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Siegel families with application to class fields

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We investigate certain families of meromorphic Siegel modular functions on which Galois groups act in a natural way. By using Shimura's reciprocity law we construct some algebraic numbers in the ray class fields of CM-fields in terms of special values of functions in these Siegel families.

Keywords: abelian varieties; class field theory; Shimura's reciprocity law; CM-fields; Siegel modular functions

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1. Introduction

For a positive integer N let $\mathfrak{F}_N$ be the field of meromorphic modular functions of level N (defined on $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$) whose Fourier coefficients belong to the Nth cyclotomic field. As is well known, $\mathfrak{F}_N$ is a Galois extension of $\mathfrak{F}_1$ whose Galois group is isomorphic to $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}$ (see [11, § 6.1–6.2]). Now, let $N \geqslant 2$ and consider a set

$$V_N = \{\boldsymbol{v} \in \mathbb{Q}^2 \mid N \text{ is the smallest positive integer for which } N\boldsymbol{v} \in \mathbb{Z}^2\}$$

as the index set. We call a family $\{f_{\boldsymbol{v}}(\tau)\}_{\boldsymbol{v}\in V_N}$ of functions in $\mathfrak{F}_N$ a *Fricke family* of level N if each $f_{\boldsymbol{v}}(\tau)$ depends only on $\pm\boldsymbol{v} \pmod{\mathbb{Z}^2}$ and satisfies

$$f_{\boldsymbol{v}}(\tau)^{\alpha} = f_{\alpha^{\mathrm{T}}\boldsymbol{v}}(\tau) \quad (\alpha \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\{\pm I_2\}),$$

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where α^{T} means the transpose of α. For example, Siegel functions of one variable form such a Fricke family of level N [8, ch. 2, proposition 1.3] (see also [4] or [6]).

Let K be an imaginary quadratic field with the ring of integers $\mathcal{O}_K$, and let $\mathfrak{f}$ be a proper non-trivial ideal of $\mathcal{O}_K$. We denote by $\mathrm{Cl}(\mathfrak{f})$ and $K_{\mathfrak{f}}$ the ray class group modulo $\mathfrak{f}$ and its corresponding ray class field modulo $\mathfrak{f}$, respectively. If $\{f_{\boldsymbol{v}}(\tau)\}_{\boldsymbol{v}}$ is a Fricke family of level N in which every $f_{\boldsymbol{v}}(\tau)$ is holomorphic on $\mathbb{H}$, then we can assign to each ray class $\mathcal{C} \in \mathrm{Cl}(\mathfrak{f})$ an algebraic number $f_{\mathfrak{f}}(\mathcal{C})$ as a special value of a function in $\{f_{\boldsymbol{v}}(\tau)\}_{\boldsymbol{v}}$. Furthermore, we attain by Shimura's reciprocity law that $f_{\mathfrak{f}}(\mathcal{C})$ belongs to $K_{\mathfrak{f}}$ and satisfies

$$f_{\mathfrak{f}}(\mathcal{C})^{\sigma_{\mathfrak{f}}(\mathcal{D})} = f_{\mathfrak{f}}(\mathcal{C}\mathcal{D}) \quad (\mathcal{D} \in \mathrm{Cl}(\mathfrak{f})),$$

where $\sigma_{\mathfrak{f}}$ is the Artin reciprocity map for $\mathfrak{f}$ (see [8, ch. 11, theorem 1.1]).

In this paper we shall define a Siegel family $\{h_M(Z)\}_M$ of level N consisting of meromorphic Siegel modular functions of (higher) genus g and level N, which is a generalization of a Fricke family of level N in the case when $g = 1$ (definition 3.1). It turns out that every Siegel family of level N is induced from a meromorphic Siegel modular function for the congruence subgroup $\Gamma^1(N)$ with rational Fourier coefficients (theorem 3.5).

Let K be a CM-field and let $\mathfrak{f} = N\mathcal{O}_K$. Given a Siegel family $\{h_M(Z)\}_M$ of level N, we shall introduce a number $h_{\mathfrak{f}}(\mathcal{C})$ as a special value of a function in $\{h_M(Z)\}_M$ for each ray class $\mathcal{C} \in \mathrm{Cl}(\mathfrak{f})$ (definition 5.4). Under certain assumptions on K (assumption 5.1) we shall prove that if $h_{\mathfrak{f}}(\mathcal{C})$ is finite, then it lies in the ray class field $K_{\mathfrak{f}}$ whose Galois conjugates are of the same form (theorem 7.2 and corollary 7.3). To this end, we assign a principally polarized abelian variety to each non-trivial ideal of $\mathcal{O}_K$, and apply Shimura's reciprocity law to $h_{\mathfrak{f}}(\mathcal{C})$.

On the other hand, we note that there is a remarkable paper by Grant [2] in which he generalized a classical formula of Eisenstein and obtained classes of S-units by evaluating abelian functions at the intersections of divisors on the Jacobian of the curve $y^2 = x^5 + \frac{1}{4}$. We hope that our invariant $h_{\mathfrak{f}}(\mathcal{C})$ obtained from a Siegel family in theorem 4.3 will contribute further towards finding a higher-dimensional analogue of an elliptic unit.

2. Actions on Siegel modular functions

First, we shall describe the Galois group between fields of meromorphic Siegel modular functions in a concrete way.

Let g be a positive integer and let

$$\eta_g = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$

For every commutative ring R with unity we define

$$\begin{aligned} \mathrm{GSp}_{2g}(R) &= \{\alpha \in \mathrm{GL}_{2g}(R) \mid \alpha^{\mathrm{T}}\eta_g\alpha = \nu(\alpha)\eta_g \text{ with } \nu(\alpha) \in R^{\times}\}, \\ \mathrm{Sp}_{2g}(R) &= \{\alpha \in \mathrm{GSp}_{2g}(R) \mid \nu(\alpha) = 1\}. \end{aligned}$$

Let

$$G = \mathrm{GSp}_{2g}(\mathbb{Q})$$

and let $G_{\mathbb{A}}$ be the adelization of G, let G_0 be its non-Archimedean part and let G_∞ be its Archimedean part. One can extend the multiplier map $\nu\colon G \to \mathbb{Q}^\times$ continuously to the map $\nu\colon G_{\mathbb{A}} \to \mathbb{Q}_{\mathbb{A}}^\times$ and set

$$G_{\infty+} = \{\alpha \in G_\infty \mid \nu(\alpha) > 0\}, \qquad G_{\mathbb{A}+} = G_0 G_{\infty+}, \qquad G_+ = G \cap G_{\mathbb{A}+}.$$

Furthermore, let

$$\varDelta = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \;\middle|\; s \in \prod_p \mathbb{Z}_p^\times \right\},$$
$$U_1 = \prod_p \mathrm{GSp}_{2g}(\mathbb{Z}_p) \times G_{\infty+},$$
$$U_N = \{x \in U_1 \mid x_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)} \text{ for all rational primes } p\}$$

for every positive integer N. Then we have

$$U_N \trianglelefteq U_1 \leqslant G_{\mathbb{A}+} \quad \text{and} \quad G_{\mathbb{A}+} = U_N \varDelta G_+$$

(see [13, lemma 8.3(1)]).

