to produce the two roots

$$\tan\frac{\theta}{2} = \mu\tau \text{ and } \frac{-\mu}{\tau} \tag{3}$$

where $\mu = \dfrac{1 - \tan\frac{1}{2}\alpha}{1 + \tan\frac{1}{2}\alpha}$ and $\tau = \tan\frac{1}{2}\phi$.

The root of interest here is the first one as it refers to the top half of the ellipse (as shown in the Figure). The second root is simply the first one modified, with ϕ replaced by $(\phi - \pi)$, and it refers to the bottom half of the ellipse where the radius vector can be extended to meet the ellipse again at some point.

It remains to determine expressions for the radius of curvature of the ellipse and the slope of the normal at the point P, both of which will be used later. An expression for the former can be found on the internet by googling the words *radius of curvature at a point on an ellipse*. Otherwise, it can be established from first principles. Whichever, it can be secured in a form that involves only the ellipse parameter θ together with the usual ellipse dimensions a and b, and if it is denoted by R, it is

$$R = \frac{\left(b^2\cos^2\theta + a^2\sin^2\theta\right)^{3/2}}{ab}.$$

If lengths are normalised, as proposed here, to that of the semi-major axis a, this expression can be rewritten in the form

$$\frac{R}{a} = \frac{\left(1 - \sin^2\alpha\cos^2\theta\right)^{3/2}}{\cos\alpha}.$$

Finally, by observation of Figure 1, the slope of the normal of the

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ellipse at the point P is $\tan\psi$. However, from analytical geometry, the equation to the normal is well known and the slope from using the parameters adopted here is simply $\frac{\tan\theta}{\cos\alpha}$. These two forms for the slope must be the same, and so it follows that $\tan\psi = \frac{\tan\theta}{\cos\alpha}$. As the point P moves around the ellipse, the parameters θ and ϕ will vary with time, t say, and this equivalent slope equation can be differentiated with respect to time to obtain the result

$$\frac{d\psi}{dt} = \frac{\cos\alpha}{1 - \sin^2\alpha\cos^2\theta}\frac{d\theta}{dt}.$$

This result can be combined with the above radius of curvature expression to obtain an expression V for the tangential velocity $R\frac{d\psi}{dt}$:

$$V = a\sqrt{1 - \sin^2\alpha\cos^2\theta}\,\frac{d\theta}{dt}. \qquad (4)$$

Having secured preliminary results, attention now will turn to the ramifications of Kepler's 2nd law.

Kepler's 2nd law

This law states that the radius vector (RV of Figure 1) sweeps out equal areas in equal times as the planet point P traverses its elliptic path around the Sun. Thus it is necessary to develop first a formula for the normalised area A (normalised to the square of the semi-major axis length) swept out by the radius vector of Figure 1, starting when P is at the perihelion end of the semi-major axis and finishing where it is as shown in the Figure. Symbolically, this can be written

$$A = \int_0^\phi \frac{\rho_s^2}{2}\,d\phi = \int_{LL}^{UL} \frac{\rho_s^2}{2}\frac{d\phi}{d\theta}\,d\theta$$

where the limits of integration for the variable θ, LL and UL, can be obtained from (3). For example, when $\phi = 0$, $LL = 0$ and for the upper limit, $UL = 2\tan^{-1}\left(\mu\tan\frac{1}{2}\phi\right)$. With respect to the θ-integration, the term $\frac{d\phi}{d\theta}$ in the integrand can be determined by differentiation of the first equation in (3), leading to the result

$$\frac{d\phi}{d\theta} = \frac{\cos\alpha}{1 - \sin\alpha\cos\theta}.$$

The other term in the integrand, ρ_s^2, can be replaced by its expression given in the first equation of (2) to produce the result

$$A = \frac{\cos\alpha}{2}\int_0^\theta (1 - \sin\alpha\cos\theta)^2\frac{1}{1 - \sin\alpha\cos\theta}d\theta.$$

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This integrates to

$$A = \frac{\cos\alpha}{2}\left[\theta - \sin\alpha\sin\theta\right]. \quad (5)$$

