A Forcing Axiom Deciding the Generalized Souslin Hypothesis

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Abstract. We derive a forcing axiom from the conjunction of square and diamond, and present a few applications, primary among them being the existence of super-Souslin trees. It follows that for every uncountable cardinal λ, if λ^{++} is not a Mahlo cardinal in Gödel's constructible universe, then $2^\lambda = \lambda^+$ entails the existence of a λ^+-complete λ^{++}-Souslin tree.

1 Introduction

1.1 Trees

A *tree* is a partially ordered set $(T, <_T)$ with the property that for every $x \in T$, the downward cone $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. The order type of $(x_\downarrow, <_T)$ is denoted by $\mathrm{ht}(x)$, and the *α-th level* of the tree is the set

$$T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}.$$

The tree $(T, <_T)$ is said to be *χ-complete* if for every chain $C \subseteq T$ of size less than χ, there is $x \in T$ such that $C \subseteq x_\downarrow \cup \{x\}$.

If κ is a regular uncountable cardinal, then a *κ-Aronszajn tree* is a tree of size κ having no chains or levels of size κ, and a *κ-Souslin tree* is a tree of size κ having no chains or antichains of size κ. As tree levels are antichains, any κ-Souslin tree is a κ-Aronszajn tree.

In 1920, Mikhail Souslin [24] asked whether every ccc, dense, complete linear ordering with no endpoints is isomorphic to the real line. (Here, *ccc* is a consequence of separability, asserting that every pairwise-disjoint family of open intervals is countable.) In [12], Kurepa showed that a negative answer to Souslin's question is equivalent to the existence of an $\aleph_1$-Souslin tree. Attempts to settle the question by constructing an $\aleph_1$-Souslin tree proved unsuccessful but did lead to Aronszajn's construction of an $\aleph_1$-Aronszajn tree, which is described in [12]. The question remained open until it was proved in [9,11,23,26] that, in contrast to the existence of $\aleph_1$-Aronszajn trees, the existence of $\aleph_1$-Souslin trees is independent of the usual axioms of set theory (ZFC).

As these objects proved incredibly useful and important, a systematic study of their consistency and interrelation was carried out. Following standard conventions, we let TP_κ stand for the nonexistence of κ-Aronszajn trees (the *tree property* at κ), SH_κ stand

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for the nonexistence of κ-Souslin trees (the *Souslin Hypothesis* at κ), and CH_λ stand for $2^\lambda = \lambda^+$. Two early results read as follows.

Theorem 1.1 (Specker [25])   *For every cardinal λ, CH_λ implies the failure of $TP_{\lambda^{++}}$.*

By a *cardinal*, we always mean an infinite cardinal.

Theorem 1.2 (Jensen [10])   *In Gödel's constructible universe, L, for every regular uncountable cardinal κ, the following are equivalent:*

- TP_κ;
- SH_κ;
- *κ is a weakly compact cardinal.*

We remind the reader that a cardinal κ is *weakly compact* if and only if it is uncountable and Ramsey's theorem holds at the level of κ, *i.e.,* every graph of size κ contains a clique or an anticlique of size κ.

Ever since Jensen's result, the general belief has been that the consistency of SH_κ for κ of the form λ^{++} requires the consistency of a weakly compact cardinal. This conjecture is supported by the following later results.

Theorem 1.3 (Mitchell and Silver [15])   *The existence of a regular cardinal λ for which $TP_{\lambda^{++}}$ holds is equiconsistent with the existence of a weakly compact cardinal. In particular, the consistency of a weakly compact cardinal gives the consistency of $\neg CH_\lambda$ together with $SH_{\lambda^{++}}$.*

Theorem 1.4 (Laver and Shelah [14])   *For every cardinal λ, if there is a weakly compact cardinal above λ, then there is a forcing extension by a λ^+-directed closed forcing notion in which CH_λ and $SH_{\lambda^{++}}$ both hold.*

Theorem 1.5 (Rinot [17])   *For every cardinal λ, if CH_λ, CH_{λ^+}, and $SH_{\lambda^{++}}$ all hold, then λ^{++} is a weakly compact cardinal in L.*

Whether the hypotheses of Theorem 1.5 are mutually consistent, relative to any large cardinal assumption, is a major open problem.

In this paper, we are interested in a possible converse for Theorem 1.4. As of now, the best result in this direction gives a lower bound of an inaccessible cardinal. Recall that a cardinal κ is *inaccessible* if it is regular, uncountable, and strong limit, and *Mahlo* if it is inaccessible and has stationarily-many inaccessible cardinals below it. Note also that any weakly compact cardinal has stationarily-many Mahlo cardinals below it.

Theorem 1.6 (Shelah and Stanley [20])   *For every cardinal λ, if CH_λ and $SH_{\lambda^{++}}$ both hold, then λ^{++} is an inaccessible cardinal in L.*

Here, we establish the following theorem.

Theorem A   *For every uncountable cardinal λ, if CH_λ and $SH_{\lambda^{++}}$ both hold, then λ^{++} is a Mahlo cardinal in L.*

The following table provides a clear summary of these results.

Theorem	λ	CH_λ	CH_{λ^+}	$SH_{\lambda^{++}}$	lower bound	upper bound
1.3	regular	✗	✓	✓		weakly compact
1.5	arbitrary	✓	✓	✓	weakly compact	
1.4	arbitrary	✓	✗	✓		weakly compact
1.6	arbitrary	✓	✗	✓	inaccessible	
A	uncountable	✓	✗	✓	Mahlo	

1.2 Combinatorial Constructions

In order to prove Theorem A, we develop a general framework for carrying out combinatorial constructions. It turns out that, in this and other applications, it is often desirable to be able to construct an object of size κ^+, where κ is a regular uncountable cardinal, using approximations to that object of size less than κ. When one attempts to carry out such a construction using only the axioms of ZFC, though, one naturally runs into problems: the construction seems to require κ^+ steps, but the approximations may become too large after only κ steps.

The usual way to attempt to overcome this problem is to assume, in addition to ZFC, certain nice combinatorial features of κ or κ^+. One such feature, whose definition is motivated by precisely such constructions, is the existence of a $(\kappa, 1)$-morass (see [4, §4] or [5, Chapter VIII]). Velleman [30] and Shelah and Stanley [20] present frameworks for carrying out constructions of objects of size κ^+ using a $(\kappa, 1)$-morass. In both instances, these frameworks take the form of forcing axioms that turn out to be equivalent to the existence of morasses.

Another combinatorial assumption that can be helpful in these constructions is the existence of a diamond sequence. In a series of papers on models with second order properties, culminating in a general treatment in [19], Shelah et al. developed a technique for using $\diamondsuit(\kappa)$ to build objects of size κ^+ out of approximations of size $< \kappa$. Ideas from these papers were used by Foreman, Magidor, and Shelah [7] to prove, assuming the consistency of a huge cardinal, the consistency of the existence of an ultrafilter $\mathcal{U}$ on ω_1 such that $|\omega^{\omega_1}/\mathcal{U}| = \aleph_1$, and later by Foreman [6] to prove, again assuming the consistency of a huge cardinal, the consistency of the existence of an $\aleph_1$-dense ideal on $\aleph_2$.

In this paper, we present a framework for constructions of objects of size κ^+ using $\diamondsuit(\kappa)$ and $\square_\kappa^B$, a weakening of $\square_\kappa$ that, unlike $\square_\kappa$ itself, is implied by the existence of a $(\kappa, 1)$-morass. As in [20, 30], our framework takes the form of a forcing axiom. Specifically, in Section 2, we isolate a class of forcing notions $\mathcal{P}_\kappa$, introduce the notion of a *sharply dense system*, and formulate a forcing axiom, $SDFA(\mathcal{P}_\kappa)$, that asserts that for every $\mathbb{P}$ from the class $\mathcal{P}_\kappa$ and every sequence $\langle \mathcal{D}_i \mid i < \kappa \rangle$ of sharply dense systems, there is a filter G on $\mathbb{P}$ that meets each $\mathcal{D}_i$ everywhere.

The last two sections of the paper are devoted to the proof of the following theorem.

Theorem B   *For every regular uncountable cardinal κ, if $\diamondsuit(\kappa)$ and $\square_\kappa^B$ both hold, then so does* $SDFA(\mathcal{P}_\kappa)$.

In Section 3, we give a few simple applications of the forcing axiom $\mathrm{SDFA}(\mathcal{P}_\kappa)$. We open by pointing out that the Cohen forcing $\mathrm{Add}(\kappa, \kappa^+)$ is a member of the class $\mathcal{P}_\kappa$. Then we show that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ entails $\kappa^{<\kappa} = \kappa$ and $\square_\kappa^B$. This has three consequences. First, it shows that our square hypothesis in Theorem B is optimal.

Theorem B′   *Suppose that κ is a regular uncountable cardinal and $\diamond(\kappa)$ holds. Then the following are equivalent:*

- $\square_\kappa^B$ *holds;*
- $\mathrm{SDFA}(\mathcal{P}_\kappa)$ *holds.*

Second, by Shelah's theorem [18] stating that CH_λ entails $\diamond(\lambda^+)$ for every uncountable cardinal λ, it gives cases in which the diamond hypothesis is optimal, as well:

Theorem B″   *For every successor cardinal $\kappa > \aleph_1$, the following are equivalent:*

- $\diamond(\kappa)$ *and $\square_\kappa^B$ both hold;*
- $\mathrm{SDFA}(\mathcal{P}_\kappa)$ *holds.*

Third, it implies that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ entails the existence of a strong stationary coding set, *i.e.,* a stationary subset of $[\kappa^+]^{<\kappa}$ on which the map $x \mapsto \sup(x)$ is injective. This is of interest because the existence of such a set was previously obtained by Shelah and Stanley [21] from their forcing axiom $S_\kappa(\diamond)$, which is equivalent to the existence of a $(\kappa, 1)$-morass with a built-in diamond sequence, and later (though earlier in terms of publication date) by Velleman [29] from the existence of a stationary simplified $(\kappa, 1)$-morass.

Section 4 is dedicated to the study of super-Souslin trees. For a cardinal λ, a λ^{++}-*super-Souslin tree* is a λ^{++}-tree $(T, <_T)$ with a certain highly absolute combinatorial property that ensures that $(T, <_T)$ has a λ^{++}-Souslin subtree in any ZFC extension W of the universe V that satisfies $\mathcal{P}^W(\lambda) = \mathcal{P}^V(\lambda)$ and $(\lambda^{++})^W = (\lambda^{++})^V$. These trees were introduced in a paper by Shelah and Stanley [20], where the existence of super-Souslin trees provided the primary application of the forcing axiom isolated in that paper. In particular, they proved that the existence of a λ^{++}-super-Souslin tree follows from the existence of a $(\lambda^+, 1)$-morass together with CH_λ. In [21,30] the same hypotheses are shown to entail the existence of a λ^{++}-super-Souslin tree that is also λ^+-complete. Here, we prove the following analogous result.

Theorem C   *For every cardinal λ, $\mathrm{SDFA}(\mathcal{P}_{\lambda^+})$ entails the existence of a λ^+-complete λ^{++}-super-Souslin tree.*

By Theorems B and C and the fact that for any super-Souslin tree $(T, <_T)$, there exists some $x \in T$ such that $(x^\uparrow, <_T)$ is Souslin, we obtain the following corollary.

Corollary 1.7   *For every cardinal λ, if $\diamond(\lambda^+)$ and $\square_{\lambda^+}^B$ both hold, then there is a λ^+-complete λ^{++}-Souslin tree.*

Recalling Jensen's theorem [10] stating that if $\square_\kappa$ fails, then κ^+ is a Mahlo cardinal in L, and Shelah's theorem [18] stating that CH_λ entails $\diamondsuit(\lambda^+)$ for every uncountable cardinal λ, we see that Theorem A follows from Corollary 1.

We also obtain a corollary concerning partition relations. Recall that, for ordinals α, β, and γ, the statement $\alpha \to (\beta, \gamma)^2$ asserts that for every coloring $c: [\alpha]^2 \to \{0, 1\}$, either there exists $B \subseteq \alpha$ of order type β that is 0-monochromatic, or there exists $C \subseteq \alpha$ of order type γ that is 1-monochromatic. By a recent theorem of Raghavan and Todorcevic [16], the existence of a κ^+-Souslin tree entails $\kappa^+ \not\to (\kappa^+, \log_\kappa(\kappa^+) + 2)^2$, where $\log_\kappa(\kappa^+)$ denotes the least cardinal ν such that $\kappa^\nu > \kappa$. We thus obtain the following corollary.

Corollary 1.8 *Suppose that λ is an uncountable cardinal. If CH_λ and*

$$\lambda^{++} \to (\lambda^{++}, \lambda^+ + 2)^2$$

both hold, then λ^{++} is a Mahlo cardinal in L.

Note that by a theorem of Erdős and Rado, CH_λ entails $\lambda^{++} \to (\lambda^{++}, \lambda^+ + 1)^2$.

1.3 Notation and Conventions

If x is a set, π is a function, and $x \subseteq \mathrm{dom}(\pi)$, then $\pi``x$ denotes the set $\{\pi(y) \mid y \in x\}$. We write c.o.i. as a shorthand for "continuous, order-preserving injection." In particular, a c.o.i. is a map π from a set of ordinals into the ordinals such that π is continuous, order-preserving, and injective, and, moreover, $\mathrm{dom}(\pi)$ is closed in its supremum. Thus, when we write, for example, "$\pi: y \to \kappa^+$ is a c.o.i.", it is implicit that y is closed in its supremum. For ordinals $\theta < \mu$, let $\binom{\mu}{\theta} := \{\mathrm{Im}(\pi) \mid \pi: \theta \to \mu \text{ is a c.o.i.}\}$; *i.e.,* $\binom{\mu}{\theta}$ consists of all closed copies of θ in μ.

For a set of ordinals x, $\mathrm{otp}(x)$ denotes the order type of x and, for all $i < \mathrm{otp}(x)$, $x(i)$ denotes the unique element α of x such that $\mathrm{otp}(x \cap \alpha) = i$. We write

$$\mathrm{ssup}(x) := \sup\{\alpha + 1 \mid \alpha \in x\}, \quad \mathrm{acc}(x) := \{\alpha \in x \mid \sup(x \cap \alpha) = \alpha > 0\},$$

$$\mathrm{nacc}(x) := x \smallsetminus \mathrm{acc}(x), \quad \mathrm{acc}^+(x) := \{\alpha < \mathrm{ssup}(x) \mid \sup(x \cap \alpha) = \alpha > 0\},$$

$$\mathrm{cl}(x) := x \cup \mathrm{acc}^+(x).$$

By convention, $\mathrm{ssup}(\varnothing) = \sup(\varnothing) = 0$. For sets of ordinals x and y, we write $x \sqsubseteq y$ if and only if y is an end-extension of x, *i.e.,* $y \cap \mathrm{ssup}(x) = x$. For cardinals $\lambda < \mu$, let $E_\lambda^\mu := \{\alpha < \mu \mid \mathrm{cf}(\alpha) = \lambda\}$, let $E_{<\lambda}^\mu := \{\alpha < \mu \mid \mathrm{cf}(\alpha) < \lambda\}$, let $[\mu]^{<\lambda} := \{x \subseteq \mu \mid |x| < \lambda\}$, and let $[\mu]^2 := \{(\alpha, \beta) \mid \alpha < \beta < \mu\}$. Also, let H_μ denote the collection of all sets of hereditary cardinality less than μ.

Throughout the paper, κ stands for an arbitrary regular uncountable cardinal. For simplicity, the reader can assume that $\kappa = \aleph_1$.

2 The Forcing Axiom

We begin by introducing the class $\mathcal{P}_\kappa$ of forcing notions that will be of interest.

Definition 2.1 $\mathcal{P}_\kappa$ consists of all triples $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{Q})$ such that $(\mathbb{P}, \leq_\mathbb{P})$ is a forcing notion, $\mathbb{1}_\mathbb{P} \in \mathbb{Q} \subseteq \mathbb{P}$, and all of the following requirements hold.

(1) **(Realms)** For all $p \in \mathbb{P}$, there is a unique $x_p \in [\kappa^+]^{<\kappa}$, which we refer to as the *realm* of p. The map $p \mapsto x_p$ is a projection from $(\mathbb{P}, \leq_\mathbb{P})$ to $([\kappa^+]^{<\kappa}, \supseteq)$:

    (a) $x_{\mathbb{1}_\mathbb{P}} = \varnothing$;

    (b) for all $q \leq_\mathbb{P} p$, we have $x_q \supseteq x_p$;

    (c) for all $p \in \mathbb{P}$ and $x \in [\kappa^+]^{<\kappa}$ with $x \supseteq x_p$, there is $q \leq_\mathbb{P} p$ with $x_q = x$.

(2) **(Scope)** For all $y \subseteq \kappa^+$, let $\mathbb{P}_y := \{p \in \mathbb{P} \mid x_p \subseteq y\}$ and $\mathbb{Q}_y := \mathbb{Q} \cap \mathbb{P}_y$. Then $\mathbb{P}_\varnothing = \{\mathbb{1}_\mathbb{P}\}$ and $\mathbb{P}_\kappa \subseteq H_\kappa$.

(3) **(Actions of c.o.i.'s)** For every $y \subseteq \kappa^+$ and every c.o.i. $\pi : y \to \kappa^+$, π acts on $\mathbb{P}_y$ in such a way that, for all $p, q \in \mathbb{P}_y$:

    (a) $\pi.p$ is in $\mathbb{P}$ with $x_{\pi.p} = \pi``x_p$, and if $p \in \mathbb{Q}_y$, then $\pi.p$ is in $\mathbb{Q}$;

    (b) $\pi.q \leq_\mathbb{P} \pi.p$ if and only if $q \leq_\mathbb{P} p$;

    (c) if π is the identity map, then $\pi.p = p$;

    (d) if $\pi' : y' \to \kappa^+$ is a c.o.i. with $\mathrm{Im}(\pi) \subseteq y'$, then $\pi'.(\pi.p) = (\pi' \circ \pi).p$;

    (e) if $\pi' : y' \to \kappa^+$ is a c.o.i. with $x_p \subseteq y'$, then $\pi \restriction x_p = \pi' \restriction x_p$ implies that $\pi.p = \pi'.p$.

(4) **(Restrictions)** For all $p \in \mathbb{P}$ and $\alpha < \kappa^+$, there is a unique $\leq_\mathbb{P}$-least condition r such that $x_r = x_p \cap \alpha$ and $p \leq_\mathbb{P} r$. This condition r is referred to as $p \restriction \alpha$. Moreover:

    (a) if $p \in \mathbb{Q}$, then $p \restriction \alpha \in \mathbb{Q}$;

    (b) if $q \leq_\mathbb{P} p$, then $q \restriction \alpha \leq_\mathbb{P} p \restriction \alpha$.