Note that the group $G_{\infty+}$ acts on the Siegel upper half-space

$$\mathbb{H}_g = \{Z \in M_g(\mathbb{C}) \mid Z^{\mathrm{T}} = Z,\ \mathrm{Im}(Z) \text{ is positive definite}\}$$

by

$$\alpha(Z) = (AZ + B)(CZ + D)^{-1} \quad (\alpha \in G_{\infty+},\ Z \in \mathbb{H}_g),$$

where A, B, C, D are $g \times g$ block matrices of α. Let $\mathcal{F}_N$ be the field of meromorphic Siegel modular functions of genus g for the congruence subgroup

$$\varGamma(N) = \{\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \mid \gamma \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z})}\}$$

of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ whose Fourier coefficients belong to the Nth cyclotomic field $\mathbb{Q}(\zeta_N)$ with $\zeta_N = \mathrm{e}^{2\pi \mathrm{i}/N}$. That is, if $f \in \mathcal{F}_N$, then

$$f(Z) = \frac{\sum_h c(h) e(\mathrm{tr}(hZ)/N)}{\sum_h d(h) e(\mathrm{tr}(hZ)/N)} \quad \text{for some } c(h), d(h) \in \mathbb{Q}(\zeta_N),$$

where the denominator and numerator of f are Siegel modular forms of the same weight, h runs over all $g \times g$ positive semi-definite symmetric matrices over half integers with integral diagonal entries, and $e(w) = \mathrm{e}^{2\pi \mathrm{i} w}$ for $w \in \mathbb{C}$ [5, § 4, theorem 1]. Let

$$\mathcal{F} = \bigcup_{N=1}^{\infty} \mathcal{F}_N.$$

Proposition 2.1. *There exists a homomorphism $\tau\colon G_{\mathbb{A}+} \to \mathrm{Aut}(\mathcal{F})$ satisfying the following properties. Let*

$$f(Z) = \frac{\sum_h c(h) e(\mathrm{tr}(hZ)/N)}{\sum_h d(h) e(\mathrm{tr}(hZ)/N)} \in \mathcal{F}_N.$$

(i) *If $\alpha \in G_+ = \{\alpha \in G \mid \nu(\alpha) > 0\}$, then*

$$f^{\tau(\alpha)} = f \circ \alpha.$$

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(ii) *If*

$$\beta = \begin{bmatrix} I_g & O_g \\ O_g & sI_g \end{bmatrix} \in \Delta$$

and t is a positive integer such that $t \equiv s_p \pmod{N\mathbb{Z}_p}$ *for all rational primes p, then*

$$f^{\tau(\beta)} = \frac{\sum_h c(h)^\sigma e(\mathrm{tr}(hZ)/N)}{\sum_h d(h)^\sigma e(\mathrm{tr}(hZ)/N)},$$

where σ *is the automorphism of* $\mathbb{Q}(\zeta_N)$ *given by* $\zeta_N^\sigma = \zeta_N^t$.

(iii) *For every positive integer N we have*

$$\mathcal{F}_N = \{f \in \mathcal{F} \mid f^{\tau(x)} = f \text{ for all } x \in U_N\}.$$

(iv) *We have* $\ker(\tau) = \mathbb{Q}^\times G_{\infty+}$.

Proof. See [13, theorem 8.10]. □

Since

$$U_N(\mathbb{Q}^\times G_{\infty+})/\mathbb{Q}^\times G_{\infty+} \simeq U_N/(U_N \cap \mathbb{Q}^\times G_{\infty+}) \simeq \begin{cases} U_1/\pm G_{\infty+} & \text{if } N = 1, \\ U_N/G_{\infty+} & \text{if } N > 1, \end{cases}$$

we see by proposition 2.1(iii) and (iv) that $\mathcal{F}_N$ is a Galois extension of $\mathcal{F}_1$ with

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N. \tag{2.1}$$

Proposition 2.2. *We have*

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

Proof. Let $\alpha \in U_1$. Take a matrix A in $M_{2g}(\mathbb{Z})$ for which $A \equiv \alpha_p \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes p. Define a matrix $\psi(\alpha) \in M_{2g}(\mathbb{Z}/N\mathbb{Z})$ by the image of A under the natural reduction $M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z})$. Then, by the Chinese remainder theorem, $\psi(\alpha)$ is well defined and independent of the choice of A. Furthermore, let t be an integer relatively prime to N such that $t \equiv \nu(\alpha_p) \pmod{N\mathbb{Z}_p}$ for all rational primes p. We then derive that

$$t\eta_g \equiv \nu(\alpha_p)\eta_g \equiv \alpha_p^{\mathrm{T}}\eta_g\alpha_p \equiv A^{\mathrm{T}}\eta_g A \equiv \psi(\alpha)^{\mathrm{T}}\eta_g\psi(\alpha) \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$$

for all rational primes p, and hence $\psi(\alpha) \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Thus, we obtain a group homomorphism

$$\psi\colon U_1 \to \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}).$$

Let $\beta \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and take a preimage B of β under the natural reduction $M_{2g}(\mathbb{Z}) \to M_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since $\nu(\beta) \in (\mathbb{Z}/N\mathbb{Z})^\times$ and

$$B^{\mathrm{T}}\eta_g B \equiv \beta^{\mathrm{T}}\eta_g\beta \equiv \nu(\beta)\eta_g \pmod{N \cdot M_{2g}(\mathbb{Z})},$$

B belongs to $\mathrm{GSp}_{2g}(\mathbb{Z}_p)$ for every rational prime p dividing N. Let $\alpha = (\alpha_p)_p$ be the element of $\prod_p \mathrm{GSp}_{2g}(\mathbb{Z}_p)$ given by

$$\alpha_p = \begin{cases} B & \text{if } p \mid N, \\ I_{2g} & \text{otherwise.} \end{cases}$$

We then see that $\alpha \in U_1$ and $\psi(\alpha) = \beta$. Thus, ψ is surjective.