Since $\theta = 2\tan^{-1}\left(\mu\tan\left(\frac{1}{2}\phi\right)\right)$, this equation can also be presented in an alternative form that involves implicitly the radius vector angle ϕ. It allows for the calculation of the ellipse parameter angle θ (and hence ϕ) relevant to a specified swept out area. For example, the area of an ellipse, here normalised to the square of its semi-major axis, is simply $\pi\cos\alpha$. Thus, for the Earth say, the area swept out in one sidereal day is $\left(\frac{\pi\cos\alpha}{365.256}\right)$, according to Kepler's 2nd law. Therefore, after t such days the area swept out is $\left(\frac{\pi t\cos\alpha}{365.256}\right)$ and the corresponding θ-value can be determined from (5), written now in the more general form

$$\left[\theta - \sin\alpha\sin\theta\right] = \frac{2\pi t}{N}, \quad (6)$$

where N denotes the number of days in the sidereal year for whichever planet is under consideration. In astrophysics and orbital dynamics, equation (6) is known as Kepler's time equation. It relates the terms involving the eccentric anomaly on the left-hand side to a term on the right-hand side involving the time variable t with, in this case, a planet's number of days in a sidereal year.

If t is a uniformly spaced input, this equation can be solved numerically to find the corresponding non-uniformly spaced θ-value. The business is tedious but straightforward for a given planet, and it is well suited to manipulations on a spreadsheet, where numerical routines such as a Newton-Raphson method or any other iterative method can be employed. In this way it is possible to compile a day-by-day picture of θ-values that correspond to regular t-values throughout the year. It seems also to be the route that astrophysicists adopt in their investigations. However, if θ $\left(0° \leqslant \theta \leqslant 360°\right)$ is the regularly spaced input, a corresponding value for t is determined directly from (6) without recourse to iterative procedures. In this way it is possible to compile a picture of t-values throughout the year but such values will not generally be uniformly spaced.

If the ellipse eccentricity is small, it is possible to develop small argument approximations to the parameters of interest. These are quoted in the appendix but only some of them will be used in the later commentary.

Velocity considerations

In order to use (4), it will be necessary to determine the time rate of change of the ellipse parameter angle θ. This can be obtained by differentiating equation (6) with respect to time, to obtain the result

$$\frac{d\theta}{dt} = \frac{2\pi}{N}\cdot\frac{1}{1 - \sin\alpha\cos\theta}.$$

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If this is substituted into (4), the square of the tangential velocity can then be written in the form

$$V^2 = \left(\frac{2\pi a}{N}\right)^2 \left(\frac{1 + \sin\alpha\cos\theta}{1 - \sin\alpha\cos\theta}\right) = \left(\frac{2\pi a}{N}\right)^2 \left(\frac{2}{1 - \sin\alpha\cos\theta} - 1\right) \quad (7)$$

which, from the first equation in (2), can be rewritten as

$$V^2 = \left(\frac{2\pi a}{N}\right)^2 \left(\frac{2}{\rho_s} - 1\right), \quad (8)$$

where a is in km and the velocity V is in km/day. It can be expressed in km/s by dividing by 24×3600. If a ϕ-dependent result for the velocity is required, it will suffice to replace θ in (7) by $\theta = 2\tan^{-1}\left(\frac{\mu\tan\phi}{2}\right)$. Equation (8) has been developed without recourse to energy considerations, and its form is identical to that of (1). On reconciling the two equations, it follows that $GM = 4\pi^2 a^3 / N^2$. Typically, a and N are in metres and seconds respectively. This result encapsulates Kepler's 3rd law, namely, that N^2 is directly proportional to a^3. It reduces to

$$GM = \frac{3125}{5832} \cdot \frac{\pi^2 a^3}{N^2} \quad (9)$$

where a is in kilometres and when N is the number of sidereal days in the orbit year. This equation can serve as a useful check on the accuracy of values assumed for the various input parameters and scientific constants. For example, in the case of the Earth, if inputs for M, N and a are as prescribed above, it follows that the numerical value for G should be 6.673981×10^{-11}.

Comparisons of outputs from the two different equations

Data on the orbit parameters of Earth and Mercury can be found on the internet [2]. This data is shown in Table 1.

Planet	Perihelion (km)	Aphelion (km)	Sidereal Period
Earth	1.47095×10^8	1.521×10^8	365.256
Mercury	4.6×10^7	6.9818×10^7	87.969

TABLE 1: Planet parameters

The eccentricities $e\,(\equiv \sin\alpha)$ and semi-major axis lengths a are given by $a = \frac{1}{2}(d_A + d_P)$ and $e = \dfrac{d_A - d_P}{d_A + d_P}$, where d_A and d_P denote respectively the distances of aphelion and perihelion shown in Table 1.

Curves showing the calculated orbit speed for each planet throughout the sidereal year, based on equations (7) or (8), are shown in Figures 2 and 3 below.