(5) **(Vertical limits)** Suppose that $\xi < \kappa$ and $\langle p_\eta \mid \eta < \xi \rangle$ is a sequence of conditions from $\mathbb{P}$ such that, for all $\eta < \eta' < \xi$, we have $p_\eta = p_{\eta'} \restriction \mathrm{ssup}(x_{p_\eta})$. Then there is a unique condition $p \in \mathbb{P}$ such that $x_p = \bigcup_{\eta < \xi} x_{p_\eta}$ and, for all $\eta < \xi$, $p_\eta = p \restriction \mathrm{ssup}(x_{p_\eta})$. Moreover, if $p_\eta \in \mathbb{Q}$ for all $\eta < \xi$, then $p \in \mathbb{Q}$.

(6) **(Sharpness)** For all $q \in \mathbb{Q}$, x_q is closed in its supremum. Moreover, for all $p \in \mathbb{P}$, there is $q \leq_\mathbb{P} p$ with $x_q = \mathrm{cl}(x_p)$ such that $q \in \mathbb{Q}$.

(7) **(Controlled closure)** Suppose that $\xi < \kappa$ and $\langle q_\eta \mid \eta < \xi \rangle$ is a decreasing sequence of conditions from $\mathbb{Q}$. Let $x := \bigcup_{\eta < \xi} x_{q_\eta}$. Suppose that $\alpha < \mathrm{ssup}(x)$ and that $r \in \mathbb{Q}_{\mathrm{ssup}(x \cap \alpha)}$ is a lower bound for $\langle q_\eta \restriction \alpha \mid \eta < \xi \rangle$. Then there is $q \in \mathbb{Q}$ such that

    (a) $q \restriction \mathrm{ssup}(x \cap \alpha) = r$;

    (b) $x_q = \mathrm{cl}(x_r \cup x)$;

    (c) q is a lower bound for $\langle q_\eta \mid \eta < \xi \rangle$.

(8) **(Amalgamation)** For all $p \in \mathbb{Q}$, $\alpha < \mathrm{ssup}(x_p)$, and $q \in \mathbb{P}_\alpha$ with $q \leq_\mathbb{P} p \restriction \alpha$, we have that p and q have a unique $\leq_\mathbb{P}$-greatest lower bound r. Moreover, it is the case that $x_r = x_q \cup x_p$ and $r \restriction \alpha = q$.

We now introduce the class of families of dense sets that we will be interested in meeting.

Definition 2.2 (Sharply dense set)   Suppose that $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{Q}) \in \mathcal{P}_\kappa$ and D is a non-empty subset of $\mathbb{P}$. Denote $x_D := \bigcap\{x_p \mid p \in D\}$. We say that D is *sharply dense* if and only if for every $p \in \mathbb{P}$, there is $q \in D$ with $q \leq_\mathbb{P} p$ such that $x_q = \mathrm{cl}(x_p \cup x_D)$.

Definition 2.3 (Sharply dense system)   Suppose that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$. We say that $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$ is a *sharply dense system* if and only if there exists an ordinal $\theta_\mathcal{D} < \kappa$ such that $\mathcal{D}$ is of the form $\{D_x \mid x \in \binom{\kappa^+}{\theta_\mathcal{D}}\}$, where for all $x \in \binom{\kappa^+}{\theta_\mathcal{D}}$:

- D_x is sharply dense with $x_{D_x} = x$;
- for every $p \in \mathbb{P}$, and every c.o.i. $\pi \colon y \to \kappa^+$ with $x \subseteq x_p \subseteq y$, we have $p \in D_x$ if and only if $\pi.p \in D_{\pi``x}$.

Definition 2.4   Suppose that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$ and $\mathcal{D}$ is a sharply dense system. We say that a filter G on $\mathbb{P}$ *meets $\mathcal{D}$ everywhere* if and only if , for all $D \in \mathcal{D}$, $G \cap D \neq \varnothing$.

We are now ready to formulate our forcing axiom for sharply dense systems.

Definition 2.5   SDFA$(\mathcal{P}_\kappa)$ is the assertion that, for every $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$ and every collection $\{\mathcal{D}_i \mid i < \kappa\}$ of sharply dense systems, there exists a filter G on $\mathbb{P}$ such that, for all $i < \kappa$, G meets $\mathcal{D}_i$ everywhere.

3 Applications

In this section we present a few applications of SDFA$(\mathcal{P}_\kappa)$. First, let us point out two features of members of the class $\mathcal{P}_\kappa$.

Proposition 3.1   *Suppose that* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$.

(i)   *Then* $(\mathbb{Q}, \leq_{\mathbb{P}})$ *is κ-closed.*

(ii)   *For all* $x \subseteq \kappa^+$, *denote* $D_x := \{q \in \mathbb{Q} \mid x_q \supseteq x\}$. *Then for all* $\theta < \kappa$, $\{D_x \mid x \in \binom{\kappa^+}{\theta}\}$ *is a sharply dense system.*

Proof   (i) Suppose that $\xi < \kappa$ and $\vec{q} = \langle q_\eta \mid \eta < \xi \rangle$ is a decreasing sequence of conditions from $\mathbb{Q}$. Note that if $x := \bigcup_{\eta < \xi} x_{q_\eta}$ is empty, then $\mathbb{1}_{\mathbb{P}}$ is a lower bound for $\vec{q}$, so we may assume that x is nonempty. Since $\mathbb{1}_{\mathbb{P}} \in \mathbb{Q}$ and $\mathbb{P}_0 = \{\mathbb{1}_{\mathbb{P}}\}$, we infer from clause (4) (of Definition 2.1) that $\{q_\eta \upharpoonright 0 \mid \eta < \xi\} = \mathbb{Q}_0 = \{\mathbb{1}_{\mathbb{P}}\}$. So, by clause (7), using $\alpha := 0$ and $r := \mathbb{1}_{\mathbb{P}}$, we infer that $\vec{q}$ admits a lower bound.

Part (ii) can be inferred from clauses (3a) and (6) of Definition 2.1.   ∎

Next, we show that the actions of c.o.i.'s behave as expected with respect to the restriction operation.

Proposition 3.2   *Suppose that* $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$, $p \in \mathbb{P}$, $\alpha \in x_p$, *and* $\pi \colon y \to \kappa^+$ *is a c.o.i. with* $x_p \subseteq y \subseteq \kappa^+$. *Then* $\pi.(p \upharpoonright \alpha) = (\pi.p) \upharpoonright \pi(\alpha)$.

Proof   Let $r := \pi.(p \upharpoonright \alpha)$. Since $p \leq_{\mathbb{P}} p \upharpoonright \alpha$, clause (3b) (of Definition 2.1) implies that $\pi.p \leq_{\mathbb{P}} r$. In addition, by clauses (2a) and (4), and since $\alpha \in y$, we have:

$$x_r = \pi``x_{p \upharpoonright \alpha} = \pi``(x_p \cap \alpha) = \pi``x_p \cap \pi``\alpha = \pi``x_p \cap \pi(\alpha) = x_{\pi.p} \cap \pi(\alpha).$$

This shows that r is a candidate for being $(\pi.p) \upharpoonright \pi(\alpha)$. To finish the proof, fix an arbitrary $q \in \mathbb{P}$ such that $x_q = x_{\pi.p} \cap \pi(\alpha)$ and $\pi.p \leq_{\mathbb{P}} q$. We have to verify that $r \leq_{\mathbb{P}} q$.

Let $\pi' := \{(\delta, \varepsilon) \mid (\varepsilon, \delta) \in \pi\}$, so that π' is a c.o.i. and $\pi' \circ \pi$ and $\pi \circ \pi'$ are the identity maps on their respective domains. Since $\pi.p \leq_{\mathbb{P}} q$ and $x_q \subseteq \mathrm{Im}(\pi)$, and by

clauses (3c) and (3d), we have $p = \pi'.(\pi.p) \leq_{\mathbb{P}} \pi'.q$. Moreover, $x_{\pi'.q} = x_p \cap \alpha$, so, by clause (4), $p \restriction \alpha \leq_{\mathbb{P}} \pi'.q$. Now another application of clauses (3b) and (3c) yields $r \leq_{\mathbb{P}} q$.   ∎

3.1 A Warm-up Example

Let us point out that $\mathbb{P} := \mathrm{Add}(\kappa, \kappa^+)$ belongs to the class $\mathcal{P}_\kappa$. Specifically, $p \in \mathbb{P}$ if and only if p is a function from a subset of $\kappa^+ \times \kappa$ to 2 and $|p| < \kappa$. Let $p \leq_{\mathbb{P}} q$ if and only if $p \supseteq q$. Let $x_p := \{\beta \in \kappa^+ \mid \exists \eta[(\beta, \eta) \in \mathrm{dom}(p)]\}$. Let $\mathbb{Q} := \{p \in \mathbb{P} \mid x_p = \mathrm{cl}(x_p)\}$. Whenever π is a c.o.i. from a subset of κ^+ to κ^+ and $p \in \mathbb{P}_{\mathrm{dom}(\pi)}$, we let $\pi.p := \{((\pi(\beta), \eta), i) \mid ((\beta, \eta), i) \in p\}$. We also let $p \restriction \alpha := \{((\beta, \eta), i) \in p \mid \beta < \alpha\}$. The reader is now encouraged to verify that, with this definition, $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$.

3.2 Cardinal Arithmetic

In this subsection, we identify a simple member of $\mathcal{P}_\kappa$ and use it to prove that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ implies $\kappa^{<\kappa} = \kappa$.

Definition 3.3 $\mathbb{P}$ consists of all pairs $p = (x, f)$ such that the following hold:

(1) $x \in [\kappa^+]^{<\kappa}$;
(2) f is a function satisfying

   (a) $|f| < \kappa$;
   (b) $\mathrm{dom}(f) \subseteq x \times \kappa$;
   (c) $f(\beta, \eta) \subseteq \beta \cap x$ for all $(\beta, \eta) \in \mathrm{dom}(f)$.

The coordinates of a condition $p \in \mathbb{P}$ will often be identified as x_p and f_p, respectively.

Definition 3.4   For all $p, q \in \mathbb{P}$, we let $q \leq_{\mathbb{P}} p$ if and only if $x_q \supseteq x_p$ and $f_q \supseteq f_p$.

Definition 3.5 $\mathbb{Q} := \{p \in \mathbb{P} \mid x_p = \mathrm{cl}(x_p)\}$.

Definition 3.6   Suppose that π is a c.o.i. from a subset of κ^+ to κ^+. For each $p \in \mathbb{P}_{\mathrm{dom}(\pi)}$, we let $\pi.p$ be the condition (x, f) such that

(1) $x = \pi``x_p$;
(2) $f = \{((\pi(\beta), \eta), \pi``z) \mid ((\beta, \eta), z) \in f_p\}$.

Definition 3.7   Suppose that $p \in \mathbb{P}$ and $\alpha < \kappa^+$. Then we define $p \restriction \alpha$ to be the condition (x, f) such that

(1) $x = x_p \cap \alpha$;
(2) $f = \{((\beta, \eta), z) \in f_p \mid \beta < \alpha\}$.

It is readily verified that, with these definitions, $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q})$ is a member of $\mathcal{P}_\kappa$.

Theorem 3.8   *Suppose $\kappa^{<\kappa} > \kappa$. Then $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q})$ witnesses that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ fails.*

Proof   We commence with a simple observation.

Claim 3.8.1   *There exists a cardinal $\lambda < \kappa$ for which $|\binom{\lambda}{\lambda}| > \kappa$.*

Proof   Since κ is regular, we have $\kappa^{<\kappa} = \sum_{\lambda < \kappa} \lambda^\lambda$. So, since $\kappa^{<\kappa} \geq \kappa^+$ and κ^+ is regular, we can fix a cardinal $\lambda < \kappa$ such that $\lambda^\lambda \geq \kappa^+$. For every $A \subseteq \lambda$, let

$$C_A := \mathrm{acc}(\lambda) \cup \{\alpha + 1 \mid \alpha \in A\}.$$

Then $A \mapsto C_A$ is an injection from $\mathcal{P}(\lambda)$ to $\binom{\lambda}{\lambda}$, and we are done.   ■

Fix a cardinal $\lambda < \kappa$ such that $|\binom{\lambda}{\lambda}| > \kappa$. For each $x \in \binom{\kappa^+}{\lambda+1}$, let D_x be the set of all conditions $(x_p, f_p) \in \mathbb{P}$ satisfying

- $x \subseteq x_p$;
- there is $\eta < \kappa$ with $(\max(x), \eta) \in \mathrm{dom}(f_p)$ such that

$$f_p(\max(x), \eta) = x \cap \max(x).$$

Evidently, $\mathcal{D} := \{D_x \mid x \in \binom{\kappa^+}{\lambda+1}\}$ is a sharply dense system.

Towards a contradiction, suppose that $\mathrm{SDFA}(\mathcal{P}_\kappa)$ holds. In particular, there exists a filter G on $\mathbb{P}$ that meets $\mathcal{D}$ everywhere. Let $f := \bigcup_{p \in G} f_p$ so that f is a function from a (possibly proper) subset of $\kappa^+ \times \kappa$ to $\mathcal{P}(\kappa^+)$. Put

$$\Lambda := \{f(\lambda, \eta) \mid \exists \eta < \kappa[(\lambda, \eta) \in \mathrm{dom}(f)]\}.$$

Clearly, $|\Lambda| \leq \kappa$. Finally, let $C \in \binom{\lambda}{\lambda}$ be arbitrary. Since $C \cup \{\lambda\} \in \binom{\kappa^+}{\lambda+1}$, we have $G \cap D_{C \cup \{\lambda\}} \neq \varnothing$, and hence $C \in \Lambda$. It follows that $\binom{\lambda}{\lambda} \subseteq \Lambda$, contradicting the fact that $|\binom{\lambda}{\lambda}| > \kappa \geq |\Lambda|$.   ■

Corollary 3.9 $\mathrm{SDFA}(\mathcal{P}_\kappa)$ *entails* $\kappa^{<\kappa} = \kappa$.

3.3 Baumgartner's Square

In unpublished work, Baumgartner introduced the principle $\square_\kappa^B$, which is a natural weakening of Jensen's $\square_\kappa$ principle.

Definition 3.10   A $\square_\kappa^B$*-sequence* is a sequence $\langle C_\beta \mid \beta \in \Gamma \rangle$ that satisfies

(1) $E_\kappa^{\kappa^+} \subseteq \Gamma \subseteq \mathrm{acc}(\kappa^+)$;
(2) for all $\beta \in \Gamma$, C_β is club in β and $\mathrm{otp}(C_\beta) \leq \kappa$;
(3) for all $\beta \in \Gamma$ and all $\alpha \in \mathrm{acc}(C_\beta)$, we have $\alpha \in \Gamma$ and $C_\alpha = C_\beta \cap \alpha$.

The principle $\square_\kappa^B$ asserts the existence of a $\square_\kappa^B$-sequence.[1]

Some basic facts about $\square_\kappa^B$ can be found in [30], where it goes by the name "weak $\square_\kappa$." In particular, it is shown in [30] that $\square_\kappa^B$ follows from the existence of a $(\kappa, 1)$-morass.

Theorem 3.11   *Suppose that* $\mathrm{SDFA}(\mathcal{P}_\kappa)$ *holds. Then so does* $\square_\kappa^B$.

---

[1]Note that $\square_\kappa^B$ is equivalent to the principle $\square_\kappa(\kappa^+, \sqsubseteq_\kappa)$ from [1, §1].

The rest of this subsection is devoted to proving Theorem 3.11. We must first identify a relevant member of $\mathcal{P}_\kappa$, which will be a slight modification of the poset used to add $\square_\kappa^B$ in [30, §1.3].

Definition 3.12 $\mathbb{P}$ consists of all pairs $p = (x, f)$ satisfying the following:

(1) $x \in [\kappa^+]^{<\kappa}$;
(2) f is a function from x to $\mathcal{P}(x)$ such that for all $\beta \in x$:
    (a) $f(\beta)$ is a closed subset of β (we say that c is a *closed subset of β* if and only if $c \subseteq \beta$ and for every $\alpha < \beta$, $c \cap \alpha \neq \varnothing \Rightarrow \sup(c \cap \alpha) \in c$);
    (b) for all $\alpha \in \mathrm{acc}(f(\beta))$, we have $f(\alpha) = f(\beta) \cap \alpha$.

The coordinates of a condition $p \in \mathbb{P}$ will often be identified as x_p and f_p, respectively.

Definition 3.13   For all $p, q \in \mathbb{P}$, we let $q \leq_{\mathbb{P}} p$ if and only if :

- $x_p \subseteq x_q$;
- for all $\beta \in x_p$, we have $f_p(\beta) \sqsubseteq f_q(\beta)$;
- for all $\beta \in x_p$, if $\sup(x_q \cap \beta) = \beta$, then $f_q(\beta) = f_p(\beta)$.

Definition 3.14 $\mathbb{Q}$ is the set of all conditions $p \in \mathbb{P}$ such that

(1) $x_p = \mathrm{cl}(x_p)$;
(2) for all $\beta \in \mathrm{nacc}(x_p) \smallsetminus \{\min(x_p)\}$, we have $\max(f_p(\beta)) = \max(x_p \cap \beta)$.

In order to show that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$, we must define the actions of c.o.i.'s on $\mathbb{P}$ and a restriction operation.

Definition 3.15   Suppose that π is a c.o.i. from a subset of κ^+ to κ^+. For each $p \in \mathbb{P}_{\mathrm{dom}(\pi)}$, we define $\pi.p$ to be the condition $(x, f) \in \mathbb{P}$ such that

(1) $x = \pi``x_p$;
(2) for all $\alpha \in x_p$, we have $f(\pi(\alpha)) = \pi``f_p(\alpha)$.

Definition 3.16   Suppose that $p \in \mathbb{P}$ and $\alpha < \kappa^+$. Then $p \upharpoonright \alpha$ is the condition $(x, f) \in \mathbb{P}$ such that $x = x_p \cap \alpha$ and $f = f_p \upharpoonright x$.

Naturally, for each $p \in \mathbb{P}$, we let x_p denote the realm of p. With these definitions, it is immediate that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q})$ satisfies clauses (1)–(5) of Definition 2.1. We now verify clauses (6)–(8), in order.

Lemma 3.17   *If $p \in \mathbb{P}$, then there is $q \in \mathbb{Q}$ with $q \leq_{\mathbb{P}} p$ such that $x_q = \mathrm{cl}(x_p)$.*

Proof   Set $x_q := \mathrm{cl}(x_p)$, so that $\mathrm{nacc}(x_q) = \mathrm{nacc}(x_p)$ and $\mathrm{acc}(x_q) \supseteq \mathrm{acc}(x_p)$. Next, define $f_q \colon x_q \to \mathcal{P}(x_q)$ by

$$
f_q(\alpha) := \begin{cases} f_p(\alpha) \cup \{\max(x_q \cap \alpha)\} & \text{if } \alpha \in \mathrm{nacc}(x_q) \smallsetminus \{\min(x_q)\}, \\ \varnothing & \text{if } \alpha \in x_q \smallsetminus x_p, \\ f_p(\alpha) & \text{otherwise.} \end{cases}
$$

It is clear that $q := (x_q, f_q)$ is as desired. ∎

Lemma 3.18 *Suppose that $\xi < \kappa$ and $\langle q_\eta \mid \eta < \xi \rangle$ is a decreasing sequence of conditions from $\mathbb{Q}$. Let $x := \bigcup_{\eta < \xi} x_{q_\eta}$, and suppose that $\alpha < \mathrm{ssup}(x)$ and $r \in \mathbb{Q}_{\mathrm{ssup}(x \cap \alpha)}$ is a lower bound for $\langle q_\eta \restriction \alpha \mid \eta < \xi \rangle$. Then there is $q \in \mathbb{Q}$ such that*

(1) $q \restriction \mathrm{ssup}(x \cap \alpha) = r$;
(2) $x_q = \mathrm{cl}(x_r \cup x)$;
(3) q *is a lower bound for* $\langle q_\eta \mid \eta < \xi \rangle$.