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Clearly, U_N is contained in $\ker(\psi)$. Let $\gamma \in \ker(\psi)$. Since $\gamma_p \equiv I_{2g} \pmod{N \cdot M_{2g}(\mathbb{Z}_p)}$ for all rational primes p, we get $\gamma \in U_N$, and hence $\ker(\psi) = U_N$. Therefore, ψ induces an isomorphism $U_1/U_N \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, from which we achieve, by (2.1),

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq U_1/\pm U_N \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

□

REMARK 2.3. We have the decomposition

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1) \simeq \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq G_N \cdot \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\},$$

where

$$G_N = \left\{ \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix} \;\middle|\; \nu \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}.$$

By proposition 2.1 one can describe the action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\mathcal{F}_N$ as follows.

Let

$$f(Z) = \frac{\sum_h c(h) e(\mathrm{tr}(hZ)/N)}{\sum_h d(h) e(\mathrm{tr}(hZ)/N)} \in \mathcal{F}_N.$$

(i) An element

$$\beta = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}$$

of G_N acts on f by

$$f^\beta = \frac{\sum_h c(h)^\sigma e(\mathrm{tr}(hZ)/N)}{\sum_h d(h)^\sigma e(\mathrm{tr}(hZ)/N)},$$

where σ is the automorphism of $\mathbb{Q}(\zeta_N)$ satisfying $\zeta_N^\sigma = \zeta_N^\nu$.

(ii) An element γ of $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ acts on f by

$$f^\gamma = f \circ \gamma',$$

where γ' is any preimage of γ under the natural reduction

$$\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}.$$

3. Siegel families of level N

By making use of the description of $\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$ in § 2 we shall introduce a generalization of a Fricke family in higher dimensional cases.

Let $N \geqslant 2$. For $\alpha \in M_{2g}(\mathbb{Z})$ we denote by $\tilde{\alpha}$ its reduction modulo N. Define a set

$$\mathcal{V}_N = \left\{ (1/N) \begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} \;\middle|\; \alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z}) \text{ such that } \tilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z}) \right\}.$$

Let α, β be elements of $M_{2g}(\mathbb{Z})$ satisfying $\alpha, \beta \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. If M is an element of $\mathcal{V}_N$ induced from α, then it is straightforward that $\beta^{\mathrm{T}} M$ is also an element of $\mathcal{V}_N$ given by the product $\alpha\beta$.

Definition 3.1. We call a family $\{h_M(Z)\}_{M\in\mathcal{V}_N}$ a *Siegel family* of level N if it satisfies the following:

(S1) each $h_M(Z)$ belongs to $\mathcal{F}_N$;

(S2) $h_M(Z)$ depends only on $\pm M \pmod{M_{2g\times g}(\mathbb{Z})}$;

(S3) $h_M(Z)^\sigma = h_{\sigma^{\mathrm{T}}M}(Z)$ for all $\sigma \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$.

By $\mathcal{S}_N$ we mean the set of such Siegel families of level N.

Remark 3.2. Let $\{h_M(Z)\}_M \in \mathcal{S}_N$.

(i) The property (S3) yields a right action of the group $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\{h_M(Z)\}_M$.

(ii) We let

$$M = (1/N)\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix} \in \mathcal{V}_N,$$

and so there is a matrix

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Considering $\tilde{\alpha}$ as an element of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ we obtain

$$h_{(1/N)\left[\begin{smallmatrix} I_g \\ O_g \end{smallmatrix}\right]}(Z)^{\tilde{\alpha}} = h_{(1/N)\alpha^{\mathrm{T}}\left[\begin{smallmatrix} I_g \\ O_g \end{smallmatrix}\right]}(Z) = h_M(Z).$$

Thus, the action of $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$ on $\{h_M(Z)\}_M$ is transitive.

Let

$$\Gamma^1(N) = \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\},$$

and let $\mathcal{F}_N^1(\mathbb{Q})$ be the field of meromorphic Siegel modular functions for $\Gamma^1(N)$ with rational Fourier coefficients.

Lemma 3.3. *If* $\{h_M(Z)\}_M \in \mathcal{S}_N$, *then*

$$h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) \in \mathcal{F}_N^1(\mathbb{Q}).$$

Proof. For any

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma^1(N)$$

we deduce by (S2) and (S3) that

$$\begin{aligned} h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(\gamma(Z)) &= h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z)^{\tilde{\gamma}} = h_{\gamma^{\mathrm{T}}\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) \\ &= h_{(1/N)\left[\begin{smallmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{smallmatrix}\right]}(Z) = h_{\left[\begin{smallmatrix} (1/N)I_g \\ O_g \end{smallmatrix}\right]}(Z) \end{aligned}$$

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because

$$A \equiv I_g, \qquad B \equiv O_g \pmod{N \cdot M_g(\mathbb{Z})}.$$

Thus, $h_{\left[\begin{smallmatrix}(1/N)I_g \\ O_g\end{smallmatrix}\right]}(Z)$ is modular for $\Gamma^1(N)$.

For every $\nu \in (\mathbb{Z}/N\mathbb{Z})^\times$ we see by (S2) and (S3) that

$$h_{\left[\begin{smallmatrix}(1/N)I_g \\ O_g\end{smallmatrix}\right]}(Z)^{\left[\begin{smallmatrix}I_g & O_g \\ O_g & \nu I_g\end{smallmatrix}\right]} = h_{\left[\begin{smallmatrix}I_g & O_g \\ O_g & \nu I_g\end{smallmatrix}\right]\left[\begin{smallmatrix}(1/N)I_g \\ O_g\end{smallmatrix}\right]}(Z) = h_{\left[\begin{smallmatrix}(1/N)I_g \\ O_g\end{smallmatrix}\right]}(Z),$$

which implies that $h_{\left[\begin{smallmatrix}(1/N)I_g \\ O_g\end{smallmatrix}\right]}(Z)$ has rational Fourier coefficients. This proves the lemma. □

One can consider $\mathcal{S}_N$ as a field under the binary operations

$$\{h_M(Z)\}_M + \{k_M(Z)\}_M = \{(h_M + k_M)(Z)\}_M,$$
$$\{h_M(Z)\}_M \cdot \{k_M(Z)\}_M = \{(h_M k_M)(Z)\}_M.$$