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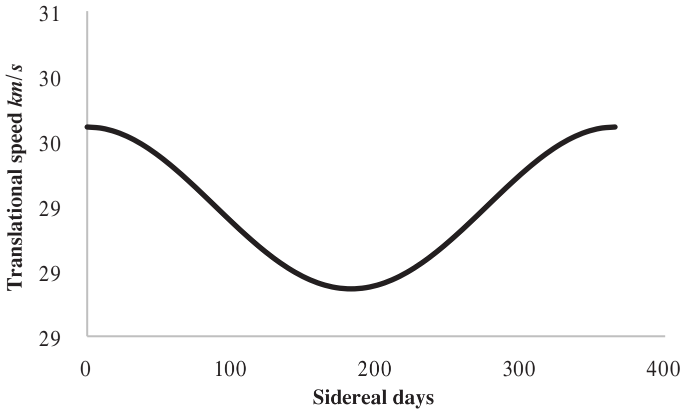

FIGURE 2: Earth orbit speed per day

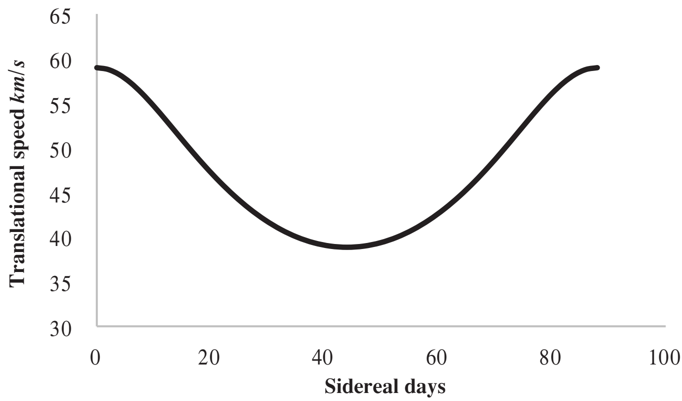

FIGURE 3: Mercury orbit speed per day

Approximations

On replacing ρ_s in (8) by the small argument approximation for it given in the appendix, it is possible to secure an approximate equation for the velocity on any given sidereal day provided the orbit eccentricity is small. Quite simply,

$$V \approx \left(\frac{2\pi a}{N}\right)\left(1 + \alpha \cos\lambda\right) \tag{10}$$

where $\lambda = \dfrac{2\pi t}{N}$. It is a straightforward matter to verify on a spreadsheet that results from the approximation of (10) serve the purpose very well in the case of an Earth orbit velocity, because the eccentricity is very small; the results are in close agreement with those obtained from either (1) or (8). In

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the case of Mercury's orbit, it can be appreciated in the same way that the approximations are not as good but can nonetheless produce “back of the envelope” results, suitable perhaps when great accuracy is not required. The fine details are left as an exercise for the reader to appreciate.

Discussion

A classic example drawn from the realms of applied mathematics related to the velocity of planets orbiting the Sun has been examined, and two expressions for orbit speeds were presented. The first, the Vis-viva equation, was derived elsewhere from energy considerations, and was assumed. The other equation, which led to a similar result, was developed differently using mathematics that should be familiar to undergraduates and possibly sixth formers. This Article has highlighted too the relationship between various parameters (equation (9)) that can serve as a useful check on values ascribed to them. It has been pointed out also that, in the case of small orbit eccentricities, useful approximations can be employed to simplify the sums without significantly compromising accuracy. Whilst consideration has been given to just two planets in the solar system, it would be a relatively simple matter to extend the arguments to all planets. To this end it would be necessary to secure input information about them, such as that which can be found, for example, in [2].

References

1. https://en.wikipedia.org/wiki/Earth's_orbit, accessed November 2023.
2. https://nssdc.gsfc.nasa.gov/planetary/factsheet, accessed November 2023.

Appendix

Approximations to the parameters ϕ, ρ_s (normalised) and θ (see the main body of the text for the expressions involving them and other associated terms), when the eccentricity of the associated ellipse is assumed to be small, can be obtained in the usual way by exploiting small argument expansions. Specifically, writing $\lambda = \frac{2\pi t}{N}$, it can be shown that, neglecting terms $O\left(\alpha^2\right)$.

$$\theta \approx \lambda + \alpha \sin\lambda, \ \phi \approx 2\theta - \lambda, \text{ and } \rho_s \approx 1 - \alpha \cos\lambda.$$

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