Proof We will construct a condition $q = (x_q, f_q)$ as desired. We are required to let $x_q := \mathrm{cl}(x_r \cup x)$ and to ensure that $f_q \restriction x_r := f_r$. As $x_r \sqsubseteq x_q$, it remains to determine $f_q \restriction (x_q \smallsetminus \mathrm{ssup}(x \cap \alpha))$. We will define $f_q(\beta)$ by recursion on $\beta \in (x_q \smallsetminus \mathrm{ssup}(x \cap \alpha))$, maintaining the hypothesis that $(x_q \cap (\beta+1), f_q \restriction (\beta+1))$ is an element of $\mathbb{Q}$ and a lower bound for $\langle q_\eta \restriction (\beta + 1) \mid \eta < \xi \rangle$. For notational ease, if $\beta \in \mathrm{nacc}(x_q) \smallsetminus \{\min(x_q)\}$, then let $\beta^- := \max(x_q \cap \beta)$.

▸ If $\beta \in \mathrm{acc}(x)$, then fix $\eta_\beta < \xi$ such that $\beta \in x_{q_{\eta_\beta}}$, and let $f_q(\beta) := \bigcup_{\eta \in [\eta_\beta, \xi)} f_{q_\eta}(\beta)$. There are two possibilities to consider here. If there is $\eta^* \in [\eta_\beta, \xi)$ such that $\sup(x_{q_{\eta^*}} \cap \beta) = \beta$, then it follows from Definition 3.13 that $f_q(\beta) = f_{q_{\eta^*}}(\beta)$.

If, on the other hand, there is no such η^*, then the fact that each x_{q_η} is closed in its supremum implies that, for all $\eta \in [\eta_\beta, \xi)$, we have $\beta \in \mathrm{nacc}(x_{q_\eta})$ and hence $\max(f_{q_\eta}(\beta)) = \max(x_{q_\eta} \cap \beta)$. Since $\beta \in \mathrm{acc}(x)$, it then follows that $f_q(\beta)$ is club in β.

▸ If $\beta \in \mathrm{acc}^+(x) \smallsetminus x$, then let $\gamma := \min(x \smallsetminus (\beta+1))$. There is $\eta_\beta < \xi$ such that for all $\eta \in [\eta_\beta, \xi)$, we have $\gamma \in x_{q_\eta}$ and $x_{q_\eta} \cap \beta \neq \varnothing$. For all such η, let $\delta_\eta := \max(x_{q_\eta} \cap \beta)$. It follows that $\sup\{\delta_\eta \mid \eta \in [\eta_\beta, \xi)\} = \beta$ and, for all $\eta \in [\eta_\beta, \xi)$, we have $\max(f_{q_\eta}(\gamma)) = \delta_\eta$. We can therefore let $f_q(\beta) := \bigcup_{\eta \in [\eta_\beta, \xi)} f_{q_\eta}(\gamma)$.

▸ If $\beta \in \mathrm{nacc}(x)$ and $\beta^- \notin x$, then, by the construction in the previous case, we have $f_q(\beta^-) = \bigcup_{\eta < \xi} f_{q_\eta}(\beta)$. We can therefore let $f_q(\beta) := f_q(\beta^-) \cup \{\beta^-\}$.

▸ If $\beta \in \mathrm{nacc}(x)$ and $\beta^- \in x$, then there is $\eta_\beta < \xi$ such that $\{\beta, \beta^-\} \subseteq x_{q_{\eta_\beta}}$. But then, for all $\eta \in [\eta_\beta, \xi)$, we have $f_{q_\eta}(\beta) = f_{q_{\eta_\beta}}(\beta)$ and $\max(f_{q_\eta}(\beta)) = \beta^-$. We can therefore let $f_q(\beta) := f_{q_{\eta_\beta}}(\beta)$.

It is easily verified that q, constructed in this manner, is as desired. ∎

Lemma 3.19 *Suppose that $p \in \mathbb{Q}$, $\alpha < \mathrm{ssup}(x_p)$, $q \in \mathbb{P}_\alpha$, and $q \leq_{\mathbb{P}} p \restriction \alpha$. Then p and q have a $\leq_{\mathbb{P}}$-greatest lower bound, r. Moreover, we have $x_r = x_p \cup x_q$ and $r \restriction \alpha = q$.*

Proof Let $x_r := x_p \cup x_q$, so that $x_r \cap \alpha = x_q$. Define $f_r \colon x_r \to \mathcal{P}(x_r)$ by

$$f_r(\beta) := \begin{cases} f_q(\beta) & \text{if } \beta < \alpha, \\ f_p(\beta) & \text{otherwise.} \end{cases}$$

To see that $r := (x_r, f_r)$ is a condition, we fix arbitrary $\beta \in x_r$ and $\gamma \in \mathrm{acc}(f_r(\beta))$, and verify that $f_r(\gamma) = f_r(\beta) \cap \gamma$. To avoid trivialities, suppose that $\beta \geq \alpha > \gamma$. Since $f_r(\beta) = f_p(\beta) \subseteq x_p$, we have $\sup(x_p \cap \gamma) = \gamma$, so, since $q \leq_{\mathbb{P}} p \restriction \alpha$, we infer that $f_q(\gamma) = f_p(\gamma) = f_p(\beta) \cap \gamma$, i.e., $f_r(\gamma) = f_r(\beta) \cap \gamma$.

It is now readily checked that r has the desired properties. ∎

It follows that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_\kappa$. For each $x \in \binom{\kappa^+}{3}$, let $D_x := \{p \in \mathbb{Q} \mid x_p \supseteq x\}$. By Proposition 3.1(ii), $\mathcal{D} := \{D_x \mid x \in \binom{\kappa^+}{3}\}$ is a sharply dense system, so we can apply SDFA($\mathcal{P}_\kappa$) to obtain a filter G on $\mathbb{P}$ that meets $\mathcal{D}$ everywhere. For all $\beta \in E^{\kappa^+}_\kappa$, let $C_\beta := \bigcup\{f_p(\beta) \mid p \in G, \ \beta \in x_p\}$. Note that for all $p \in G$ and $\beta \in x_p$, we have $|f_p(\beta)| \leq |x_p| < \kappa$.

Claim 3.20   *Suppose that $\beta, \gamma \in E^{\kappa^+}_\kappa$. Then the following hold.*

(i)   *C_β is club in β and $\mathrm{otp}(C_\beta) = \kappa$;*
(ii)   *For all $\alpha \in \mathrm{acc}(C_\beta) \cap \mathrm{acc}(C_\gamma)$, we have $C_\beta \cap \alpha = C_\gamma \cap \alpha$.*

Proof   (i) By the definition of $\mathbb{P}$ and the fact that G is a filter, it follows that C_β is a subset of β, closed in its supremum, such that every proper initial segment of C_β has size $< \kappa$. It thus suffices to verify that C_β is unbounded in β. To this end, fix $\alpha < \beta$. Since G meets $\mathcal{D}$ everywhere, we can find $p \in G \cap D_{\{\alpha, \beta, \beta+1\}}$. Since $\mathrm{cf}(\beta) = \kappa$ and $|x_p| < \kappa$, we have $\beta \in \mathrm{nacc}(x_p)$. Therefore, since $p \in \mathbb{Q}$, we have $\max(f_p(\beta)) = \max(x_p \cap \beta) \geq \alpha$, so $C_\beta \cap [\alpha, \beta) \neq \varnothing$.

(ii) Given $\alpha \in \mathrm{acc}(C_\beta) \cap \mathrm{acc}(C_\gamma)$, we fix $p \in G \cap D_{\{\alpha, \beta, \gamma\}}$. As in the previous case, we have $\max(f_p(\beta)) \geq \alpha$ and $\max(f_p(\gamma)) \geq \alpha$. Consequently, $C_\beta \cap \alpha = f_p(\beta) \cap \alpha$ and $C_\gamma \cap \alpha = f_p(\gamma) \cap \alpha$. By the definition of $\mathbb{P}$, it then follows that $C_\beta \cap \alpha = f_p(\alpha) = C_\gamma \cap \alpha$. ∎

Let $\Gamma := E^{\kappa^+}_\kappa \cup \bigcup\{\mathrm{acc}(C_\beta) \mid \beta \in E^{\kappa^+}_\kappa\}$. For each $\alpha \in \Gamma \smallsetminus E^{\kappa^+}_\kappa$, find $\beta \in E^{\kappa^+}_\kappa$ such that $\alpha \in \mathrm{acc}(C_\beta)$, and let $C_\alpha := C_\beta \cap \alpha$. By the preceding Claim, this is independent of the choice of β. It follows that $\langle C_\alpha \mid \alpha \in \Gamma \rangle$ is a $\square^B_\kappa$-sequence, thus completing the proof of Theorem 3.11.

3.4   Strong Stationary Coding Sets

In [21], Shelah and Stanley derive a stationary coding set from the existence of a $(\kappa, 1)$-morass with built-in $\diamond$. Specifically, they obtain a stationary subset $\mathcal{S}$ of $[\kappa^+]^{<\kappa}$ on which the map $x \mapsto \sup(x)$ is one-to-one. By Theorem 3.11 and the next proposition, this also follows from the forcing axiom SDFA($\mathcal{P}_\kappa$).

Proposition 3.21 (Folklore)   *If $\square^B_\kappa$ holds, then there exists a stationary subset of $[\kappa^+]^{<\kappa}$ on which the map $x \mapsto \sup(x)$ is one-to-one.*

Proof   Let $\langle C_\beta \mid \beta \in \Gamma \rangle$ be a $\square^B_\kappa$-sequence. Enlarge it to a sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ by letting, for all limit $\beta \in \kappa \smallsetminus \Gamma$, C_β be an arbitrary club in β of order type $\mathrm{cf}(\beta)$, and letting $C_{\beta+1} := \{\beta\}$ for all $\beta < \kappa$.

Let $\rho^{\vec{C}}_1 : [\kappa^+]^2 \to \kappa$ denote the associated *maximal weight* function from [27, §6.2]. For each $\beta < \kappa^+$, let $\rho_{1\beta} : \beta \to \kappa$ denote the fiber map $\rho^{\vec{C}}_1(\cdot, \beta)$. Note that

- for all $\beta < \kappa^+$, $\rho_{1\beta}[C_\beta] = \mathrm{otp}(C_\beta)$;
- for all $\beta < \kappa^+$, $\rho_{1\beta}$ is $(<\kappa)$-to-1;
- for all $\beta \in \Gamma$ and $\alpha \in \mathrm{acc}(C_\beta)$, we have $\rho_{1\alpha} \subseteq \rho_{1\beta}$.

In particular, for every $\beta \in E_{<\kappa}^{\kappa^+}$, we have that

$$x_\beta := (\rho_{1\beta})^{-1}[\text{otp}(C_\beta)]$$

is a cofinal subset of β of size $< \kappa$. Thus, we are left with proving the following claim.

Claim 3.22 $\{x_\beta \mid \beta \in E_{<\kappa}^{\kappa^+}\}$ *is stationary in* $[\kappa^+]^{<\kappa}$.

Proof Given a function $f: [\kappa^+]^{<\omega} \to \kappa^+$, let us fix some $\gamma \in E_\kappa^{\kappa^+}$ such that $f``[\gamma]^{<\omega} \subseteq \gamma$. Define $g: \kappa \to \kappa$ by letting

$$g(\varepsilon) := \sup(\rho_{1\gamma}``f``[\rho_{1\gamma}^{-1}[\varepsilon]]^{<\omega}) \text{ for all } \varepsilon < \kappa.$$

Fix $\epsilon \in \text{acc}(\kappa)$ such that $g[\epsilon] \subseteq \epsilon$. Put $\beta := C_\gamma(\epsilon)$, so that $\text{otp}(C_\beta) = \epsilon$ and $\rho_{1\beta} \subseteq \rho_{1\gamma}$. To see that $f``[x_\beta]^{<\omega} \subseteq x_\beta$, let $\{\alpha_i \mid i < n\} \in [x_\beta]^{<\omega}$ be arbitrary. Since $x_\beta = (\rho_{1\beta})^{-1}[\epsilon]$ and $\rho_{1\beta} \subseteq \rho_{1\gamma}$, we have

$$\{\rho_{1\gamma}(\alpha_i) \mid i < n\} = \{\rho_{1\beta}(\alpha_i) \mid i < n\} \in [\epsilon]^{<\omega}.$$

Fix a large enough $\varepsilon < \epsilon$ such that $\{\alpha_i \mid i < n\} \in [\rho_{1\gamma}^{-1}[\varepsilon]]^{<\omega}$. Since $g(\varepsilon) < \epsilon$, we then have $f(\{\alpha_i \mid i < n\}) \in x_\beta$. ∎

This completes the proof of Proposition 3.21. ∎

Note that, by [8, §3], strong stationary coding sets can be seen as a GCH-free version of $\diamondsuit$. For more information on stationary coding sets, see [31].

4 Super-Souslin Trees

Throughout this section, λ denotes an arbitrary cardinal.

The notion of a λ^{++}-super-Souslin tree was isolated by Shelah in response to work by Laver on trees with ascent paths. Ascent paths provide obstacles to a tree being special; super-Souslin trees are designed to present a similar obstacle that entails the existence not only of a nonspecial tree but of a Souslin one. In Subsection 4.1, we provide, as a means of helping to motivate and provide context for the definition of super-Souslin trees, some remarks on the connection between these notions. In Subsection 4.2, we will prove the following theorem.

Theorem C *Suppose that* $\text{SDFA}(\mathcal{P}_{\lambda^+})$ *holds. Then there exists a* λ^+-*complete* λ^{++}-*super-Souslin tree.*

4.1 Introduction to Super-Souslin Trees

A tree $(T, <_T)$ is said to be a κ-*tree* if for every $\alpha < \kappa$, T_α is a nonempty set of size less than κ and $T_\kappa = \varnothing$. The tree is said to be *splitting* if every node in the tree admits at least two immediate successors. It is said to be *normal* if, for all $\alpha < \beta < \kappa$ and all $u \in T_\alpha$, there is $v \in T_\beta$ such that $u <_T v$. It is said to be *Hausdorff* if for all limit $\alpha < \kappa$ and all $u, v \in T_\alpha$, the equality $u_\downarrow = v_\downarrow$ implies $u = v$. For convenience, we will not require that a tree be Hausdorff. Note, however, that any splitting (resp. normal) tree $(T, <_T)$ can easily be turned into a splitting (resp. normal) Hausdorff tree $(T', <_{T'})$

by shifting all levels T_α to be $T'_{\alpha+1}$ and, for limit $\alpha < \kappa$, letting T'_α consist of unique limits of all branches through $\bigcup_{\beta<\alpha} T_\beta$ that are continued in T_α.

Definition 4.1   Let θ be an arbitrary cardinal. For each $\alpha < \kappa$, let T^θ_α denote the collection of all injections $a: \theta \to T_\alpha$. Let T^θ denote $\bigcup_{\alpha<\kappa} T^\theta_\alpha$.

An element of T^θ will be referred to as a *θ-level sequence* from T (or, simply, a *level sequence* from T). For $a, b \in T^\theta$, we abuse notation and write $a <_T b$ if and only if, for all $i < \theta$, $a(i) <_T b(i)$. Likewise, $a \leq_T b$ if and only if $a(i) \leq_T b(i)$ for all $i < \theta$.

Definition 4.2 $[T^\theta]^2 := \{(a, b) \in T^\theta \times T^\theta \mid a <_T b\}$.

Definition 4.3 (Shelah [20])   A λ^{++}-super-Souslin tree is a normal, splitting λ^{++}-tree $(T, <_T)$ for which there exists a function $F: [T^\lambda]^2 \to \lambda^+$ satisfying the following condition: for all $a, b, c \in T^\lambda$ with $a <_T b, c$, if $F(a, b) = F(a, c)$, then there is $i < \lambda$ such that $b(i)$ and $c(i)$ are $<_T$-comparable.

Fact 4.4 (Shelah [20])   *Suppose $(T, <_T)$ is a λ^{++}-super-Souslin tree. If W is an outer model of V with the same $\mathcal{P}(\lambda)$ and λ^{++}, then, in W, there exists some $x \in T$ such that $(x^\uparrow, <_T)$ is a λ^{++}-Souslin tree.*

The next lemma shows that the two-dimensional function F witnessing that a tree $(T, <_T)$ is λ^{++}-super-Souslin cannot be replaced by a one-dimensional function.

Lemma 4.5   *Suppose that $(T, <_T)$ is a normal splitting κ-tree, and θ, μ are cardinals less than κ (e.g., $\kappa = \lambda^{++}$, $\mu = \lambda^+$, and $\theta = \lambda$.) There exists no function $F: T^\theta \to \mu$ such that, for every $a, b \in T^\theta$, if $F(a) = F(b)$, then there is $i < \theta$ such that $a(i)$ and $b(i)$ are $<_T$-comparable.*

Proof   Suppose for sake of contradiction that there is such a function F. We first argue that $(T, <_T)$ is a κ-Souslin tree.

Claim 4.5.1   *Suppose W is an outer model of V in which κ is not collapsed. Then $(T, <_T)$ is a κ-Souslin tree in W.*

Proof   Work in V. As the proof of [3, Claim A.7.1] makes clear, the fact that $(T, <_T)$ is normal and splitting implies that for every $u \in T$, we can find some $a_u \in T^\theta$ such that $u <_T a_u(i)$ for all $i < \theta$. Next, let us work in W, where W is an outer model of V in which κ is not collapsed. Since $(T, <_T)$ is a splitting κ-tree, to show that $(T, <_T)$ is κ-Souslin, it suffices to show that it has no antichains of size κ. Towards a contradiction, suppose that $U := \{u_\alpha \mid \alpha < \kappa\}$ is an antichain. While it is possible that $U \in W \smallsetminus V$, we nevertheless have $\{a_{u_\alpha} \mid \alpha < \kappa\} \subseteq V$. Since κ is not collapsed, we can find ordinals $\alpha < \beta < \kappa$ such that $F(a_{u_\alpha}) = F(a_{u_\beta})$. Pick $i < \theta$ such that $a_{u_\alpha}(i)$ and $a_{u_\beta}(i)$ are $<_T$-comparable. Then u_α and u_β are $<_T$-comparable. This is a contradiction. ∎

Now force over V with the forcing notion $\mathbb{P} := (T, >_T)$ (*i.e.*, the order of $\mathbb{P}$ is the reverse of the tree order). As $(T, <_T)$ is a κ-Souslin tree in V, we have that $\mathbb{P}$ has the κ-c.c. and does not collapse κ. Therefore, the preceding claim implies that $(T, <_T)$ is a κ-Souslin tree in $V^{\mathbb{P}}$, contradicting the fact that $\mathbb{P}$ adds a cofinal branch through $(T, <_T)$. ∎

The next lemma shows that the range of the function F witnessing that a tree $(T, <_T)$ is λ^{++}-super-Souslin cannot be smaller than λ^+. In particular, there is no straightforward generalization of the notion of super-Souslin tree to inaccessible cardinals.