By lemma 3.3 we get the ring homomorphism

$$\phi_N : \mathcal{S}_N \to \mathcal{F}_N^1(\mathbb{Q})$$
$$\{h_M(Z)\}_M \mapsto h_{\left[\begin{smallmatrix}(1/N)I_g \\ O_g\end{smallmatrix}\right]}(Z).$$

LEMMA 3.4. *If $M \in \mathcal{V}_N$, then there exists*

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\gamma} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N)\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}.$$

Proof. Let

$$\alpha = \begin{bmatrix} A & B \\ U & V \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\alpha} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N)\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}.$$

In $M_{2g}(\mathbb{Z}/N\mathbb{Z})$, decompose $\tilde{\alpha}$ as

$$\tilde{\alpha} = \begin{bmatrix} I_g & O_g \\ O_g & \nu I_g \end{bmatrix}\begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix} \quad \text{with } \nu = \nu(\tilde{\alpha}) \in (\mathbb{Z}/N\mathbb{Z})^\times$$

so that

$$\begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}$$

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belongs to $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Since the reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ is surjective (see [10]), we can take $\gamma \in M_{2g}(\mathbb{Z})$ satisfying

$$\tilde{\gamma} = \begin{bmatrix} A & B \\ \nu^{-1}U & \nu^{-1}V \end{bmatrix}.$$

□

THEOREM 3.5. *$\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via ϕ_N.*

Proof. Since $\mathcal{S}_N$ and $\mathcal{F}_N^1(\mathbb{Q})$ are fields, it suffices to show that ϕ_N is surjective.

Let $h(Z) \in \mathcal{F}_N^1(\mathbb{Q})$. For each $M \in \mathcal{V}_N$, take any

$$\gamma = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\tilde{\gamma} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N) \begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}$$

by using lemma 3.4. We set

$$h_M(Z) = h(Z)^{\tilde{\gamma}}.$$

We claim that $h_M(Z)$ is independent of the choice of γ. Indeed, if

$$\gamma' = \begin{bmatrix} A & B \\ C' & D' \end{bmatrix} \in M_{2g}(\mathbb{Z})$$

such that $\widetilde{\gamma'} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$, then we attain in $M_{2g}(\mathbb{Z}/N\mathbb{Z})$ that

$$\widetilde{\gamma'}\tilde{\gamma}^{-1} = \begin{bmatrix} A & B \\ C' & D' \end{bmatrix} \begin{bmatrix} D^{\mathrm{T}} & -B^{\mathrm{T}} \\ -C^{\mathrm{T}} & A^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix}$$

by the fact $\tilde{\gamma}, \widetilde{\gamma'} \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let δ be an element of $\mathrm{Sp}_{2g}(\mathbb{Z})$ such that $\tilde{\delta} = \widetilde{\gamma'}\tilde{\gamma}^{-1}$. We then achieve

$$h(Z)^{\widetilde{\gamma'}} = (h(Z)^{\widetilde{\gamma'}\tilde{\gamma}^{-1}})^{\tilde{\gamma}} = h(\delta(Z))^{\tilde{\gamma}} = h(Z)^{\tilde{\gamma}}$$

because $h(Z)$ is modular for $\Gamma^1(N)$ and $\delta \in \Gamma^1(N)$.

Now, for any

$$\sigma = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\}$$

with $\nu = \nu(\sigma)$ we derive that

$$\begin{aligned} h_M(Z)^{\sigma} &= h(Z)^{\tilde{\gamma}\sigma} \\ &= h(Z)^{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]\left[\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right]} \\ &= h(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g \end{smallmatrix}\right]\left[\begin{smallmatrix} AP+BR & AQ+BS \\ \nu^{-1}(CP+DR) & \nu^{-1}(CQ+DS) \end{smallmatrix}\right]} \end{aligned}$$

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$$= h(Z)^{\begin{bmatrix} AP+BR & AQ+BS \\ \nu^{-1}(CP+DR) & \nu^{-1}(CQ+DS) \end{bmatrix}}$$

since $h(Z)$ has rational Fourier coefficients

$$= h_{\begin{bmatrix} (AP+BR)^{\mathrm{T}} \\ (AQ+BS)^{\mathrm{T}} \end{bmatrix}}(Z)$$

$$= h_{\begin{bmatrix} P^{\mathrm{T}} & R^{\mathrm{T}} \\ Q^{\mathrm{T}} & S^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}}(Z)$$

$$= h_{\sigma^{\mathrm{T}} M}(Z).$$

This shows that the family $\{h_M(Z)\}_M$ belongs to $\mathcal{S}_N$. Furthermore, since

$$\phi_N(\{h_M(Z)\}_M) = h_{\begin{bmatrix} (1/N)I_g \\ O_g \end{bmatrix}}(Z) = h(Z)^{\begin{bmatrix} I_g & O_g \\ O_g & I_g \end{bmatrix}} = h(Z),$$

ϕ is surjective as desired. □

REMARK 3.6. (i) By proposition 2.2 and remark 2.3 we obtain

$$\mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_N^1(\mathbb{Q})) \simeq G_N \cdot \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \;\middle|\; \gamma = \pm \begin{bmatrix} I_g & O_g \\ * & I_g \end{bmatrix} \right\}.$$

(ii) Let $\mathcal{F}_{1,N}(\mathbb{Q})$ be the field of meromorphic Siegel modular functions for

$$\Gamma_1(N) = \left\{ \gamma \in \mathrm{Sp}_{2g}(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{bmatrix} I_g & * \\ O_g & I_g \end{bmatrix} \pmod{N \cdot M_{2g}(\mathbb{Z})} \right\}$$

with rational Fourier coefficients. If we set

$$\omega = \begin{bmatrix} (1/\sqrt{N})I_g & O_g \\ O_g & \sqrt{N} I_g \end{bmatrix},$$

then we know that $\omega \in \mathrm{Sp}_{2g}(\mathbb{R})$ and

$$\omega \begin{bmatrix} A & B \\ C & D \end{bmatrix} \omega^{-1} = \begin{bmatrix} A & (1/N)B \\ NC & D \end{bmatrix} \quad \text{for all } \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}).$$

This implies

$$\omega \Gamma^1(N) \omega^{-1} = \Gamma_1(N),$$

and so $\mathcal{F}_{1,N}(\mathbb{Q})$ and $\mathcal{F}_N^1(\mathbb{Q})$ are isomorphic via

$$\begin{aligned} \mathcal{F}_{1,N}(\mathbb{Q}) &\to \mathcal{F}_N^1(\mathbb{Q}) \\ h(Z) &\mapsto (h \circ \omega)(Z) = h((1/N)Z). \end{aligned}$$

4. An example of a Siegel family

In this section, we shall give a concrete example of a Siegel family by means of theta constants.

Let g be a positive integer. For

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_u \\ \boldsymbol{v}_\ell \end{bmatrix} \in \mathbb{Q}^{2g}$$

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with $\boldsymbol{v}_u, \boldsymbol{v}_\ell \in \mathbb{Q}^g$, the *theta constant* $\theta_{\boldsymbol{v}}(Z)$ is given by

$$\theta_{\boldsymbol{v}}(Z) = \sum_{\boldsymbol{n}\in\mathbb{Z}^g} e(\tfrac{1}{2}(\boldsymbol{n}+\boldsymbol{v}_u)^{\mathrm{T}} Z(\boldsymbol{n}+\boldsymbol{v}_u) + (\boldsymbol{n}+\boldsymbol{v}_u)^{\mathrm{T}}\boldsymbol{v}_\ell) \quad (Z \in \mathbb{H}_g).$$

It was shown by Igusa (see [3, theorem 2]) that $\theta_{\boldsymbol{v}}(Z)$ is identically zero if and only if every entry of the vector $\boldsymbol{v}$ is in $(1/2)\mathbb{Z}$ and $e(2\boldsymbol{v}_u^{\mathrm{T}}\boldsymbol{v}_\ell) = -1$. Let

$$S_- = \left\{ \boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_u \\ \boldsymbol{a}_\ell \end{bmatrix} \in \{0, 1/2\}^{2g} \;\middle|\; e(2\boldsymbol{a}_u^{\mathrm{T}}\boldsymbol{a}_\ell) = -1 \right\} \quad \text{and} \quad S_+ = \{0,1/2\}^{2g} \setminus S_-.$$

Now, let $\boldsymbol{v} \in \mathbb{Q}^{2g}$ with exact denominator $N \geqslant 2$. We define

$$\Theta_{\boldsymbol{v}}(Z) = 2^{4N} e(-2^g N(2^g-1)(2^g+1)\boldsymbol{v}_u^{\mathrm{T}}\boldsymbol{v}_\ell) \frac{\prod_{\boldsymbol{a}\in S_-} \theta_{\boldsymbol{a}-\boldsymbol{v}}(Z)^{4N(2^g+1)}}{\prod_{\boldsymbol{b}\in S_+} \theta_{\boldsymbol{b}}(Z)^{4N(2^g-1)}} \quad (Z \in \mathbb{H}_g)$$

(see [7, definition 4.2]).

Proposition 4.1. *The function $\Theta_{\boldsymbol{v}}(Z)$ depends only on $\pm\boldsymbol{v} \pmod{\mathbb{Z}^{2g}}$. Moreover, it belongs to $\mathcal{F}_N$ and satisfies that*

$$\Theta_{\boldsymbol{v}}(Z)^\sigma = \Theta_{\sigma^{\mathrm{T}}\boldsymbol{v}}(Z)$$

for every $\sigma \in \mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})/\{\pm I_{2g}\} \simeq \mathrm{Gal}(\mathcal{F}_N/\mathcal{F}_1)$.

Proof. See [7, lemma 4.4]. □

Remark 4.2. One can readily verify that if $g \geqslant 2$, then $\Theta_{\boldsymbol{v}}(Z)$ is identically zero if and only if $N = 2$.

Theorem 4.3. *If $\boldsymbol{r} \in \mathbb{Q}^g$ with exact denominator $N \geqslant 3$, then $\{\Theta_{M(N\boldsymbol{r})}\}_{M\in\mathcal{V}_N}$ is a Siegel family of level N.*

Proof. For any $\gamma \in \Gamma^1(N)$ we derive by proposition 4.1 that

$$\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(\gamma(Z)) = \Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)^{\tilde{\gamma}} = \Theta_{\gamma^{\mathrm{T}}\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z) = \Theta_{\left[\begin{smallmatrix} I_g & * \\ O_g & I_g\end{smallmatrix}\right]\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z) = \Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z).$$

This shows that $\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)$ is modular for $\Gamma^1(N)$. Furthermore, for any $\nu \in (\mathbb{Z}/N\mathbb{Z})^\times$, by proposition 4.1 we see that

$$\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)^{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g\end{smallmatrix}\right]} = \Theta_{\left[\begin{smallmatrix} I_g & O_g \\ O_g & \nu I_g\end{smallmatrix}\right]\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z) = \Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z).$$

Thus, $\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)$ has rational Fourier coefficients, and hence $\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)$ belongs to $\mathcal{F}_N^1(\mathbb{Q})$.

For each $M \in \mathcal{V}_N$, we can take an element

$$\gamma_M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

of $M_{2g}(\mathbb{Z})$ such that $\tilde{\gamma}_M \in \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$ and

$$M = (1/N)\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}$$

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by lemma 3.4. Then, by the proof of theorem 3.5, the family $\{\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)^{\tilde{\gamma}_M}\}_{M\in\mathcal{V}_N}$ turns out to be a Siegel family of level N. Lastly, we obtain by proposition 4.1 that

$$\Theta_{\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z)^{\tilde{\gamma}_M} = \Theta_{\gamma_M^{\mathrm{T}}\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z) = \Theta_{\left[\begin{smallmatrix}A^{\mathrm{T}} & C^{\mathrm{T}}\\ B^{\mathrm{T}} & D^{\mathrm{T}}\end{smallmatrix}\right]\left[\begin{smallmatrix}\boldsymbol{r}\\ \boldsymbol{0}\end{smallmatrix}\right]}(Z) = \Theta_{\left[\begin{smallmatrix}A^{\mathrm{T}}\\ B^{\mathrm{T}}\end{smallmatrix}\right]\boldsymbol{r}}(Z) = \Theta_{M(N\boldsymbol{r})}(Z).$$

This completes the proof. □

5. Special values associated with a Siegel family

As an application of a Siegel family of level N we shall construct a number associated with each ray class modulo N of a CM-field.