Lemma 4.6   *Suppose that $(T, <_T)$ is a normal, splitting κ-tree, and θ, μ are cardinals less than κ. If $\mu^+ < \kappa$, then there exists no function $F: [T^\theta]^2 \to \mu$ such that, for all $a, b, c \in T^\theta$ with $a <_T b, c$, if $F(a, b) = F(a, c)$, then there is $i < \theta$ such that $b(i)$ and $c(i)$ are $<_T$-comparable.*

Proof   Suppose that F is a counterexample. Fix an arbitrary $a \in T^\theta$. As the proof of [3, Claim A.7.1] makes clear, the fact that $(T, <_T)$ is normal and splitting implies that there exists some large enough $\beta < \kappa$ and an injection $b: \mu^+ \times \theta \to T_\beta$ such that for all $\eta < \mu^+$ and all $i < \theta, a(i) <_T b(\eta, i)$. For each $\eta < \mu^+$, define $b_\eta: \theta \to T_\beta$ by stipulating $b_\eta(i) := b(\eta, i)$. Now, find $\eta < \zeta < \mu^+$ such that $F(a, b_\eta) = F(a, b_\zeta)$. Then there must exist some $i < \theta$ such that $b_\eta(i)$ and $b_\zeta(i)$ are $<_T$-comparable, contradicting the fact that $b_\eta(i)$ and $b_\zeta(i)$ are two distinct elements of T_β. ∎

Now, we move on to deal with the notion of an ascent path.

Definition 4.7 (Laver)   Suppose that θ is a cardinal $< \kappa$ and $\mathcal{F}$ is a family satisfying $\theta \in \mathcal{F} \subseteq \mathcal{P}(\theta)$. An $\mathcal{F}$-*ascent path* through a κ-tree $(T, <_T)$ is a sequence $\vec{f} = \langle f_\alpha \mid \alpha < \kappa \rangle$ such that for all $\alpha < \beta < \kappa$:
(1) f_α is a function from θ to T_α;
(2) $\{i < \theta \mid f_\alpha(i) <_T f_\beta(i)\} \in \mathcal{F}$.

Definition 4.8   For every cardinal θ, write $\mathcal{F}_\theta^{\mathrm{fin}} := \{Z \subseteq \theta \mid |\theta \smallsetminus Z| < \omega\}$, $\mathcal{F}_\theta^{\mathrm{bd}} := \{Z \subseteq \theta \mid \sup(\theta \smallsetminus Z) < \theta\}$, and $\mathcal{F}_\theta := \mathcal{P}(\theta) \smallsetminus \{\varnothing\}$.

By [22], if $(T, <_T)$ is a special λ^+-tree that admits an $\mathcal{F}_\theta^{\mathrm{bd}}$-ascent path, then $\mathrm{cf}(\theta) = \mathrm{cf}(\lambda)$. By [28], if λ is regular and $(T, <_T)$ is a special λ^+-tree that admits an $\mathcal{F}_\theta$-ascent path, then $\theta = \lambda$. A construction of a special λ^+-tree with an $\mathcal{F}_{\mathrm{cf}(\lambda)}^{\mathrm{bd}}$-ascent path may be found in [13]. Constructions of κ-Souslin trees with $\mathcal{F}_\theta^{\mathrm{fin}}$-ascent paths may be found in [3].

Proposition 4.9 (Folklore)   *Any λ^{++}-super-Souslin tree $(T, <_T)$ admits an $\mathcal{F}_\lambda$-ascent path.*

Proof   Suppose $(T, <_T)$ is a λ^{++}-super-Souslin tree and $F: [T^\lambda]^2 \to \lambda^+$ is a witnessing map. Fix an arbitrary $a \in T^\lambda$. Let ϵ be such that $a \in T_\epsilon^\lambda$. By normality of $(T, <_T)$, for each $\beta \in \lambda^{++} \smallsetminus \epsilon$, we may fix $a_\beta \in T_\beta^\lambda$ with $a \leq_T a_\beta$. Pick a cofinal

subset $B \subseteq \lambda^{++} \setminus \epsilon$ on which the map $\beta \mapsto F(a, a_\beta)$ is constant. Then $\langle a_\beta \mid \beta \in B \rangle$ induces an $\mathcal{F}_\theta$-ascent path $\vec{f} = \langle f_\alpha \mid \alpha < \lambda^{++} \rangle$, as follows. For every $\alpha < \lambda^{++}$, let $\beta(\alpha) := \min(B \setminus \alpha)$, and define $f_\alpha: \lambda \to T_\alpha$ by letting $f_\alpha(i)$ be the unique element of T_α such that $f_\alpha(i) \leq_T a_{\beta(\alpha)}(i)$.   ∎

Aiming for an $\mathcal{F}_\lambda^{\mathrm{bd}}$-ascent path, one may want to strengthen Definition 4.3 to assert that for all $a, b, c \in T^\lambda$ with $a <_T b, c$, if $F(a, b) = F(a, c)$, then

$$I(b, c) := \left\{ i < \lambda \mid b(i) \text{ and } c(i) \text{ are } <_T\text{-comparable} \right\}$$

is in $\mathcal{F}_\lambda^{\mathrm{bd}}$. However, this is impossible, by the following result.

Lemma 4.10 *Suppose $(T, <_T)$ is a normal, splitting λ^{++}-tree, $F: [T^\lambda]^2 \to \lambda^+$, and $\mathcal{F}$ is a proper filter on λ. Then there are $(a, b), (a, c) \in [T^\lambda]^2$ with $F(a, b) = F(a, c)$ such that $I(b, c) \notin \mathcal{F}$.*

Proof Towards a contradiction, suppose that for all $(a, b), (a, c) \in [T^\lambda]^2$ with $F(a, b) = F(a, c)$, we have $I(b, c) \in \mathcal{F}$. For all $a \in T^\lambda$ and $\eta < \lambda^+$, let $U_a := \{b \in T^\lambda \mid a \leq_T b\}$ and $U_a^\eta := \{b \in U_a \mid F(a, b) = \eta\}$. Now, fix some $a \in T^\lambda$ arbitrarily, and, for every $\eta < \lambda^+$, let $U^\eta := \{b \in U_a \mid U_b \cap U_a^\eta \neq \varnothing\}$ be the downward closure of U_a^η within U_a.

Claim 4.10.1 *Suppose that $\eta < \lambda^+$ and $b, c \in U^\eta$. Then $I(b, c) \in \mathcal{F}$.*

Proof Pick $b' \in U_b \cap U_a^\eta$ and $c' \in U_c \cap U_a^\eta$. Since $F(a, b') = \eta = F(a, c')$, and by assumption, we have that $I(b', c') \in \mathcal{F}$.

Let $\beta, \beta', \gamma, \gamma'$ be such that $b \in T_\beta^\lambda$, $b' \in T_{\beta'}^\lambda$, $c \in T_\gamma^\lambda$, and $c' \in T_{\gamma'}^\lambda$. Without loss of generality, $\beta' \leq \gamma'$. Now, there are two relevant configurations of the other ordinals to consider.

Case 1: $\beta \leq \beta' < \gamma$. In this case, for all $i \in I(b', c')$, we have $b(i) \leq_T b'(i)$ and $b'(i), c(i) \leq_T c'(i)$, so $b(i)$ and $c(i)$ are $<_T$-comparable.

Case 2: $\beta, \gamma \leq \beta'$. In this case, for all $i \in I(b', c')$, we have $b(i), c(i) \leq_T b'(i)$ and again, $b(i)$ and $c(i)$ are $<_T$-comparable.   ∎

For any two distinct ordinals η, ζ below λ^+, let $\delta_{\eta, \zeta}$ denote the least ordinal δ below λ^{++} such that there are $b \in U^\eta \cap T_\delta^\lambda$ and $c \in U^\zeta \cap T_\delta^\lambda$ for which $I(b, c) = \varnothing$, if such an ordinal exists; otherwise, leave $\delta_{\eta, \zeta}$ undefined.

Claim 4.10.2 *Suppose $\delta_{\eta, \zeta}$ is defined. Then $U^\eta \cap U^\zeta \subseteq \bigcup \{T_\beta^\lambda \mid \beta < \delta_{\eta, \zeta}\}$.*

Proof Towards a contradiction, suppose that $d \in U^\eta \cap U^\zeta \cap T_\beta^\lambda$ for some $\beta \geq \delta_{\eta, \zeta}$. Since U^η and U^ζ are downward closed, we can simply assume that $\beta = \delta_{\eta, \zeta}$.

Using the fact that $\beta = \delta_{\eta, \zeta}$, fix $b \in U^\eta \cap T_\beta^\lambda$ and $c \in U^\zeta \cap T_\beta^\lambda$ such that $I(b, c) = \varnothing$. By Claim 4.10.1, since $b, d \in U^\eta$ and $c, d \in U^\zeta$, we have that $I(b, d)$ and $I(c, d)$ are in $\mathcal{F}$. In particular, $I(b, d) \cap I(c, d) \neq \varnothing$, contradicting the fact that $I(b, c) = \varnothing$.   ∎

As $\lambda^+ < \lambda^{++}$, let $\beta < \lambda^{++}$ be large enough so that, if η, ζ are two distinct ordinals below λ^+ and $\delta_{\eta,\zeta}$ is defined, then $\delta_{\eta,\zeta} < \beta$. By increasing β if necessary, we can assume that $U_a \cap T_\beta^\lambda \neq \varnothing$. Fix $d \in U_a \cap T_\beta^\lambda$. By the fact that $(T, <_T)$ is splitting, for each $i < \lambda$ we can fix $e_0(i) \neq e_1(i)$, both in $T_{\beta+1}$, with $d(i) <_T e_0(i), e_1(i)$. Let $\eta := F(a, e_0)$ and $\zeta := F(a, e_1)$. Clearly, $I(e_0, e_1) = \varnothing$, so that $\delta_{\eta,\zeta}$ is defined. So, by our choice of β, we have $\delta_{\eta,\zeta} < \beta$. However, since $d <_T e_0, e_1$, we have $d \in U^\eta \cap U^\zeta \cap T_\beta^\lambda$, contradicting Claim 4.10.2. ∎

4.2 Proof of Theorem C

The rest of this section is devoted to proving Theorem C. We will define a poset $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{Q}) \in \mathcal{P}_{\lambda^+}$ and a collection $\{\mathcal{D}_i \mid i < \lambda^+\}$ of sharply dense systems such that any filter that meets each $\mathcal{D}_i$ everywhere gives rise to a λ^+-complete λ^{++}-super-Souslin tree. We intend to construct a tree $(T, <_T)$ with underlying set $\lambda^{++} \times \lambda^+$, such that, furthermore, $T_\alpha = \{\alpha\} \times \lambda^+$ for all $\alpha < \lambda^{++}$. We start by defining $\mathbb{P}$.

Definition 4.11 $\mathbb{P}$ consists of all quintuples $p = (x, <^0, t, <^1, f)$ satisfying the following requirements.

(1) $x \in [\lambda^{++}]^{<\lambda^+}$.
(2) $<^0$ is a partial ordering on x such that for all $\beta \in x$, $\mathrm{pred}_p^0(\beta) := \{\alpha \in x \mid \alpha <^0 \beta\}$ is a closed subset of β that is well-ordered by $<^0$.
(3) $t \in [x \times \lambda^+]^{<\lambda^+}$. In a slight abuse of notation and anticipating the generic object, for all $\alpha \in x$, we let t_α denote $t \cap (\{\alpha\} \times \lambda^+)$ and t_α^λ denote the set of injective functions from λ to t_α. For each a in $t^\lambda := \bigcup_{\alpha \in x} t_\alpha^\lambda$, we let $\mathrm{Lev}(a)$ denote the unique ordinal α such that $a \in t_\alpha^\lambda$.
(4) $<^1$ is a tree order on t such that, for all $\beta \in x$ and all $v \in t_\beta$, letting $\mathrm{pred}_p^1(v) := \{u \in t \mid u <^1 v\}$, we have $\{\alpha \in x \mid \mathrm{pred}_p^1(v) \cap t_\alpha \neq \varnothing\} = \mathrm{pred}_p^0(\beta)$. Let $[t^\lambda]^2 := \{(a, b) \mid a, b \in t^\lambda, a <^1 b\}$, where for $a, b \in t^\lambda$, we write $a <^1 b$ if and only if $a(i) <^1 b(i)$ for all $i < \lambda$.
(5) f is a partial function from $[t^\lambda]^2$ to $[\lambda^+]^{<\lambda^+} \setminus \{\varnothing\}$ and $|f| \leq \lambda$.
(6) Suppose that $(a, b), (a, c) \in \mathrm{dom}(f)$. If $f(a, b) \cap f(a, c) \neq \varnothing$ and $\mathrm{Lev}(b) \leq^0 \mathrm{Lev}(c)$, then $|\{i < \lambda \mid b(i) \leq^1 c(i)\}| = \lambda$.
(7) For all $(a, c) \in \mathrm{dom}(f)$ and all $b \in t^\lambda$ such that $a <^1 b <^1 c$, we have $(a, b) \in \mathrm{dom}(f)$ and $f(a, b) \supseteq f(a, c)$.

The coordinates of a condition $p \in \mathbb{P}$ will often be identified as $x_p, <_p^0, t_p, <_p^1$, and f_p, respectively.

Definition 4.12   For all $p, q \in \mathbb{P}$, we let $q \leq_\mathbb{P} p$ if and only if the following hold:

- $x_q \supseteq x_p$;
- $t_q \supseteq t_p$;
- $\mathrm{dom}(f_q) \supseteq \mathrm{dom}(f_p)$;
- for all $(a, b) \in \mathrm{dom}(f_p)$, we have $f_q(a, b) \supseteq f_p(a, b)$.
- $<_q^0 \supseteq <_p^0$;
- $<_q^1 \supseteq <_p^1$;

Definition 4.13 $\mathbb{Q}$ is the set of all conditions $p \in \mathbb{P}$ such that

(1) $x_p = \mathrm{cl}(x_p)$;
(2) $<_p^0$ is the usual ordinal ordering on x_p.

We now show that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q})$ is in $\mathcal{P}_{\lambda^+}$. For $p \in \mathbb{P}$, x_p is the realm of p. We next describe how c.o.i.'s act on $\mathbb{P}$. In order to make it easier to refer to and manipulate level sequences in our conditions, we introduce the following notation.

Notation 4.14   By SDFA$(\mathcal{P}_{\lambda^+})$ and Corollary 3.9, CH$_\lambda$ holds. Therefore, we can let $\langle \sigma_\delta \mid \delta < \lambda^+ \rangle$ injectively enumerate $^\lambda\lambda^+$. For all $\alpha < \lambda^{++}$ and $\delta < \lambda^+$, let $a_{\alpha, \delta} \colon \lambda \to \{\alpha\} \times \lambda^+$ be defined by stipulating $a_{\alpha, \delta}(i) := (\alpha, \sigma_\delta(i))$. Note that every level sequence in our desired tree $(T, <_T)$ will be of the form $a_{\alpha, \delta}$ for a unique pair $(\alpha, \delta) \in \lambda^{++} \times \lambda^+$.

Definition 4.15   Suppose that π is a c.o.i. from a subset of λ^{++} to λ^{++}. For each $p \in \mathbb{P}_{\mathrm{dom}(\pi)}$, we define $\pi.p$ to be the condition $(x, <^0, t, <^1, f) \in \mathbb{P}$ such that

(1) $x = \pi``x_p$;
(2) $<^0 = \{(\pi(\alpha), \pi(\beta)) \mid (\alpha, \beta) \in <_p^0\}$;
(3) $t = \{(\pi(\alpha), \eta) \mid (\alpha, \eta) \in t_p\}$;
(4) $<^1 = \{((\pi(\alpha), \eta), (\pi(\beta), \zeta)) \mid ((\alpha, \eta), (\beta, \zeta)) \in <_p^1\}$;
(5) $f = \{((a_{\pi(\alpha), \delta}, a_{\pi(\beta), \epsilon}), z) \mid ((a_{\alpha, \delta}, a_{\beta, \epsilon}), z) \in f_p\}$.

Finally, we describe the restriction operation.

Definition 4.16   Suppose that $p \in \mathbb{P}$ and $\alpha < \lambda^{++}$. Then $p \restriction \alpha$ is the condition $(x, <^0, t, <^1, f)$ such that

• $x = x_p \cap \alpha$;
• $<^0 = <_p^0 \cap x^2$;
• $t = t_p \cap (\alpha \times \lambda^+)$;
• $<^1 = <_p^1 \cap t^2$;
• $f = \{((a, b), z) \in f_p \mid (a, b) \in [t^\lambda]^2\}$.

With these definitions, it follows easily that $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q})$ satisfies clauses (1)–(5) of Definition 2.1. We now verify clauses (6)–(8), in order.

Lemma 4.17   *Suppose $p \in \mathbb{P}$. Then there is $q \in \mathbb{Q}$ with $q \leq_{\mathbb{P}} p$ such that $x_q = \mathrm{cl}(x_p)$.*

Proof   We need to define $q = (x_q, <_q^0, t_q, <_q^1, f_p)$. Of course, we let $x_q := \mathrm{cl}(x_p)$ and let $<_q^0$ be the usual ordinal ordering on x_q. Thus, the main task is in finding suitable t_q, $<_q^1$ and f_q. Our strategy is to define the first two and then derive f_q by minimally extending f_p so as to satisfy Definition 4.11(7). To be precise, once t_q and $<_q^1$ are determined, we will let

$$\mathrm{dom}(f_q) := \mathrm{dom}(f_p)$$
$$\cup \left\{ (a, b) \in [t_q^\lambda]^2 \mid \exists c \in t_p^\lambda \big(a <_q^1 b <_q^1 c \text{ and } (a, c) \in \mathrm{dom}(f_p) \big) \right\},$$

and for all $(a, b) \in \mathrm{dom}(f_q)$, we will let

$$ f_q(a, b) := f_p(a, b) \cup \bigcup \left\{ f_p(a, c) \mid (a, c) \in \mathrm{dom}(f_p) \text{ and } a <_q^1 b <_q^1 c \right\}. $$

We now turn to defining t_q and $<_q^1$ to ensure that clauses (4) and (6) of Definition 4.11 hold. By our intended definition of f_q and $<_q^0$, these clauses dictate that, for all $\beta \in x_q$ and $\alpha \in x_q \cap (\beta + 1)$, the following hold:

(4′) for all $v \in (t_q)_\beta$, there is some $u \in (t_q)_\alpha$ with $u \leq_q^1 v$;

(6′) for all $(a, b), (a, c) \in \mathrm{dom}(f_p)$ with $f_p(a, b) \cap f_p(a, c) \neq \varnothing$, if $b \in (t_p)_\alpha^\lambda$ and $c \in (t_p)_\beta^\lambda$, then $|\{i < \lambda \mid b(i) \leq_q^1 c(i)\}| = \lambda$.