Let n be a positive integer, K be a CM-field with $[K:\mathbb{Q}]=2n$ and $\{\varphi_1,\dots,\varphi_n\}$ be a set of embeddings of K into $\mathbb{C}$ such that $(K,\{\varphi_i\}_{i=1}^n)$ is a CM-type. We fix a finite Galois extension L of $\mathbb{Q}$ containing K, and set

$$\begin{aligned} S &= \{\sigma\in \mathrm{Gal}(L/\mathbb{Q}) \mid \sigma|_K=\varphi_i \text{ for some } i\in\{1,2,\dots,n\}\},\\ S^* &= \{\sigma^{-1}\mid \sigma\in S\},\\ H^* &= \{\gamma\in\mathrm{Gal}(L/\mathbb{Q})\mid \gamma S^*=S^*\}.\end{aligned}$$

Let K^* be the subfield of L corresponding to the subgroup H^* of $\mathrm{Gal}(L/\mathbb{Q})$, and let $\{\psi_1,\dots,\psi_g\}$ be the set of all embeddings of K^* into $\mathbb{C}$ arising from the elements of S^*. Then we know that $(K^*,\{\psi_j\}_{j=1}^g)$ is a primitive CM-type and

$$K^*=\mathbb{Q}\left(\sum_{i=1}^n a^{\varphi_i}\;\middle|\; a\in K\right)$$

(see [12, § 8.3, proposition 28]). We call this CM-type $(K^*,\{\psi_j\}_{j=1}^g)$ the reflex of $(K,\{\varphi_i\}_{i=1}^n)$. Using this CM-type we define an embedding

$$\begin{aligned}\Psi\colon K^* &\to \mathbb{C}^g\\ a &\mapsto \begin{bmatrix} a^{\psi_1}\\ \vdots\\ a^{\psi_g}\end{bmatrix}.\end{aligned}$$

For each purely imaginary element c of K^* we associate an $\mathbb{R}$-bilinear form

$$\begin{aligned} E_c\colon \mathbb{C}^g\times\mathbb{C}^g &\to \mathbb{R}\\ (\boldsymbol{u},\boldsymbol{v}) &\mapsto \sum_{j=1}^g c^{\psi_j}(u_j\bar{v}_j-\bar{u}_jv_j) \quad \left(\boldsymbol{u}=\begin{bmatrix}u_1\\ \vdots\\ u_g\end{bmatrix},\ \boldsymbol{v}=\begin{bmatrix}v_1\\ \vdots\\ v_g\end{bmatrix}\right).\end{aligned}$$

Then, one can readily check that

$$E_c(\Psi(a),\Psi(b)) = \mathrm{Tr}_{K^*/\mathbb{Q}}(c a\bar{b}) \quad \text{for all } a,b\in K^* \tag{5.1}$$

by using the fact $\overline{a^{\psi_j}}=\bar{a}^{\psi_j}$ for all $a\in K^*$ $(1\leqslant j\leqslant g)$.

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ASSUMPTION 5.1. In what follows we assume the following conditions.

(i) $(K^*)^* = K$.

(ii) There is a purely imaginary element ξ of K^* and a $\mathbb{Z}$-basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2g}\}$ of the lattice $\Psi(\mathcal{O}_{K^*})$ in $\mathbb{C}^g$ for which

$$\left[E_\xi(\boldsymbol{a}_i, \boldsymbol{a}_j)\right]_{1\leqslant i,j\leqslant 2g} = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}.$$

In this case, we say that the complex torus $(\mathbb{C}^g/\Psi(\mathcal{O}_{K^*}), E_\xi)$ is a principally polarized abelian variety with a symplectic basis $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{2g}\}$. See [12, § 6.2].

(iii) $\mathfrak{f} = N\mathcal{O}_K$ for an integer $N \geqslant 2$.

REMARK 5.2. The assumption 5.1(i) is equivalent to saying that $(K, \{\varphi_i\}_{i=1}^n)$ is a primitive CM-type, namely, the abelian varieties of this CM-type are simple [12, § 8.2, proposition 26].

By assumption 5.1(i) one can define a group homomorphism

$$\begin{aligned} \mathfrak{g} \colon K^\times &\to (K^*)^\times \\ d &\mapsto \prod_{i=1}^n d^{\varphi_i}, \end{aligned}$$

and extend it continuously to the homomorphism $\mathfrak{g} \colon K_{\mathbb{A}}^\times \to (K^*)_{\mathbb{A}}^\times$ of idele groups. It is also known that for each fractional ideal $\mathfrak{a}$ of K there is a fractional ideal $\mathcal{G}(\mathfrak{a})$ of K^* such that [12, § 8.3]

$$\mathcal{G}(\mathfrak{a})\mathcal{O}_L = \prod_{i=1}^n (\mathfrak{a}\mathcal{O}_L)^{\varphi_i}.$$

Let $\mathcal{C}$ be a given ray class in $\mathrm{Cl}(\mathfrak{f})$. Take any integral ideal $\mathfrak{c}$ in $\mathcal{C}$, and let

$$\mathcal{N}(\mathfrak{c}) = \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{c}) = |\mathcal{O}_K/\mathfrak{c}|.$$

LEMMA 5.3. $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ *is also a principally polarized abelian variety.*

Proof. It follows from (5.1) that

$$\begin{aligned} E_{\xi\mathcal{N}(\mathfrak{c})}(\Psi(\mathcal{G}(\mathfrak{c})^{-1}), \Psi(\mathcal{G}(\mathfrak{c})^{-1})) &= \mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c})\mathcal{G}(\mathfrak{c})^{-1}\overline{\mathcal{G}(\mathfrak{c})^{-1}}) \\ &= \mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{O}_{K^*}) \\ &= E_\xi(\Psi(\mathcal{O}_{K^*}), \Psi(\mathcal{O}_{K^*})) \\ &\subseteq \mathbb{Z} \end{aligned}$$

because E_ξ is a Riemann form on $\mathbb{C}^g/\Psi(\mathcal{O}_{K^*})$. Thus, $E_{\xi\mathcal{N}(\mathfrak{c})}$ defines a Riemann form on $\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1})$.