In order to satisfy clause (4′), it is possible that we will have to add new nodes to t_q, *i.e.*, that $t_q \setminus t_p \neq \varnothing$. However, we will do so in such a way that each element of $t_q \setminus t_p$ will be a $<_q^1$-predecessor of an element of t_p. Consequently, to define t_q and $<_q^1$, it suffices to specify $\mathrm{pred}_q^1(v)$ for all $v \in t_p$.

Now, by recursion on $\beta \in x_p$, we define $\mathrm{pred}_q^1(v)$ for all $v \in (t_p)_\beta$ in a way that ensures that clauses (4′) and (6′) hold for all $\alpha \in x_q \cap (\beta + 1)$. Suppose that $\beta \in x_p$ and, for all $\alpha \in x_p \cap \beta$, we have specified $\mathrm{pred}_q^1(u)$ for every $u \in (t_p)_\alpha$. Let $t_{<\beta}$ denote the underlying set of the tree we have defined thus far, *i.e.*,

$$ \bigcup_{\alpha \in x_p \cap \beta} \bigcup_{u \in (t_p)_\alpha} \left(\mathrm{pred}_q^1(u) \cup \{u\} \right). $$

If $\mathrm{pred}_p^0(\beta) = \varnothing$ and $v \in (t_p)_\beta$, then let B be a maximal branch through $t_{<\beta}$. It might be the case that B is bounded below β, *i.e.*, there is $\gamma \in x_q \cap \beta$ with $B \cap (\{\gamma\} \times \lambda^+) = \varnothing$. If this is the case, then, for each such γ, add a new element from $\{\gamma\} \times \lambda^+$ to t_q and require that these new elements, together with B, form a branch whose levels are unbounded in $x_q \cap \beta$. Let this unbounded branch be denoted by B^*, and set $\mathrm{pred}_q^1(v) := B^*$.

If $\mathrm{pred}_p^0(\beta)$ is unbounded in β, then, for all $v \in (t_p)_\beta$, we are obliged to let $\mathrm{pred}_q^1(v)$ be precisely $\bigcup_{u \in \mathrm{pred}_p^1(v)}(\mathrm{pred}_q^1(u) \cup \{u\})$.

It remains to consider the case in which $\mathrm{pred}_p^0(\beta)$ is nonempty and bounded in β. Put $\beta' := \sup(\mathrm{pred}_p^0(\beta))$. Since $\mathrm{pred}_p^0(\beta)$ is a closed nonempty subset of β, we have $\beta' \in x_p$. If there is no $\gamma \in x_p$ with $\beta' < \gamma < \beta$, then, for all $v \in (t_p)_\beta$, we are again obliged to let $\mathrm{pred}_q^1(v) := \bigcup_{u \in \mathrm{pred}_p^1(v)}(\mathrm{pred}_q^1(u) \cup \{u\})$. Thus, from now on, suppose that $x_p \cap (\beta', \beta) \neq \varnothing$.

Let $\langle (a_\ell, b_\ell, c_\ell) \mid \ell < \lambda \rangle$ enumerate all triples (a, b, c) such that

- $(a, b), (a, c) \in \mathrm{dom}(f_p)$;
- $f_p(a, b) \cap f_p(a, c) \neq \varnothing$;
- $c \in (t_p)_\beta^\lambda$ and there is $\alpha \in x_p \cap (\beta', \beta)$ such that $b \in (t_p)_\alpha^\lambda$.

Moreover, assume that each such triple is enumerated as (a_ℓ, b_ℓ, c_ℓ) for λ-many $\ell < \lambda$. (If there are no such triples, then simply define $\mathrm{pred}_q^1(v)$ arbitrarily for each $v \in (t_p)_\beta$ subject to the constraint $\mathrm{pred}_q^1(v) \supseteq \mathrm{pred}_p^1(v)$.)

Now, by recursion on $\ell < \lambda$, we choose nodes $v_\ell \in (t_p)_\beta$ and specify $\mathrm{pred}^1_q(v_\ell)$. Suppose that $\ell < \lambda$ and we have chosen $v_{\ell'}$ and $\mathrm{pred}^1_q(v_{\ell'})$ for all $\ell' < \ell$. Consider the triple (a_ℓ, b_ℓ, c_ℓ).

Suppose first that $a_\ell \in (t_p)_{\beta'}$. We have that, for all $i < \lambda$, $a_\ell(i) <^1_p b_\ell(i), c_\ell(i)$. In particular, since $\beta' = \max(\mathrm{pred}^0_p(\beta))$, we have, for all $i < \lambda$, $\mathrm{pred}^1_q(b_\ell(i)) \supseteq \mathrm{pred}^1_p(c_\ell(i))$. Choose $i < \lambda$ such that $c_\ell(i) \notin \{v_{\ell'} \mid \ell' < \ell\}$, set $v_\ell := c_\ell(i)$, and let B be a maximal branch through $t_{<\beta}$ with $b_\ell(i) \in B$. As in the case in which $\mathrm{pred}^0_p(\beta) = \varnothing$, extend B, by adding nodes if necessary, to a branch B^* whose levels are unbounded in $x_q \cap \beta$, and set $\mathrm{pred}^1_q(v_\ell) := B^*$.

Suppose next that $a_\ell \in (t_p)_{<\beta'}$. Let $c' \in (t_p)^\lambda_{\beta'}$ be the unique level sequence such that $a_\ell <^1_p c' <^1_p c_\ell$. Since $p \in \mathbb{P}$, we have

$$(a_\ell, c') \in \mathrm{dom}(f_p) \quad \text{and} \quad f_p(a_\ell, c') \supseteq f_p(a_\ell, c_\ell).$$

In particular, $f_p(a_\ell, c') \cap f_p(a_\ell, b_\ell) \neq \varnothing$, so, by our inductive hypothesis, we know that, for λ-many $i < \lambda$, we have $c'(i) <^1_q b_\ell(i)$. Choose such an i with $c_\ell(i) \notin \{v_{\ell'} \mid \ell' < \ell\}$ and let $v_\ell := c_\ell(i)$. As in the previous case, by adding nodes if necessary, fix a branch B^* whose levels are unbounded in $x_q \cap \beta$ with $b_\ell(i) \in B^*$, and set $\mathrm{pred}^1_q(v_\ell) := B^*$.

At the end of this process, if there are nodes in $(t_p)_\beta \setminus \{v_\ell \mid \ell < \lambda\}$, then assign their $<^1_q$-predecessors arbitrarily. We must verify that we have maintained the inductive hypothesis. To this end, fix (a, b, c) such that

- $(a, b), (a, c) \in \mathrm{dom}(f_p)$;
- $f_p(a, b) \cap f_p(a, c) \neq \varnothing$;
- $c \in (t_p)^\lambda_\beta$ and there is $\alpha \in x_p \cap \beta$ such that $b \in (t_p)^\lambda_\alpha$.

Suppose first that $\alpha \leq \beta'$. This implies that $a \in (t_p)^\lambda_{<\beta'}$. Therefore, we can let $c' \in (t_p)^\lambda_{\beta'}$ be the unique level sequence such that $a <^1_p c' <^1_p c$. Then $f_p(a, c') \supseteq f_p(a, c)$, so, by the inductive hypothesis applied at β', we have that, for λ-many $i < \lambda$, $b(i) \leq^1_q c'(i) \leq^1_q c(i)$, so we are done.

Next, suppose $\beta' < \alpha < \beta$. In this case, for λ-many $\ell < \lambda$, we have $(a, b, c) = (a_\ell, b_\ell, c_\ell)$. For each such ℓ, at stage ℓ of the construction, we chose a distinct $i < \lambda$ and ensured that $b_\ell(i) <^1_q c_\ell(i)$, so, for λ-many $i < \lambda$, we have $b(i) <^1_q c(i)$, as desired. ∎

Lemma 4.18   *Suppose that $\xi < \lambda^+$ and $\langle q_\eta \mid \eta < \xi \rangle$ is a decreasing sequence from $\mathbb{Q}$. Let $x := \bigcup_{\eta < \xi} x_{q_\eta}$. Suppose that $\alpha < \mathrm{ssup}(x)$ and that $r \in \mathbb{Q}_{\mathrm{ssup}(x \cap \alpha)}$ is a lower bound for $\langle q_\eta \restriction \alpha \mid \eta < \xi \rangle$. Then there is $q \in \mathbb{Q}$ such that*

- *q is a lower bound for $\langle q_\eta \mid \eta < \xi \rangle$;*
- *$q \restriction \mathrm{ssup}(x \cap \alpha) = r$;*
- *$x_q = \mathrm{cl}(x_r \cup x)$.*

Proof x_q and $<^0_q$ are determined by the requirements of the lemma. We now specify t_q, $<^1_q$, and f_q. We must let $q \restriction \mathrm{ssup}(x \cap \alpha) = r$, so we only deal with the parts of t_q, $<^1_q$, and f_q related to levels at $\mathrm{ssup}(x \cap \alpha)$ or higher.

Fix $\beta \in x_q \smallsetminus \mathrm{ssup}(x \cap \alpha)$. If $\beta \in x$, then let $(t_q)_\beta := \bigcup_{\eta < \xi} (t_{q_\eta})_\beta$. If $\beta \notin x$, then let $\gamma := \min(x \smallsetminus \beta)$, and let $(t_q)_\beta := \{ (\beta, \zeta) \mid (\gamma, \zeta) \in \bigcup_{\eta < \xi} (t_{q_\eta})_\gamma \}$.

We define $<^1_q$ by specifying $\mathrm{pred}^1_q(v)$ for all $v \in t_q$. This is already done for all $v \in (t_q)_{<\mathrm{ssup}(x \cap \alpha)}$. We take care of the $v \in (t_q)_{\geq \mathrm{ssup}(x \cap \alpha)}$ by recursion on the $\beta \in x_q$ such that $v \in (t_q)_\beta$. Thus, suppose $\beta \in x_q \smallsetminus \mathrm{ssup}(x \cap \alpha)$, and we have defined $\mathrm{pred}^1_q(u)$ for all $u \in (t_q)_{<\beta}$.

Suppose first that $\beta \notin x$, and let $\gamma := \min(x \smallsetminus \beta)$. If $v = (\beta, \zeta) \in (t_q)_\beta$, let $v' := (\gamma, \zeta) \in (t_q)_\gamma$ and let $\mathrm{pred}^1_q(v)$ be the $<^1_q$-downward closure of $\bigcup_{\eta < \xi} \mathrm{pred}^1_{q_\eta}(v')$.

Suppose next that $\beta \in x$ and $\beta' := \sup(x_q \cap \beta) \notin x$. If $v = (\beta, \zeta) \in (t_q)_\beta$, then let $v' := (\beta', \zeta) \in (t_q)_{\beta'}$, and $\mathrm{pred}^1_q(v) := \{v'\} \cup \mathrm{pred}^1_q(v')$.

Finally, suppose that $\beta \in x$ and $\sup(x_q \cap \beta) \in x$. Then, for all $v \in (t_q)_\beta$, let $\mathrm{pred}^1_q(v)$ be the $<^1_q$-downward closure of $\bigcup_{\eta < \xi} \mathrm{pred}^1_{q_\eta}(v)$.

To finish, we define f_q. Suppose that $\beta \in x_q \smallsetminus (\mathrm{ssup}(x \cap \alpha) \cup x)$ and $b \in (t_q)^\lambda_\beta$. Let $\gamma_\beta := \min(x \smallsetminus \beta)$, and let $b' \in (t_q)^\lambda_{\gamma_\beta}$ be given by letting $b'(i)$ be the unique (γ_β, ζ) such that $b(i) = (\beta, \zeta)$. Note that $b <^1_q b'$. We set

$$\mathrm{dom}(f_q) := \mathrm{dom}(f_r) \cup \bigcup_{\eta < \xi} \mathrm{dom}(f_{q_\eta}) \cup$$

$$\left\{ (a, b) \;\middle|\; \exists \beta \in x_q \smallsetminus (\mathrm{ssup}(x \cap \alpha) \cup x) \big(b \in (t_q)^\lambda_\beta \text{ and } (a, b') \in \bigcup_{\eta < \xi} \mathrm{dom}(f_{q_\eta}) \big) \right\}.$$

If $(a, b) \in \mathrm{dom}(f_r)$, then we set $f_q(a, b) := f_r(a, b)$. If $(a, b) \in \bigcup_{\eta < \xi} \mathrm{dom}(f_{q_\eta}) \smallsetminus \mathrm{dom}(f_r)$, then we let $f_q(a, b) := \bigcup_{\eta < \xi} f_{q_\eta}(a, b)$. If (a, b) is such that $b \in (t_p)^\lambda_\beta$ for some $\beta \in x_p \smallsetminus (\mathrm{ssup}(x \cap \alpha) \cup x)$ and $(a, b') \in \bigcup_{\eta < \xi} \mathrm{dom}(f_{q_\eta})$, then let $f_q(a, b) = \bigcup_{\eta < \xi} f_{q_\eta}(a, b') = f_q(a, b')$. It is easily verified that q is as desired. ∎

Lemma 4.19 *Suppose $p \in \mathbb{Q}$, $\alpha < \mathrm{ssup}(x_p)$, and $q \leq p \upharpoonright \alpha$ with $q \in \mathbb{P}_\alpha$. Then there is $r \in \mathbb{P}$ that is a greatest lower bound for p and q. Moreover, we have $x_r = x_p \cup x_q$ and $r \upharpoonright \alpha = q$.*

Proof We construct such an r by doing as little as possible while still satisfying Definition 4.11 and extending both p and q. Let $x_r := x_p \cup x_q$, and require that $r \upharpoonright \alpha = q$. Suppose that $\beta \in x_p \smallsetminus \alpha$. Let

$$\mathrm{pred}^0_r(\beta) := \mathrm{pred}^0_p(\beta) \cup \bigcup_{\gamma \in \mathrm{pred}^0_p(\beta) \cap \alpha} \mathrm{pred}^0_q(\gamma).$$

Let $t_r := t_p \cup t_q$. If $v \in t_p \smallsetminus t_q$, then let $\mathrm{pred}^1_r(v) := \mathrm{pred}^1_p(v) \cup \bigcup_{u \in \mathrm{pred}^1_p(v) \cap t_q} \mathrm{pred}^1_q(u)$. Finally, let $\mathrm{dom}(f_r) := \mathrm{dom}(f_p) \cup \mathrm{dom}(f_q)$. If $(a, b) \in \mathrm{dom}(f_q)$, then let $f_r(a, b) := f_q(a, b)$. If $(a, b) \in \mathrm{dom}(f_p) \smallsetminus \mathrm{dom}(f_q)$, then let $f_r(a, b) := f_p(a, b)$.

The only clauses of Definition 4.11 that are nontrivial to check are (6) and (7). Let us first deal with clause (6). To this end, fix $a, b, c \in t^\lambda_r$ such that $(a, b), (a, c) \in \mathrm{dom}(f_r)$ and $f_r(a, b) \cap f_r(a, c) \neq \varnothing$. If we have either $(a, b), (a, c) \in \mathrm{dom}(f_q)$ or $(a, b), (a, c) \in \mathrm{dom}(f_p) \smallsetminus \mathrm{dom}(f_q)$, then the conclusion of clause (6) follows from the fact that $p, q \in \mathbb{P}$. Thus, we can assume without loss of generality that $(a, b) \in$

$\mathrm{dom}(f_q)$ and $(a, c) \in \mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q)$. Let $\beta, \gamma \in x_r$ be such that $b \in (t_r)_\beta^\lambda$ and $c \in (t_r)_\gamma^\lambda$. By assumption, we have $\beta < \alpha \leq \gamma$.

If $\beta \not\leq_r^0 \gamma$, then there is nothing to check. Thus, assume that $\beta \leq_r^0 \gamma$. By the definition of $\leq_r^0$, it follows that there is $\beta' \in (x_p \cap \alpha)$ such that $\beta \leq_q^0 \beta'$ and $\beta' \leq_p^0 \gamma$. Let $c' \in (t_p)_{\beta'}^\lambda$ be the unique level sequence such that $a <_p^1 c' <_p^1 c$. Since $p \in \mathbb{P}$, it follows that $(a, c') \in \mathrm{dom}(f_p)$ and $f_p(a, c') \supseteq f_p(a, c)$. Since $q \leq p \restriction \alpha$, we must have $(a, c') \in \mathrm{dom}(f_q)$ and $f_q(a, c') \supseteq f_p(a, c)$. Thus, we have $f_q(a, c') \cap f_q(a, b) \neq \varnothing$. Since $q \in \mathbb{P}$ and $\beta \leq_q^0 \beta'$, we have that, for λ-many $i < \lambda$, $b(i) \leq_q^1 c'(i)$. But then, for all such $i < \lambda$, we also have $b(i) \leq_r^1 c(i)$, as required.

Finally, we check clause (7). Suppose that $(a, c) \in \mathrm{dom}(f_r)$ and $b \in t_r^\lambda$ is such that $a <_r^1 b <_r^1 c$. If $(a, c) \in \mathrm{dom}(f_q)$, then the conclusion follows from the fact that $q \in \mathbb{P}$. Thus, suppose that $(a, c) \in \mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q)$. Let $\beta \in x_r$ be such that $b \in (t_r)_\beta^\lambda$, and let $\gamma \in x_p$ be such that $c \in (t_p)_\gamma^\lambda$. If $\beta \in x_p$, then we have $a <_p^1 b <_p^1 c$, and the conclusion follows from the fact that $p \in \mathbb{P}$. Thus, assume that $\beta \in x_q \setminus x_p$. Then there is $\beta' \in x_p \cap \alpha$ such that $\beta \leq_q^0 \beta'$ and $\beta' \leq_p^0 \gamma$. Let $c' \in (t_p)_{\beta'}^\lambda$ be the unique level sequence such that $a <_p^1 c' <_p^1 c$. Since $p \in \mathbb{P}$, we have $(a, c') \in \mathrm{dom}(f_p)$ and $f_p(a, c') \supseteq f_p(a, c)$. Since $q \leq_\mathbb{P} p \restriction \alpha$, we have $f_q(a, c') \supseteq f_p(a, c)$. Finally, since $q \in \mathbb{P}$ and $a <_q^1 b <_q^1 c'$, we have $(a, b) \in \mathrm{dom}(f_q)$ and $f_q(a, b) \supseteq f_q(a, c')$. Thus, $(a, b) \in \mathrm{dom}(f_r)$ and $f_r(a, b) \supseteq f_r(a, c)$, as required. ∎

It now follows that $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{Q})$ is in $\mathcal{P}_{\lambda^+}$. We are thus left with isolating the relevant sharply dense systems. The following are all straightforward.

Lemma 4.20 (Normal and splitting) *Suppose $\eta < \lambda^+$. For every $\alpha < \beta < \lambda^{++}$, let $D_{\eta, \{\alpha, \beta\}}^{ns}$ be the set of all conditions $p \in \mathbb{Q}$ such that*

- $\{\alpha, \beta\} \subseteq x_p$;
- $(\alpha, \eta), (\beta, \eta) \in t_p$;
- (α, η) *has at least two $<_p^1$-successors in $(t_p)_\beta$.*

  Then $\mathcal{D}_\eta^{ns} := \{D_{\eta, x}^{ns} \mid x \in \binom{\lambda^{++}}{2}\}$ is a sharply dense system. ∎

Lemma 4.21 (Complete) *Suppose that $\mu < \lambda^+$ is a regular cardinal and $g : \mu \to \lambda^+$. For every $x \in \binom{\lambda^{++}}{\mu+1}$, let $D_{g, x}^{com}$ be the set of all conditions $p \in \mathbb{Q}$ such that*

- $x \subseteq x_p$;
- *for all $i < \mu$, we have $(x(i), g(i)) \in t_p$;*
- *if $\{(x(i), g(i)) \mid i < \mu\}$ forms a $<_p^1$-chain, then it has a $<_p^1$-upper bound in $(t_p)_{x(\mu)}$.*

  Then $\mathcal{D}_g^{com} := \{D_{g, x}^{com} \mid x \in \binom{\lambda^{++}}{\mu+1}\}$ is a sharply dense system. ∎

Lemma 4.22 (Super-Souslin) *Suppose $\delta, \epsilon < \lambda^+$. For all $\alpha < \beta < \lambda^{++}$, let $E_{\delta, \epsilon, \{\alpha, \beta\}}$ be the set of all conditions $p \in \mathbb{Q}$ such that*

- $\{\alpha, \beta\} \subseteq x_p$;
- $a_{\alpha, \delta}, a_{\beta, \epsilon} \in t_p^\lambda$;
- *if $a_{\alpha, \delta} <_p^1 a_{\beta, \epsilon}$, then $(a_{\alpha, \delta}, a_{\beta, \epsilon}) \in \mathrm{dom}(f_p)$.*

Then $\mathcal{E}_{\delta,\epsilon} := \{E_{\delta,\epsilon,x} \mid x \in \binom{\lambda^{++}}{2}\}$ *is a sharply dense system.* ∎

By SDFA($\mathcal{P}_{\lambda^+}$), we can find a filter G on $\mathbb{P}$ such that

- for every $\eta < \lambda^+$, G meets $\mathcal{D}_\eta^{ns}$ everywhere;
- for every regular cardinal $\mu < \lambda^+$ and every function $g : \mu \to \lambda^+$, G meets $\mathcal{D}_g^{com}$ everywhere (recall that by Corollary 3.9, SDFA($\mathcal{P}_{\lambda^+}$) implies $|^{<\lambda^+}\lambda^+| = \lambda^+$) ;
- for all $\delta, \epsilon < \lambda^+$, G meets $\mathcal{E}_{\delta,\epsilon}$ everywhere.

Now define a tree $(T, <_T)$ as follows. Let $T := \lambda^{++} \times \lambda^+$. Let $(\alpha, \eta) <_T (\beta, \xi)$ if and only if there is $p \in G$ such that $(\alpha, \eta), (\beta, \xi) \in t_p$ and $(\alpha, \eta) <_p^1 (\beta, \xi)$. The fact that G meets $\mathcal{D}_\eta^{ns}$ everywhere for all $\eta < \lambda^+$ ensures that $(T, <_T)$ is a normal, splitting tree and $T_\alpha = \{\alpha\} \times \lambda^+$ for all $\alpha < \lambda^{++}$. The fact that G meets $\mathcal{D}_g^{com}$ everywhere for all regular $\mu \leq \lambda$ and $g : \mu \to \lambda^+$ ensures that $(T, <_T)$ is λ^+-complete.

Finally, we define a function $F : [T^\lambda]^2 \to \lambda^+$ witnessing that $(T, <_T)$ is a super-Souslin tree. Fix $\alpha < \beta < \lambda^{++}$ and $\delta, \epsilon < \lambda^+$ such that $a_{\alpha,\delta} <_T a_{\beta,\epsilon}$. Find $p \in G \cap E_{\delta,\epsilon,\{\alpha,\beta\}}$. Since $p \in \mathbb{Q}$ and $a_{\alpha,\delta} <_T a_{\beta,\epsilon}$, it follows that $a_{\alpha,\delta} <_p^1 a_{\beta,\epsilon}$. Therefore, $(a_{\alpha,\delta}, a_{\beta,\epsilon}) \in \text{dom}(f_p)$. Let $F(a_{\alpha,\delta}, a_{\beta,\epsilon})$ be an arbitrary element of $f_p(a_{\alpha,\delta}, a_{\beta,\epsilon})$.

To verify that F is as sought, fix $a, b, c \in T^\lambda$ such that $(a, b), (a, c) \in [T^\lambda]^2$ and $F(a, b) = F(a, c)$. Without loss of generality, suppose there are $\beta \leq \gamma < \lambda^{++}$ such that $b \in T_\beta^\lambda$, and $c \in T_\gamma^\lambda$. Find $p_b \in G$ such that $(a, b) \in \text{dom}(f_{p_b})$ and $F(a, b) \in f_{p_b}(a, b)$. Similarly, find $p_c \in G$ such that $(a, c) \in \text{dom}(f_{p_c})$ and $F(a, c) \in f_{p_c}(a, c)$. Find $q \in G \cap \mathbb{Q}$ with $q \leq_{\mathbb{P}} p_b, p_c$. Then $(a, b), (a, c) \in \text{dom}(f_q)$, $F(a, b) \in f_q(a, b)$, and $F(a, c) \in f_q(a, c)$. In particular, $f_q(a, b) \cap f_q(a, c) \neq \varnothing$. Since $q \in \mathbb{Q}$ it follows that there are λ-many $i < \lambda$ such that $b(i) \leq_q^1 c(i)$. But then, for all such $i < \lambda$, we have $b(i) \leq_T c(i)$. Thus, F witnesses that $(T, <_T)$ is a λ^{++}-super-Souslin tree, so our proof of Theorem C is now complete. ∎

5 Square and Diamond

In this section, we use $\square_\kappa^B$ and $\diamondsuit(\kappa)$ to construct combinatorial objects that will help us prove Theorem B in Section 6.

5.1 Enlarged Direct Limit

In this short subsection, we introduce an "enlarged direct limit" operator. This operator motivates our application of $\square_\kappa^B$ that will be carried out in the next subsection.

Definition 5.1 For a linearly ordered set $(Y, \triangleleft)$ and a subset $Z \subseteq Y$, we define $\text{double}_Z(Y, \triangleleft)$ as a linearly ordered set whose underlying set is $(Z \times \{0\}) \uplus (Y \times \{1\})$, ordered lexicographically by letting $(y, i) \triangleleft_l (y', i')$ if and only if one of the following holds:

- $y \triangleleft y'$;
- $y = y'$ and $(i, i') = (0, 1)$.

The linearly ordered set $(Y, \lhd)$ we have in mind is a direct limit of a system of well-ordered sets, and the choice of the subset $Z \subseteq Y$ (to be doubled) will be defined momentarily. The following is obvious.

Lemma 5.2 *For all $Z \subseteq Y$, if Y is well-ordered by $\lhd$, then $\mathrm{double}_Z(Y, \lhd)$ is well-ordered by $\lhd_l$.* ∎

We start with a certain system of well-ordered sets. Suppose that $\vec{\theta} = \langle \theta_\eta \mid \eta < \xi \rangle$ and $\vec{\pi} = \langle \pi_{\eta, \eta'} \mid \eta < \eta' < \xi \rangle$ are such that

- ξ is a limit ordinal,
- $\vec{\theta}$ is a nondecreasing sequence of ordinals,
- for all $\eta < \eta' < \xi$, $\pi_{\eta, \eta'} \colon \theta_\eta \to \theta_{\eta'}$ is a c.o.i.,
- for all $\eta < \eta' < \eta'' < \xi$, we have $\pi_{\eta, \eta''} = \pi_{\eta', \eta''} \circ \pi_{\eta, \eta'}$.

As is well known, the direct limit of the system $(\vec{\theta}, \vec{\pi})$ is defined as follows.

- Put $X := \{ (\eta, \gamma) \mid \eta < \xi, \gamma < \theta_\eta \}$.
- For $(\eta, \gamma), (\eta', \gamma') \in X$ with $\eta < \eta'$, let $(\eta, \gamma) \sim (\eta', \gamma')$ if and only if $\pi_{\eta, \eta'}(\gamma) = \gamma'$.
- Let Y consists of all equivalence classes $[(\eta, \gamma)]$ for $(\eta, \gamma) \in X$.
- Order Y by letting $[(\eta_0, \gamma_0)] \lhd [(\eta_1, \gamma_1)]$ if and only if there exists some $\eta \geq \max\{\eta_0, \eta_1\}$ and $\gamma_0' < \gamma_1'$ such that $(\eta_0, \gamma_0) \sim (\eta, \gamma_0')$ and $(\eta_1, \gamma_1) \sim (\eta, \gamma_1')$.
- For each $\eta < \xi$, define a map $\pi_\eta \colon \theta_\eta \to Y$ by stipulating $\pi_\eta(\gamma) := [(\eta, \gamma)]$.

Definition 5.3 (Direct limit)   Let $\lim(\vec{\theta}, \vec{\pi})$ denote $(Y, \lhd, \langle \pi_\eta \mid \eta < \xi \rangle)$.

Next, we let Z be the set of equivalence classes in Y such that, for every representative (η, γ) from the equivalence class, $\pi_\eta \restriction \gamma$ is bounded below $\pi_\eta(\gamma)$, i.e.,

$$Z := \left\{ z \in Y \mid \forall (\eta, \gamma) \in z \, \exists y \in Y \, \forall \beta < \gamma [\pi_\eta(\beta) \lhd y \lhd \pi_\eta(\gamma)] \right\}.$$

Let $W := \mathrm{double}_Z(Y, \lhd)$, and let ϖ denote the map from Y to its canonical copy inside W, i.e., $\varpi(y) = (y, 1)$.

Definition 5.4 (Enlarged direct limit)   Let $\lim^*(\vec{\theta}, \vec{\pi})$ denote $(W, \lhd_l, \langle \pi_\eta^* \mid \eta < \xi \rangle)$, where $\pi_\eta^* := \varpi \circ \pi_\eta$ for each $\eta < \xi$.

Finally, by Lemma 5.2, in the special case that $(Y, \lhd)$ is well-ordered, we know that $(W, \lhd_l)$ is well-ordered. In this case, we put $\theta := \mathrm{otp}(W, \lhd_l)$, and let $\pi^* \colon W \to \theta$ be the collapse map. Then, we define the following.

Definition 5.5 (Ordinal enlarged direct limit) $\lim^+(\vec{\theta}, \vec{\pi})$ stands for

$$\left(\theta, \in, \langle \pi_\eta^+ \mid \eta < \xi \rangle \right),$$

where $\pi_\eta^+ := \pi^* \circ \pi_\eta^*$ for all $\eta < \xi$.

5.2   Square

Fix a $\square_\kappa^B$-sequence, $\langle C_\beta \mid \beta \in \Gamma \rangle$. Enlarge the preceding to a sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$ by letting, for all limit $\beta \in \kappa \smallsetminus \Gamma$, C_β be an arbitrary club in β of order type $\mathrm{cf}(\beta)$, and

letting $C_{\beta+1} := \{0, \beta\}$ for all $\beta < \kappa$. In particular, for every $\beta \in E_\omega^\kappa \smallsetminus \Gamma$, we have $\mathrm{acc}(C_\beta) = \varnothing$. Thus, without loss of generality, we can assume that $E_\omega^\kappa \subseteq \Gamma$. For convenience, assume also that $0 \in C_\beta$ for all nonzero $\beta < \kappa$.

We now turn to constructing a matrix $\vec{B} = \langle B_\eta^\beta \mid \beta < \kappa^+, \eta < \kappa \rangle$ such that $\bigcup_{\eta < \kappa} B_\eta^\beta = \beta + 1$ for all $\beta < \kappa^+$. From this matrix, for each $\beta < \kappa^+$, we shall derive the following additional objects:

- We let η_β denote the least $\eta < \kappa$ such that $B_\eta^\beta \neq \varnothing$.
- For each $\xi \in \mathrm{acc}(\kappa \smallsetminus \eta_\beta)$, we write $B_{<\xi}^\beta := \bigcup_{\eta < \xi} B_\eta^\beta$.
- For each $\eta < \kappa$, we set $\theta_\eta^\beta := \mathrm{otp}(B_\eta^\beta)$ and let $\pi_\eta^\beta \colon B_\eta^\beta \to \theta_\eta^\beta$ denote the unique order-preserving bijection.
- For each $\eta < \xi < \kappa$, $\pi_{\eta,\xi}^\beta \colon \theta_\eta^\beta \to \theta_\xi^\beta$ will denote the order-preserving injection indicating how B_η^β "sits inside" B_ξ^β, i.e., $\pi_{\eta,\xi}^\beta := \pi_\xi^\beta \circ (\pi_\eta^\beta)^{-1}$.

We shall also derive a "distance function" $d \colon [\kappa^+]^2 \to \kappa$ by letting

$$ d(\alpha, \beta) := \min\{\eta < \kappa \mid \alpha \in B_\eta^\beta\} \quad (\alpha < \beta < \kappa^+). $$

Lemma 5.6 *There exists a matrix $\vec{B} = \langle B_\eta^\beta \mid \beta < \kappa^+, \eta < \kappa \rangle$ such that for each $\beta < \kappa^+$, the following hold:*

(1) $\langle B_\eta^\beta \mid \eta < \kappa \rangle$ *is a $\subseteq$-increasing sequence of closed sets, each of size $< \kappa$, that converges to $\beta + 1$, and $\beta \in B_{\eta_\beta}^\beta$;*

(2) *for all $\eta < \kappa$ and $\alpha \in B_\eta^\beta$, we have $B_\eta^\alpha = B_\eta^\beta \cap (\alpha + 1)$ and $\pi_\eta^\alpha = \pi_\eta^\beta \restriction (\alpha + 1)$;*

(3) *for all $\eta < \kappa$, if $\mathrm{cf}(\beta) = \kappa$, then $\max(B_\eta^\beta \cap \beta) = C_\beta(\omega\eta)$;*

(4) *if $\beta \in \Gamma \cap E_{<\kappa}^{\kappa^+}$, then $\eta_\beta = \mathrm{otp}(\mathrm{acc}(C_\beta))$ and $\mathrm{acc}(C_\beta) \subseteq B_{\eta_\beta}^\beta$;*

(5) *for all $\xi \in \mathrm{acc}(\kappa \smallsetminus \eta_\beta)$, all of the following hold:*

    (a) B_ξ^β *is the ordinal closure of $B_{<\xi}^\beta$;*

    (b) *for every $\alpha \in B_\xi^\beta \smallsetminus B_{<\xi}^\beta$, letting $\gamma := \min(B_\xi^\beta \smallsetminus (\alpha + 1))$, we have $\mathrm{cf}(\gamma) = \kappa$ and $\alpha = C_\gamma(\omega\xi)$;*

    (c) $\mathrm{cf}(\beta) = \kappa$ *if and only if $\mathrm{ssup}(\pi_{\eta,\xi}^\beta \text{``} \pi_\eta^\beta(\beta)) < \pi_\xi^\beta(\beta)$ for all $\eta \in [\eta_\beta, \xi)$.*

Proof The construction is by recursion on $\beta < \kappa^+$.

    Case 0: $\beta = 0$. Set $B_\eta^\beta := \{0\}$ for all $\eta < \kappa$. It is trivial to see that (1)–(5) all hold.

    Case 1: $\beta = \alpha + 1$. For all $\eta < \eta_\alpha$, let $B_\eta^\beta := \varnothing$, and for all $\eta \in [\eta_\alpha, \kappa)$, let $B_\eta^\beta := \{\beta\} \cup B_\eta^\alpha$. It is trivial to see that (1)–(5) all hold.

    Case 2: $\beta \in \mathrm{acc}(\kappa)$ **and** $\sup(\mathrm{acc}(C_\beta)) < \beta$. In particular, $a := C_\beta \smallsetminus \sup(\mathrm{acc}(C_\beta))$ is a cofinal subset of β of order type ω. Note that since $\mathrm{cf}(\beta) = \omega$, we have $\beta \in \Gamma$. Put $\eta_\beta := \mathrm{otp}(\mathrm{acc}(C_\beta))$ and $\eta^* := \max\{\eta_\beta, \sup(d\text{``}[a]^2)\}$. Now, for all $\eta < \kappa$, define B_η^β as follows:

      ▸ If $\eta < \eta_\beta$, then let $B_\eta^\beta := \varnothing$. Clauses (2)–(5) are trivially satisfied.

      ▸ If $\eta_\beta \leq \eta \leq \eta^*$, then let $\alpha^* := \min(a)$ and put $B_\eta^\beta := \{\beta\} \cup B_\eta^{\alpha^*}$. Since $\alpha^* \in \mathrm{acc}(C_\beta) \cup \{0\}$ and $\beta \in \Gamma$, we have $C_{\alpha^*} \sqsubseteq C_\beta$, which ensures (4). As for

(5c), for all $\eta < \xi$ in $[\eta_\beta, \eta^*]$, we have

$$\mathrm{ssup}\left(\pi^\beta_{\eta,\xi} {}^{\text{``}} \pi^\beta_\eta(\beta)\right) = \pi^\beta_{\eta,\xi}\left(\pi^\beta_\eta(\alpha^*) + 1\right) = \pi^\beta_\xi(\alpha^*) + 1 = \pi^\beta_\xi(\beta).$$

The other clauses are easily seen to be satisfied.

▸ Otherwise, let $B^\beta_\eta := \{\beta\} \cup \bigcup_{\alpha \in a} B^\alpha_\eta$. Since $\eta > \eta^*$, for every pair of ordinals $\alpha < \alpha'$ from a, we have $\alpha \in B^{\alpha'}_\eta$, so that $B^\alpha_\eta = B^{\alpha'}_\eta \cap (\alpha + 1)$. It follows that $\langle B^\alpha_\eta \mid \alpha \in a \rangle$ is an $\sqsubseteq$-increasing sequence of closed sets. In particular, $B^\beta_\eta \cap \beta$ is a club in β, which takes care of (5c). So, all clauses are satisfied.

Case 3: $\mathrm{cf}(\beta) < \kappa$ **and** $\sup(\mathrm{acc}(C_\beta)) = \beta$. Put $\eta_\beta := \sup(d \text{``}[\mathrm{acc}(C_\beta)]^2)$, and, for all $\eta < \kappa$, define B^β_η as follows:

▸ If $\eta < \eta_\beta$, then let $B^\beta_\eta := \varnothing$. Clauses (2)–(5) are trivially satisfied.

▸ If $\eta \geq \eta_\beta$, then let $B^\beta_\eta := \{\beta\} \cup \bigcup_{\alpha \in \mathrm{acc}(C_\beta)} B^\alpha_\eta$. Since $\eta \geq \sup(d\text{``}[\mathrm{acc}(C_\beta)]^2)$, we have that $\langle B^\alpha_\eta \mid \alpha \in \mathrm{acc}(C_\beta)\rangle$ is an $\sqsubseteq$-increasing sequence of closed sets. So $B^\beta_\eta \cap \beta$ is a club in β, and all clauses except (4) are easily seen to be satisfied. Now, if $\beta \in \Gamma$, then, since (4) holds for all $\alpha \in \mathrm{acc}(C_\beta)$; we have $\eta_\beta = \mathrm{otp}(\mathrm{acc}(C_\beta))$, so that (4) holds for β, as well.