Now, let $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2g}\}$ be a symplectic basis of the abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ so that

$$\Psi(\mathcal{G}(\mathfrak{c})^{-1}) = \sum_{j=1}^{2g} \mathbb{Z}\boldsymbol{b}_j \quad \text{and} \quad \left[E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{b}_i, \boldsymbol{b}_j)\right]_{1\leqslant i,j\leqslant 2g} = \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix},$$

where

$$\mathcal{E} = \begin{bmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_g \end{bmatrix}$$

is a $g \times g$ diagonal matrix for some positive integers $\varepsilon_1, \ldots, \varepsilon_g$ satisfying $\varepsilon_1 | \cdots | \varepsilon_g$. Furthermore, let $b_1, \ldots, b_{2g}$ be elements of $\mathcal{G}(\mathfrak{c})^{-1}$ such that $\boldsymbol{b}_j = \Psi(b_j)$ $(1 \leqslant j \leqslant 2g)$. Since $\mathcal{O}_{K^*} \subseteq \mathcal{G}(\mathfrak{c})^{-1}$, we have

$$\begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{2g} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g} \end{bmatrix} \alpha \quad \text{for some } \alpha \in M_{2g}(\mathbb{Z}) \cap \mathrm{GL}_{2g}(\mathbb{Q}), \tag{5.2}$$

and hence

$$\begin{bmatrix} a_1^{\psi_1} & \cdots & a_{2g}^{\psi_1} \\ \vdots & & \vdots \\ a_1^{\psi_g} & \cdots & a_{2g}^{\psi_g} \\ \overline{a_1^{\psi_1}} & \cdots & \overline{a_{2g}^{\psi_1}} \\ \vdots & & \vdots \\ \overline{a_1^{\psi_g}} & \cdots & \overline{a_{2g}^{\psi_g}} \end{bmatrix} = \begin{bmatrix} b_1^{\psi_1} & \cdots & b_{2g}^{\psi_1} \\ \vdots & & \vdots \\ b_1^{\psi_g} & \cdots & b_{2g}^{\psi_g} \\ \overline{b_1^{\psi_1}} & \cdots & \overline{b_{2g}^{\psi_1}} \\ \vdots & & \vdots \\ \overline{b_1^{\psi_g}} & \cdots & \overline{b_{2g}^{\psi_g}} \end{bmatrix} \alpha.$$

Taking determinant and squaring gives rise to the identity

$$\Delta_{K^*/\mathbb{Q}}(a_1, \ldots, a_{2g}) = \Delta_{K^*/\mathbb{Q}}(b_1, \ldots, b_{2g}) \det(\alpha)^2.$$

It then follows that

$$\begin{aligned} \det(\alpha)^2 = \frac{|\Delta_{K^*/\mathbb{Q}}(a_1, \ldots, a_{2g})|}{|\Delta_{K^*/\mathbb{Q}}(b_1, \ldots, b_{2g})|} &= \frac{d_{K^*/\mathbb{Q}}(\mathcal{O}_{K^*})}{d_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c})^{-1})} \\ &= \mathcal{N}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c}))^2 \\ &= \mathcal{N}_{K^*/\mathbb{Q}}(\mathcal{G}(\mathfrak{c})\overline{\mathcal{G}(\mathfrak{c})}) \\ &= \mathcal{N}(\mathfrak{c})^{2g}, \end{aligned} \tag{5.3}$$

where $d_{K^*/\mathbb{Q}}$ stands for the discriminant of a fractional ideal of K^* [9, ch. III, proposition 13]. Furthermore, we deduce by (5.2) that

$$\begin{aligned} \mathcal{N}(\mathfrak{c}) \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix} &= [\mathcal{N}(\mathfrak{c}) E_\xi(\boldsymbol{a}_i, \boldsymbol{a}_j)]_{1 \leqslant i,j \leqslant 2g} \\ &= [E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{a}_i, \boldsymbol{a}_j)]_{1 \leqslant i,j \leqslant 2g} \\ &= \alpha^{\mathrm{T}} [E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{b}_i, \boldsymbol{b}_j)]_{1 \leqslant i,j \leqslant 2g} \alpha \\ &= \alpha^{\mathrm{T}} \begin{bmatrix} O_g & -\mathcal{E} \\ \mathcal{E} & O_g \end{bmatrix} \alpha. \end{aligned}$$

By taking the determinant we get $\mathcal{N}(\mathfrak{c})^{2g} = \det(\alpha)^2(\varepsilon_1 \ldots \varepsilon_g)^2$, which, by (5.3), yields that $\varepsilon_1 = \cdots = \varepsilon_g = 1$, and so $\mathcal{E} = I_g$. Therefore, $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$ becomes a principally polarized abelian variety. □

As in the proof of lemma 5.3 we take a symplectic basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2g}\}$ of the principally polarized abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$, and let $b_1, \ldots, b_{2g}$ be elements of $\mathcal{G}(\mathfrak{c})^{-1}$ such that $\boldsymbol{b}_j = \Psi(b_j)$ $(1 \leqslant j \leqslant 2g)$. We then have

$$\begin{bmatrix}\boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{2g}\end{bmatrix} = \begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g}\end{bmatrix}\alpha \quad \text{for some } \alpha = \begin{bmatrix}A & B\\ C & D\end{bmatrix} \in M_{2g}(\mathbb{Z}) \cap \mathrm{GSp}_{2g}(\mathbb{Q}). \tag{5.4}$$

Since $\nu(\alpha) = \mathcal{N}(\mathfrak{c})$ is relatively prime to N, the reduction $\tilde{\alpha}$ of α modulo N belongs to $\mathrm{GSp}_{2g}(\mathbb{Z}/N\mathbb{Z})$. Let $Z_\mathfrak{c}^*$ be the CM-point associated with the symplectic basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2g}\}$, namely

$$Z_\mathfrak{c}^* = \begin{bmatrix}\boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g\end{bmatrix},$$

which belongs to $\mathbb{H}_g$ [1, proposition 8.1.1].

Definition 5.4. Let $\{h_M(Z)\}_M \in \mathcal{S}_N$. For a given ray class $\mathcal{C} \in \mathrm{Cl}(\mathfrak{f})$ we define

$$h_\mathfrak{f}(\mathcal{C}) = h_{(1/N)\left[\begin{smallmatrix}B\\D\end{smallmatrix}\right]}(Z_\mathfrak{c}^*).$$

Remark 5.5. Here, the index matrix

$$(1/N)\begin{bmatrix}B\\ D\end{bmatrix}$$

is obtained using the fact that

$$\left(\begin{bmatrix}O_g & -I_g\\ I_g & O_g\end{bmatrix}\alpha\right)^{\mathrm{T}} = \begin{bmatrix}B^{\mathrm{T}} & D^{\mathrm{T}}\\ -A^{\mathrm{T}} & -C^{\mathrm{T}}\end{bmatrix}.$$

6. Well-definedness of $h_\mathfrak{f}(\mathcal{C})$

In this section we shall show that the value $h_\mathfrak{f}(\mathcal{C})$ given in definition 5.4 depends only on the ray class $\mathcal{C}$, and hence it is independent of the choice of a symplectic basis and an integral ideal in $\mathcal{C}$.