Case 4: $\beta \in E^{\kappa^+}_\kappa$. For all $\eta < \kappa$, let $\alpha_\eta := C_\beta(\omega\eta)$ and $B^\beta_\eta := \{\beta\} \cup B^{\alpha_\eta}_\eta$, so that (3) is satisfied.

Since $\mathrm{cf}(\beta) = \kappa$, we have $\beta \in \Gamma$. Hence, for all $\eta < \xi < \kappa$, we have $\alpha_\xi \in \Gamma$, so that $\alpha_\eta \in B^{\alpha_\xi}_\xi$ by (4). It follows that $\langle B^\beta_\eta \cap \beta \mid \eta < \kappa\rangle$ is $\subseteq$-increasing. In particular, (1) and (2) are satisfied. It also follows that, for all $\xi \in \mathrm{acc}(\kappa)$ and $\alpha \in B^\beta_\xi \cap \beta$, we have $B^\beta_{<\xi} \cap (\alpha + 1) = B^\alpha_{<\xi}$, so that (5a) and (5b) are satisfied.

Finally, to verify (5c), fix an arbitrary $\xi \in \mathrm{acc}(\kappa)$ and $\eta < \xi$. By (2) and (3), we have $B^\beta_\eta = \{\beta\} \cup B^{\alpha_\eta}_\eta$ and $B^\beta_\xi = \{\beta\} \cup B^{\alpha_\xi}_\xi$, so that $\pi^\beta_\eta(\beta) = \pi^\beta_\eta(\alpha_\eta) + 1$ and $\pi^\beta_\xi(\beta) = \pi^\beta_\xi(\alpha_\xi) + 1$. Therefore, we have

$$\pi^\beta_{\eta,\xi} \text{``} \pi^\beta_\eta(\beta) \subseteq \pi^\beta_{\eta,\xi}(\pi^\beta_\eta(\alpha_\eta)) + 1 = \pi^\beta_\xi(\alpha_\eta) + 1 < \pi^\beta_\xi(\alpha_\xi) < \pi^\beta_\xi(\beta). \qquad \blacksquare$$

The next lemma assumes familiarity with the previous subsection.

Lemma 5.7   *Suppose $\beta < \kappa^+$ and $\xi \in \mathrm{acc}(\kappa \setminus \eta_\beta)$. Write $\vec{\theta} := \langle \theta^\beta_\eta \mid \eta < \xi\rangle$ and $\vec{\pi} := \langle \pi^\beta_{\eta,\eta'} \mid \eta < \eta' < \xi\rangle$. Then $\lim^+(\vec{\theta}, \vec{\beta})$ is defined and, letting $(\theta, \in, \langle \pi^+_\eta \mid \eta < \xi\rangle) := \lim^+(\vec{\theta}, \vec{\pi})$, we have $\theta = \theta^\beta_\xi$ and $\langle \pi^+_\eta \mid \eta < \xi\rangle = \langle \pi^\beta_{\eta,\xi} \mid \eta < \xi\rangle$.*

Proof   Let $\langle Y, \lhd, \langle \pi_\eta \mid \eta < \xi\rangle\rangle := \lim((\vec{\theta}, \vec{\pi}))$. For every class $y \in Y$ and representatives $(\eta, \gamma), (\eta', \gamma') \in y$ with $\eta < \eta'$, we have $\pi^\beta_{\eta,\eta'}(\gamma) = \gamma'$, i.e., $(\pi^\beta_{\eta'})^{-1}(\gamma') = (\pi^\beta_\eta)^{-1}(\gamma)$. Therefore, for each $y \in Y$, we can let $\alpha_y := (\pi^\beta_\eta)^{-1}(\gamma)$ for an arbitrary choice of $(\eta, \gamma) \in y$. Note that, for all y, y' in Y, we have $y \lhd y'$ if and only if $\alpha_y < \alpha_{y'}$. Therefore, the order type of $(Y, \lhd)$ is precisely $\mathrm{otp}(B^\beta_{<\xi})$. In particular, $\lim(\vec{\theta}, \vec{\pi})$ is well-ordered, so $\lim^+(\vec{\theta}, \vec{\pi})$ is defined. Write $(\theta, \in, \langle \pi^+_\eta \mid \eta < \xi\rangle)$ for $\lim^+(\vec{\theta}, \vec{\pi})$.

Let Z be the set of equivalence classes in Y such that, for every representative (η, γ) from the class, we have that $\pi_\eta \restriction \gamma$ is bounded below $\pi_\eta(\gamma)$. By clause (5c) of Lemma 5.6 we know that $Z = \{z \in Y \mid \mathrm{cf}(\alpha_z) = \kappa\}$.

For all $\alpha \in B^\beta_{<\xi} \cap E^{\kappa^+}_\kappa$, we have $C_\alpha(\omega\xi) \in B^\beta_\xi \smallsetminus B^\beta_{<\xi}$ and $\alpha = \min(B^\beta_\xi \smallsetminus (C_\alpha(\omega\xi) + 1))$. Also, by clauses (5a) and (5b) of Lemma 5.6, we know that

$$B^\beta_\xi = B^\beta_{<\xi} \cup \{C_\alpha(\omega\xi) \mid \alpha \in B^\beta_{<\xi} \cap E^{\kappa^+}_\kappa\}.$$

Now, for all $z \in Z$, the addition of $(z, 0)$ when passing from Y to $W := \mathrm{double}_Z(Y, \lhd)$ corresponds precisely to the addition of $C_{\alpha_z}(\omega\xi)$ when passing from $B^\beta_{<\xi}$ to B^β_ξ. It follows that $\mathrm{otp}(W, \lhd_l) = \mathrm{otp}(B^\beta_\xi) = \theta^\beta_\xi$; that is, $\theta = \theta^\beta_\xi$. Letting $\pi^*: W \to \theta$ be the collapse map, we have that, for all $z \in Y$, $\pi^*(z, 1) = \pi^\beta_\xi(\alpha_z)$, so, for all $\eta < \xi$ and $\gamma < \theta^\beta_\eta$, we have

$$\pi^+_\eta(\gamma) = \pi^* \big([(\eta, \gamma)], 1 \big) = \pi^\beta_\xi \big((\pi^\beta_\eta)^{-1}(\gamma) \big) = \pi^\beta_{\eta, \xi}(\gamma),$$

so $\langle \pi^+_\eta \mid \eta < \xi \rangle = \langle \pi^\beta_{\eta, \xi} \mid \eta < \xi \rangle$. ∎

5.3 Diamond

Our next goal is to prove the following.

Lemma 5.8 *Suppose that $\diamond(\kappa)$ holds and that $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{Q}) \in \mathcal{P}_\kappa$. Then there are arrays $\langle \vartheta^\xi_\eta \mid \eta \leq \xi < \kappa \rangle$, $\langle \omega^\xi_{\eta, \eta'} \mid \eta < \eta' \leq \xi < \kappa \rangle$, and $\langle q^\xi_\eta \mid \eta < \xi < \kappa \rangle$ such that every $\beta < \kappa^+$ and every decreasing sequence $\langle p_\eta \mid \eta < \kappa \rangle \in \prod_{\eta < \kappa} \mathbb{P}_{B^\beta_\eta}$, there are stationarily many $\xi < \kappa$ such that*

- *$\langle \vartheta^\xi_\eta \mid \eta \leq \xi \rangle = \langle \theta^\beta_\eta \mid \eta \leq \xi \rangle$,*
- *$\langle \omega^\xi_{\eta, \eta'} \mid \eta < \eta' \leq \xi \rangle = \langle \pi^\beta_{\eta, \eta'} \mid \eta < \eta' \leq \xi \rangle$,*
- *$\langle q^\xi_\eta \mid \eta < \xi \rangle = \langle \pi^\beta_\eta . p_\eta \mid \eta < \xi \rangle$.*

The rest of this subsection will be devoted to proving Lemma 5.8. To avoid the use of codings, we will make use of the following equivalent version of $\diamond(\kappa)$ (see [2]).

Definition 5.9 $\diamond^-(H_\kappa)$ asserts the existence of a sequence $\langle A_\xi \mid \xi < \kappa \rangle$ such that, for every $A \subseteq H_\kappa$ and $p \in H_{\kappa^+}$, there exists an elementary submodel $\mathcal{M} \prec H_{\kappa^+}$, with $p \in \mathcal{M}$, such that $\kappa^\mathcal{M} := \mathcal{M} \cap \kappa$ is an ordinal $< \kappa$ and $A \cap \mathcal{M} = A_{\kappa^\mathcal{M}}$.

Fix a $\diamond^-(H_\kappa)$-sequence, $\langle A_\xi \mid \xi < \kappa \rangle$.

Definition 5.10 We say that $\xi < \kappa$ is *good* if $\xi \in \mathrm{acc}(\kappa)$ and

$$A_\xi = \{(\vartheta^\xi_\eta, q^\xi_\eta, \omega^\xi_{\eta, \eta'}, \eta, \eta') \mid \eta < \eta' < \xi\},$$

where, for all $\eta < \eta' < \eta'' < \xi$, we have

- $\vartheta^\xi_\eta \leq \vartheta^\xi_{\eta'} < \kappa$,
- $q^\xi_\eta \in \mathbb{P}_{\vartheta^\xi_\eta}$,

- $\varpi_{\eta,\eta'}^{\xi} \colon \vartheta_{\eta}^{\xi} \to \vartheta_{\eta'}^{\xi}$ is a c.o.i. and $q_{\eta'}^{\xi} \leq_{\mathbb{P}} \varpi_{\eta,\eta'}^{\xi}.q_{\eta}^{\xi}$,
- $\varpi_{\eta,\eta''}^{\xi} = \varpi_{\eta',\eta''}^{\xi} \circ \varpi_{\eta,\eta'}^{\xi}$,
- $\lim(\langle \vartheta_{\eta}^{\xi} \mid \eta < \xi \rangle, \langle \varpi_{\eta,\eta'}^{\xi} \mid \eta < \eta' < \xi \rangle)$ is well-ordered.

▸ If $\xi < \kappa$ is good, then $\langle \vartheta_{\eta}^{\xi} \mid \eta < \xi \rangle$, $\langle \varpi_{\eta,\eta'}^{\xi} \mid \eta < \eta' < \xi \rangle$, and $\langle q_{\eta}^{\xi} \mid \eta < \xi \rangle$ are already defined, and we let

$$\left(\vartheta_{\xi}^{\xi}, \in, \langle \varpi_{\eta,\xi}^{\xi} \mid \eta < \xi \rangle \right) := \lim{}^{+}\left(\langle \vartheta_{\eta}^{\xi} \mid \eta < \xi \rangle, \langle \varpi_{\eta,\eta'}^{\xi} \mid \eta < \eta' < \xi \rangle \right).$$

▸ If $\xi < \kappa$ is not good, then let $\langle \vartheta_{\eta}^{\xi} \mid \eta \leq \xi \rangle$, $\langle \varpi_{\eta,\eta'}^{\xi} \mid \eta < \eta' \leq \xi \rangle$, and $\langle q_{\eta}^{\xi} \mid \eta < \xi \rangle$ be arbitrary.

We claim that the arrays thus defined satisfy the conclusion of Lemma 5.8. To verify this, fix $\beta < \kappa^{+}$, a decreasing sequence $\langle p_{\eta} \mid \eta < \kappa \rangle \in \prod_{\eta < \kappa} \mathbb{P}_{B_{\eta}^{\beta}}$, and a club D in κ. Put

$$A := \left\{ (\theta_{\eta}^{\beta}, \pi_{\eta}^{\beta}.p_{\eta}, \pi_{\eta,\eta'}^{\beta}, \eta, \eta') \mid \eta < \eta' < \kappa \right\}.$$

Since $A \subseteq H_{\kappa}$ and $D \in H_{\kappa^{+}}$, we can let $p := \{A, D\}$ and fix an elementary submodel $\mathcal{M} \prec H_{\kappa^{+}}$ with $p \in \mathcal{M}$ such that $\xi := \mathcal{M} \cap \kappa$ is in κ and $A \cap \mathcal{M} = A_{\xi}$. By the fact that $D \in \mathcal{M}$ and the elementarity of $\mathcal{M}$, we have $\xi \in D$. Since $\mathcal{M} \cap \kappa = \xi$ and $A \in \mathcal{M}$, and by the elementarity of $\mathcal{M}$, we have

$$A_{\xi} = \left\{ (\theta_{\eta}^{\beta}, \pi_{\eta}^{\beta}.p_{\eta}, \pi_{\eta,\eta'}^{\beta}, \eta, \eta') \mid \eta < \eta' < \xi \right\}.$$

In particular, ξ is good. By Lemma 5.7, we have $\vartheta_{\xi}^{\xi} = \theta_{\xi}^{\beta}$ and, for all $\eta < \xi$, $\varpi_{\eta,\xi}^{\xi} = \pi_{\eta,\xi}^{\beta}$. Therefore, $\xi \in D$ satisfies the three bullet points in the statement of Lemma 5.8. Since D was arbitrary, this completes the proof of the lemma.

6   Proof of Theorem B

This section is devoted to the proof of Theorem B, which forms the main result of this paper.

Theorem B   *Suppose that $\square_{\kappa}^{B}$ and $\diamondsuit(\kappa)$ both hold. Then so does $\mathrm{SDFA}(\mathcal{P}_{\kappa})$.*

6.1   Setup

Fix an arbitrary $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q}) \in \mathcal{P}_{\kappa}$ along with a collection $\{\mathcal{D}_{i} \mid i < \kappa\}$ of sharply dense systems. For each $i < \kappa$, write $\mathcal{D}_{i} = \{D_{i,x} \mid x \in \binom{\kappa^{+}}{\theta_{\mathcal{D}_{i}}}\}$.

Let $\vec{B}$ be given by Lemma 5.6, and let $\langle \vartheta_{\eta}^{\xi} \mid \eta \leq \xi < \kappa \rangle$, $\langle \varpi_{\eta,\eta'}^{\xi} \mid \eta < \eta' \leq \xi < \kappa \rangle$, $\langle q_{\eta}^{\xi} \mid \eta < \xi < \kappa \rangle$ be given by Lemma 5.8 applied to $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{Q})$.

Definition 6.1   Let X denote the set of $\xi \in \mathrm{acc}(\kappa)$ such that
- ξ is good, in the sense of Definition 5.10;
- $\langle \varpi_{\eta,\xi}^{\xi}.q_{\eta}^{\xi} \mid \eta < \xi \rangle$ admits a lower bound in $\mathbb{P}_{\vartheta_{\xi}^{\xi}}$.

Let $\vartriangleleft_\kappa$ be some well-ordering of H_κ. Using $\kappa^{<\kappa} = \kappa$ (which follows from $\Diamond(\kappa)$), enumerate all elements of $\bigcup_{i<\kappa} \{i\} \times \kappa \times \binom{\kappa}{\theta_{\mathcal{D}_i}}$ as a sequence $\langle (i_\eta, j_\eta, z_\eta) \mid \eta < \kappa \rangle$.

Lemma 6.2 *There is a sequence of conditions $\langle s_\xi \mid \xi \in X \rangle \in \prod_{\xi \in X} \mathbb{Q}_{\vartheta_\xi}$, such that, for all $\xi \in X$*

- s_ξ *is a lower bound for $\langle \omega^\xi_{\eta,\xi}.q^\xi_\eta \mid \eta < \xi \rangle$;*
- *for all $\eta < \xi$, if $j_\eta < \xi$ and $z_\eta \subseteq \vartheta^\xi_{j_\eta}$, then there is $q \in D_{i_\eta, \omega^\xi_{j_\eta,\xi} \text{``} z_\eta}$ such that $s_\xi \leq_\mathbb{P} q$.*

Proof Let $\xi \in X$ be arbitrary. We first define a sequence $\langle s^\eta \mid \eta \leq \xi \rangle \in \prod_{\eta \leq \xi} \mathbb{Q}_{\vartheta^\xi_\xi}$ by recursion on η:

▸ For $\eta = 0$, use clauses (1c) and (6) of Definition 2.1 and the fact that $\xi \in X$ to find $s^0 \in \mathbb{Q}$ such that $x_{s^0} = \vartheta^\xi_\xi$ and s^0 is a lower bound for $\langle \omega^\xi_{\eta,\xi}.q^\xi_\eta \mid \eta < \xi \rangle$.

▸ For $\eta < \xi$, with $j_\eta < \xi$ and $z_\eta \subseteq \vartheta^\xi_{j_\eta}$, use the fact that $\mathcal{D}_{i_\eta}$ is a sharply dense system and that $\omega^\xi_{j_\eta,\xi} \text{``} z_\eta \subseteq \vartheta^\xi_\xi$ to find $s^{\eta,*} \in D_{i_\eta, \omega^\xi_{j_\eta,\xi} \text{``} z_\eta}$ such that $s^{\eta,*} \leq_\mathbb{P} s^\eta$ and $x_{s^{\eta,*}} = \vartheta^\xi_\xi$. Then, use Definition 2.1(6) to find $s^{\eta+1} \in \mathbb{Q}$ such that $s^{\eta+1} \leq_\mathbb{P} s^{\eta,*}$ and $x_{s^{\eta+1}} = \vartheta^\xi_\xi$.

▸ For $\eta < \xi$ with $j_\eta \geq \xi$ or $z_\eta \not\subseteq \vartheta^\xi_{j_\eta}$, simply let $s^{\eta+1} := s^\eta$.

▸ For $\eta \in \mathrm{acc}(\xi + 1)$, assuming that $\langle s^\zeta \mid \zeta < \eta \rangle$ has already been defined, use Definition 2.1(7) to let s^η be a lower bound for $\langle s^\zeta \mid \zeta < \eta \rangle$ in $\mathbb{Q}$ with $x_{s^\eta} = \vartheta^\xi_\xi$.

Having constructed $\langle s^\eta \mid \eta \leq \xi \rangle$, it is clear that $s_\xi := s^\xi$ is as sought. ∎

Fix a sequence $\langle s_\xi \mid \xi \in X \rangle$ as in the preceding lemma. We will construct a matrix of conditions $\langle p^\beta_\eta \mid \beta < \kappa^+,\ \eta < \kappa \rangle$ satisfying the following conditions:

(i) for all $\beta < \kappa^+$ and $\eta < \kappa$, we have $p^\beta_\eta \in \mathbb{P}_{B^\beta_\eta}$;

(ii) for all $\beta < \kappa^+$, $\langle p^\beta_\eta \mid \eta < \kappa \rangle$ is $\leq_\mathbb{P}$-decreasing;

(iii) for all $\beta < \kappa^+$, $\eta < \kappa$, and $\alpha \in B^\beta_\eta$, we have $p^\beta_\eta \restriction (\alpha + 1) = p^\alpha_\eta$;

(iv) for all $\beta < \kappa^+$, all $i < \kappa$, and all $x \in \binom{\beta+1}{\theta_{\mathcal{D}_i}}$, there is $\xi < \kappa$ and $q \in D_{i,x}$ such that $p^\beta_\xi \leq_\mathbb{P} q$;

(v) for all $\beta \in E^{\kappa^+}_\kappa$ and all $\xi \in \mathrm{acc}(\kappa)$, the sequence $\langle \pi^\beta_\xi.p^\beta_\eta \mid \eta \leq \xi \rangle$ depends only on the value of $C_\beta(\omega\xi)$.

Note that if we succeed, then letting G be the upward closure of $\{p^\beta_\eta \mid \beta < \kappa^+,\ \eta < \kappa\}$, it follows from (i)–(iv) that G is a filter on $\mathbb{P}$ that, for each $\eta < \kappa$, meets $\mathcal{D}_\eta$ everywhere. Of course, the sequence $\langle s_\xi \mid \xi \in X \rangle$, which was derived from $\Diamond$, will be a key to ensuring clause (iv).

6.2 Hypotheses

The construction of $\langle p^\beta_\eta \mid \beta < \kappa^+,\ \eta < \kappa \rangle$ will be by recursion on $\eta < \kappa$ and, for fixed η, by recursion on $\beta < \kappa^+$. We will maintain requirements (i)–(iii) and (v) as recursion hypotheses. In order to ensure that the construction will be successful, we

need to carry along some further hypotheses. Suppose that $\beta < \kappa^+$, $\xi \in \mathrm{acc}(\kappa)$, and $\langle p_\eta^\alpha \mid \alpha < \kappa^+, \ \eta < \xi \rangle$ has been constructed.

Definition 6.3   We say that the pair (β, ξ) is *active* if $\xi \in X$, $\theta_\xi^\beta \leq \vartheta_\xi^\xi$, and one of the following holds:

- $\xi > \eta_\beta$ and, for all $\eta < \xi$, $s_\xi \leq_\mathbb{P} \pi_\xi^\beta . p_\eta^\beta$;
- $\xi = \eta_\beta$ and there is $\gamma \in E_\kappa^{\kappa^+}$ such that $\beta \in \mathrm{acc}(C_\gamma)$ and (γ, ξ) is active.

In our construction, we will require that, for all active (β, ξ), we have $\pi_\xi^\beta . p_\xi^\beta = s_\xi \restriction \theta_\xi^\beta$. In particular, if (β, ξ) is active, then $p_\xi^\beta \in \mathbb{Q}$ and $x_{p_\xi^\beta} = B_\xi^\beta$. Moreover, for all $\beta < \kappa^+$, we will arrange that, if $\xi < \kappa$ is least such that $\beta \in x_{p_\xi^\beta}$, then either (β, ξ) is active or ($\xi = \eta_\beta$ and $p_\xi^\beta \in \mathbb{Q}$).

Lemma 6.4   *Suppose that $\beta < \kappa^+$, $\xi \in X$, and (β, ξ) is active. Then (α, ξ) is active for all $\alpha \in B_\xi^\beta$.*

Proof   Let $\alpha \in B_\xi^\beta$ be arbitrary. As $B_\xi^\beta \cap (\alpha + 1) = B_\xi^\alpha$, we have $\theta_\xi^\alpha < \theta_\xi^\beta \leq \vartheta_\xi^\xi = x_{s_\xi}$.

▸ If $\xi > \eta_\beta$ and $\alpha \in B_{<\xi}^\beta$, then $\xi > \eta_\alpha$ and, for all sufficiently large $\eta < \xi$, we have $p_\eta^\alpha = p_\eta^\beta \restriction (\alpha + 1)$. By Lemma 5.6(2), then, $s_\xi \leq_\mathbb{P} \pi_\eta^\beta . p_\eta^\alpha = \pi_\eta^\alpha . p_\eta^\alpha$, so (α, ξ) is active.

▸ If $\xi > \eta_\beta$ and $\alpha \in B_\xi^\beta \smallsetminus B_{<\xi}^\beta$, then let $\gamma := \min(B_\xi^\beta \smallsetminus (\alpha + 1))$. By Lemma 5.6(b), we know that $\mathrm{cf}(\gamma) = \kappa$ and $\alpha = C_\gamma(\omega\xi)$. It follows that $\alpha \in \Gamma$ and hence, by Lemma 5.6(4), we have $\xi = \eta_\alpha$. Since $|B_\xi^\beta| < \kappa$, and by Lemma 5.6(a), we know that $\gamma \in B_{<\xi}^\beta$, so, by the previous paragraph, (γ, ξ) is active. Hence, by Definition 6.3, (α, ξ) is active as well.

▸ If $\xi = \eta_\beta$ and $\gamma \in E_\kappa^{\kappa^+}$ is such that $\beta \in \mathrm{acc}(C_\gamma)$ and (γ, ξ) is active, then by clauses (2) and (3) of Lemma 5.6, $B_\xi^\gamma \cap (\beta + 1) = B_\xi^\beta$, so $\alpha \in B_\xi^\gamma$. Moreover, $\xi > \eta_\gamma = 0$, so, by the previous cases, we again conclude that (α, ξ) is active.   ∎

Our final recursion hypotheses concern nonactive pairs (β, ξ).

First, suppose that (β, ξ) is not active and $\xi = \eta_\beta$. If $\xi \in \mathrm{acc}(\kappa)$ and there is $\gamma \in E_\kappa^{\kappa^+}$ such that $\beta \in \mathrm{acc}(C_\gamma)$ and $\sup\{\eta < \xi \mid (\gamma, \eta) \text{ is active}\} = \xi$, then we will require that $p_\xi^\beta \in \mathbb{Q}$ and $x_{p_\xi^\beta} = B_\xi^\beta$.

Next, suppose that (β, ξ) is not active and $\xi > \eta_\beta$. Let

$$\eta^* := \max\{\sup\{\eta < \xi \mid (\beta, \eta) \text{ is active}\}, \eta_\beta\}.$$

▸ If $\eta^* = \xi$, then we will require that $p_\xi^\beta \in \mathbb{Q}$ and $x_{p_\xi^\beta} = B_\xi^\beta$.

▸ If $\eta^* < \xi$ and $\beta \in x_{p_{\eta^*}^\beta}$, then we will have $p_{\eta^*}^\beta \in \mathbb{Q}$ and will require that p_ξ^β is the $\leq_\mathbb{P}$-greatest condition q such that $q \leq_\mathbb{P} p_{\eta^*}^\beta$ and, for all $\alpha \in B_\xi^\beta \cap \beta$, $q \restriction (\alpha + 1) = p_\xi^\alpha$.

6.3 The Construction

We now turn to the actual construction. Suppose that $\beta < \kappa^+$, $\xi < \kappa$, and we have already constructed $\langle p_\eta^\alpha \mid \alpha < \kappa^+,\ \eta < \xi \rangle$ and $\langle p_\xi^\alpha \mid \alpha < \beta \rangle$. We now construct p_ξ^β. There are a number of cases to consider. In all cases, unless explicitly verified, it will be trivial to check that the recursion hypotheses are maintained.

Case 0: $\xi < \eta_\beta$. Let $p_\eta^\beta := \mathbb{1}_\mathbb{P}$.

Case 1: $\xi = \eta_\beta$. There are now a few subcases to consider.

> *Subcase 1a*: (β, ξ) **is active.** Then, in particular, $x_{s_\xi} = \vartheta_\xi^\xi \geq \theta_\xi^\beta$. Let p_ξ^β be the unique condition q such that $x_q = B_\xi^\beta$ and $\pi_\xi^\beta.q = s_\xi \restriction \theta_\xi^\beta$, i.e., $p_\xi^\beta = (\pi_\xi^\beta)^{-1}.(s_\xi \restriction \theta_\xi^\beta)$. Note that, for all $\alpha \in B_\xi^\beta$, Lemma 6.4 implies that (α, ξ) is active. We therefore have $\pi_\xi^\beta.p_\xi^\beta = \pi_\xi^\alpha.p_\xi^\alpha = s_\xi \restriction \theta_\xi^\beta$, so $p_\xi^\beta \restriction (\alpha + 1) = p_\xi^\alpha$, and requirement (iii) is satisfied.

> *Subcase 1b*: (β, ξ) **is not active and there is** $\gamma \in E_\kappa^{\kappa^+}$ **such that** $\beta \in \mathrm{acc}(C_\gamma)$ **and** $\sup\{\eta < \xi \mid (\gamma, \eta)$ **is active**$\} = \xi$. Fix such a γ. Note that $B_{<\xi}^\gamma \cap \beta$ is unbounded in $B_\xi^\beta \cap \beta$ and, for all $\alpha \in B_{<\xi}^\gamma \cap \beta$, $\sup\{\eta < \xi \mid (\alpha, \eta)$ is active$\} = \xi$. Therefore, by our recursion hypotheses, for all $\alpha \in B_{<\xi}^\gamma$, we know that $p_\xi^\alpha \in \mathbb{Q}$ and $x_{p_\xi^\alpha} = B_\xi^\alpha$. By clause (5) of Definition 2.1, there is a unique condition $q \in \mathbb{Q}$ such that $x_q = B_\xi^\beta \cap \beta$ and, for all $\alpha \in B_\xi^\beta$, we have $q \restriction (\alpha + 1) = p_\xi^\alpha$. By Definition 2.1(7), there is a lower bound p for $\langle p_\eta^\gamma \mid \eta < \xi \rangle$ such that

> - $p \in \mathbb{Q}$;
> - $x_p = B_\xi^\gamma = B_\xi^\beta \cup \{\gamma\}$;
> - $p \restriction \beta = q$.

> Fix such a lower bound p with a $\lhd_\kappa$-minimal possible value for $\pi_\xi^\gamma.p$, and let $p_\xi^\beta := p \restriction (\beta + 1)$. Note that, by requirement (v), the construction in this Subcase is independent of our choice of γ.

> *Subcase 1c*: *Otherwise.* Let p_ξ^β be the unique condition, given by Definition 2.1(5), such that

$$x_{p_\xi^\beta} = \bigcup \left\{ x_{p_\xi^\alpha} \mid \alpha \in (B_\xi^\beta \cap \beta) \right\}$$

> and, for all $\alpha \in (B_\xi^\beta \cap \beta)$, we have $p_\xi^\beta \restriction (\alpha + 1) = p_\xi^\alpha$.

Case 2: $\xi > \eta_\beta$. There are again a few subcases to consider.

> *Subcase 2a*: (β, ξ) *is active.* Let p_ξ^β be the unique condition q such that $x_q = B_\xi^\beta$ and $\pi_\xi^\beta.q = s_\xi \restriction \theta_\xi^\beta$. By Definition 6.3, we have that $s_\xi \leq_\mathbb{P} \pi_\xi^\beta.p_\eta^\beta$ for all $\eta < \xi$, which implies that $p_\xi^\beta \leq_\mathbb{P} p_\eta^\beta$ for all $\eta < \xi$, so requirement (i) holds.

> *Subcase 2b*: (β, ξ) *is not active,* $\sup\{\eta < \xi \mid (\beta, \eta)$ *is active*$\} = \xi$, *and* $\beta \notin E_\kappa^{\kappa^+}$. In this subcase, we have that $\xi \in \mathrm{acc}(\kappa)$ and $B_{<\xi}^\beta \cap \beta$ is unbounded in $B_\xi^\beta \cap \beta$. Since, for all $\alpha \in B_{<\xi}^\beta$, we know that $\sup\{\eta < \xi \mid (\alpha, \eta)$ is active$\} = \xi$, it follows as in Subcase 1b that there is a unique condition $q \in \mathbb{Q}$ such that $x_q = B_\xi^\beta \cap \beta$ and, for all $\alpha \in B_\xi^\beta$, we have $q \restriction (\alpha + 1) = p_\xi^\alpha$. By Definition 2.1(7), there is $p \in \mathbb{Q}$ such that

- p is a lower bound for $\langle p_\eta^\beta \mid \eta < \xi \rangle$;
- $p \in \mathbb{Q}$;
- $x_p = B_\xi^\beta$;
- $p \restriction \beta = q$.

Let p_ξ^β be such a p.

Subcase 2c: (β, ξ) *is not active,* $\sup\{\eta < \xi \mid (\beta, \eta)$ *is active*$\} = \xi$, *and* $\beta \in E_\kappa^{\kappa^+}$. Let $\alpha := C_\beta(\omega\xi)$, so that $B_\xi^\beta = B_\xi^\alpha \cup \{\beta\}$. When defining p_ξ^α, we were in Subcase 1b. In that subcase, we considered a $\gamma \in E_\kappa^{\kappa^+}$ such that $\alpha \in \mathrm{acc}(C_\gamma)$, produced a condition p with $x_p = B_\xi^\gamma$, and let $p_\xi^\alpha := p \restriction (\alpha + 1)$. Let $\pi: B_\xi^\gamma \to B_\xi^\beta$ be the unique order-preserving bijection, and let $p_\xi^\beta = \pi.p$. Since, by requirement (v), we have

$$\langle \pi_\xi^\beta.p_\eta^\beta \mid \eta < \xi \rangle = \langle \pi_\xi^\gamma.p_\eta^\gamma \mid \eta < \xi \rangle,$$

and since $\pi \restriction B_\xi^\alpha$ is the identity, the recursion hypotheses are all easily verified.

Subcase 2d: (β, ξ) *is not active and there is no* $\eta < \xi$ *such that* $\beta \in x_{p_\eta^\beta}$. Let p_ξ^β be the unique condition q such that $x_q = \bigcup\{x_{p_\xi^\alpha} \mid \alpha \in (B_\xi^\beta \cap \beta)\}$ and, for all $\alpha \in (B_\xi^\beta \cap \beta)$, we have $q \restriction (\alpha + 1) = p_\xi^\alpha$.

Subcase 2e: *Otherwise.* Let

$$\eta^* := \max\left\{ \sup\{\eta < \xi \mid (\beta, \eta)\text{ is active}\}, \eta_\beta \right\}.$$

Since we are not in any of the previous Subcases, it must be the case that $\eta^* < \eta$, $p_{\eta^*}^\beta \in \mathbb{Q}$, and $x_{p_{\eta^*}^\beta} = B_{\eta^*}^\beta$. For all $\eta \in (\eta^*, \xi]$, let q_η be the unique condition, given by Definition 2.1(5), such that

$$x_{q_\eta} = \bigcup\left\{ x_{p_\eta^\alpha} \mid \alpha \in (B_\eta^\beta \cap \beta) \right\}$$

and, for all $\alpha \in (B_\eta^\beta \cap \beta)$, we have $q \restriction (\alpha + 1) = p_\eta^\alpha$. By the recursion hypotheses, we know that, for all $\eta \in (\eta^*, \xi)$, p_η^β is the $\leq_\mathbb{P}$-greatest lower bound of $p_{\eta^*}^\beta$ and q_η, as given by Definition 2.1(8). Therefore, if we let p_ξ^β be the $\leq_\mathbb{P}$-greatest lower bound of $p_{\eta^*}^\beta$ and q_ξ, which again exists by Definition 2.1(8), it will follow that $p_\xi^\beta \leq_\mathbb{P} p_\eta^\beta$ for all $\eta < \xi$, so requirement (i) holds. The other requirements are easily verified.

This completes the construction. We have maintained requirements (i)–(iii) and (v) throughout. We now verify requirement (iv). To this end, fix $\beta < \kappa^+$, $i < \kappa$, and $x \in \binom{\beta+1}{\theta_{\mathcal{D}_i}}$. We will find $\xi < \kappa$ and $q \in D_{i,x}$ such that $p_\xi^\beta \leq_\mathbb{P} q$.

Fix $j < \kappa$ such that $x \subseteq B_j^\beta$, and fix $\eta^* < \kappa$ such that $(i_{\eta^*}, j_{\eta^*}, z_{\eta^*}) = (i, j, \pi_j^{\beta}``x)$. Find $\xi \in \mathrm{acc}(\kappa \smallsetminus (\max\{j, \eta^*, \eta_\beta\} + 1))$ such that

- $\langle \vartheta_\eta^\xi \mid \eta \leq \xi \rangle = \langle \theta_\eta^\beta \mid \eta \leq \xi \rangle$;
- $\langle \varpi_{\eta, \eta'}^\xi \mid \eta < \eta' \leq \xi \rangle = \langle \pi_{\eta, \eta'}^\beta \mid \eta < \eta' \leq \xi \rangle$;
- $\langle q_\eta^\xi \mid \eta < \xi \rangle = \langle \pi_\eta^\beta.p_\eta^\beta \mid \eta < \xi \rangle$.

The following two claims now suffice for the verification of requirement (iv).

Claim 6.5 (β, ξ) *is active.*

Proof We verify the requirements in Definition 6.3. We clearly have $\theta_\xi^\beta \leq \vartheta_\xi^\xi$ and $\xi > \eta_\beta$. Moreover, for all $\eta < \xi$, we have

$$\varpi_{\eta,\xi}^\xi . q_\eta^\xi = \pi_{\eta,\xi}^\beta . \pi_\eta^\beta . p_\eta^\beta = \pi_\xi^\beta . p_\eta^\beta.$$

Since $p_\xi^\beta \in \mathbb{P}_{B_\xi^\beta}$ is a lower bound for $\langle p_\eta^\beta \mid \eta < \xi \rangle$, it follows that $\pi_\xi^\beta . p_\xi^\beta \in \mathbb{P}_{\vartheta_\xi^\xi}$ is a lower bound for

$$\langle \varpi_{\eta,\xi}^\xi . q_\eta^\xi \mid \eta < \xi \rangle.$$

In particular, $\xi \in X$. It follows that s_ξ is a lower bound for

$$\langle \varpi_{\eta,\xi}^\xi . q_\eta^\xi \mid \eta < \xi \rangle = \langle \pi_\xi^\beta . p_\eta^\beta \mid \eta < \xi \rangle,$$

which completes the verification. ∎

Claim 6.6 *There is* $q \in D_{i,x}$ *such that* $p_\xi^\beta \leq_\mathbb{P} q$.

Proof Since (β, ξ) is active and $\theta_\xi^\beta = \vartheta_\xi^\xi$, we have $\pi_\xi^\beta . p_\xi^\beta = s_\xi$. It thus suffices to find $q' \in D_{i, \pi_\xi^\beta {}^{\prime\prime} x}$ such that $s_\xi \leq_\mathbb{P} q'$.

Note that $\eta^*, j < \xi$ and $\pi_j^\beta {}^{\prime\prime} x \subseteq \theta_j^\beta = \vartheta_j^\xi$. Therefore, since

$$(i, j, \pi_j^\beta {}^{\prime\prime} x) = (i_{\eta^*}, j_{\eta^*}, z_{\eta^*})$$

and $\langle s_\xi \mid \xi \in X \rangle$ satisfies the conclusion of Lemma 6.2, it follows that there is $q' \in D_{i, \varpi_{j,\xi}^\xi {}^{\prime\prime} \pi_j^\beta {}^{\prime\prime} x} = D_{i, \pi_\xi^\beta {}^{\prime\prime} x}$ such that $s_\xi \leq_\mathbb{P} q'$, as desired. ∎

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