Proposition 6.1. *The value $h_\mathfrak{f}(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$.*

Proof. Let $\{\widehat{\boldsymbol{b}}_1, \ldots, \widehat{\boldsymbol{b}}_{2g}\}$ be another symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$. Thus,

$$\begin{bmatrix}\widehat{\boldsymbol{b}}_1 & \cdots & \widehat{\boldsymbol{b}}_{2g}\end{bmatrix} = \begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g}\end{bmatrix}\beta \quad \text{for some } \beta = \begin{bmatrix}P & Q\\ R & S\end{bmatrix} \in \mathrm{GL}_{2g}(\mathbb{Z}). \tag{6.1}$$

We then derive

$$\begin{aligned}\begin{bmatrix}O_g & -I_g\\ I_g & O_g\end{bmatrix} &= \begin{bmatrix}E_{\xi\mathcal{N}(\mathfrak{c})}(\widehat{\boldsymbol{b}}_i, \widehat{\boldsymbol{b}}_j)\end{bmatrix}_{1\leqslant i,j\leqslant 2g}\\ &= \beta^{\mathrm{T}}\begin{bmatrix}E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{b}_i, \boldsymbol{b}_j)\end{bmatrix}_{1\leqslant i,j\leqslant 2g}\beta\\ &= \beta^{\mathrm{T}}\begin{bmatrix}O_g & -I_g\\ I_g & O_g\end{bmatrix}\beta,\end{aligned}$$

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which shows that $\beta \in \mathrm{Sp}_{2g}(\mathbb{Z})$. Since

$$\begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{2g} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g} \end{bmatrix} \alpha = \begin{bmatrix} \widehat{\boldsymbol{b}}_1 & \cdots & \widehat{\boldsymbol{b}}_{2g} \end{bmatrix} \beta^{-1}\alpha$$

by (5.4) and (6.1), the special value obtained by $\{\widehat{\boldsymbol{b}}_1, \dots, \widehat{\boldsymbol{b}}_{2g}\}$ is

$$h_{(1/N)\beta^{-1}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(\hat{Z}^*_{\mathfrak{c}}),$$

where $\hat{Z}^*_{\mathfrak{c}}$ is the CM-point corresponding to $\{\widehat{\boldsymbol{b}}_1, \dots, \widehat{\boldsymbol{b}}_{2g}\}$.

On the other hand, we attain that

$$\begin{aligned} \hat{Z}^*_{\mathfrak{c}} &= \begin{bmatrix} \widehat{\boldsymbol{b}}_{g+1} & \cdots & \widehat{\boldsymbol{b}}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \widehat{\boldsymbol{b}}_1 & \cdots & \widehat{\boldsymbol{b}}_g \end{bmatrix} \\ &= (\begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g \end{bmatrix} Q + \begin{bmatrix} \boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g} \end{bmatrix} S)^{-1} \\ &\qquad \times (\begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g \end{bmatrix} P + \begin{bmatrix} \boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g} \end{bmatrix} R) \quad \text{by (6.1)} \\ &= (P^{\mathrm{T}} \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g \end{bmatrix}^{\mathrm{T}} + R^{\mathrm{T}} \begin{bmatrix} \boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g} \end{bmatrix}^{\mathrm{T}}) \\ &\qquad \times (Q^{\mathrm{T}} \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g \end{bmatrix}^{\mathrm{T}} + S^{\mathrm{T}} \begin{bmatrix} \boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g} \end{bmatrix}^{\mathrm{T}})^{-1}, \quad \text{since } (\hat{Z}^*_{\mathfrak{c}})^{\mathrm{T}} = \hat{Z}^*_{\mathfrak{c}} \\ &= (P^{\mathrm{T}}(\begin{bmatrix} \boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g \end{bmatrix})^{\mathrm{T}} + R^{\mathrm{T}}) \\ &\qquad \times (Q^{\mathrm{T}}(\begin{bmatrix} \boldsymbol{b}_{g+1} & \cdots & \boldsymbol{b}_{2g} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_g \end{bmatrix})^{\mathrm{T}} + S^{\mathrm{T}})^{-1} \\ &= (P^{\mathrm{T}}(Z^*_{\mathfrak{c}})^{\mathrm{T}} + R^{\mathrm{T}})(Q^{\mathrm{T}}(Z^*_{\mathfrak{c}})^{\mathrm{T}} + S^{\mathrm{T}})^{-1} \\ &= (P^{\mathrm{T}} Z^*_{\mathfrak{c}} + R^{\mathrm{T}})(Q^{\mathrm{T}} Z^*_{\mathfrak{c}} + S^{\mathrm{T}})^{-1} \quad \text{because } (Z^*_{\mathfrak{c}})^{\mathrm{T}} = Z^*_{\mathfrak{c}} \\ &= \beta^{\mathrm{T}}(Z^*_{\mathfrak{c}}). \end{aligned} \tag{6.2}$$

Thus, we deduce that

$$\begin{aligned} h_{(1/N)\beta^{-1}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(\hat{Z}^*_{\mathfrak{c}}) &= h_{(1/N)\beta^{-1}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(\beta^{\mathrm{T}}(Z^*_{\mathfrak{c}})) \quad \text{by (6.2)} \\ &= (h_{(1/N)\beta^{-1}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z))^{\beta^{\mathrm{T}}}|_{Z = Z^*_{\mathfrak{c}}} \\ &= h_{(1/N)(\beta^{\mathrm{T}})^{\mathrm{T}}\beta^{-1}\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z^*_{\mathfrak{c}}) \quad \text{by the property (S3) of } \{h_M(Z)\}_M \\ &= h_{(1/N)\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z^*_{\mathfrak{c}}). \end{aligned}$$

This proves that the value $h_{\mathfrak{f}}(\mathcal{C})$ is independent of the choice of a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c})})$. □

Remark 6.2. One can analogously readily show that $h_{\mathfrak{f}}(\mathcal{C})$ does not depend on the choice of a symplectic basis $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_{2g}\}$ of $(\mathbb{C}^g/\Psi(\mathcal{O}_K), E_\xi)$.

Proposition 6.3. *$h_{\mathfrak{f}}(\mathcal{C})$ does not depend on the choice of an integral ideal $\mathfrak{c}$ in $\mathcal{C}$.*

Proof. Let $\mathfrak{c}'$ be another integral ideal in the class $\mathcal{C}$, and hence

$$\mathfrak{c}'\mathfrak{c}^{-1} = (1+a)\mathcal{O}_K \quad \text{for some } a \in \mathfrak{f}\mathfrak{a}^{-1}, \tag{6.3}$$

where $\mathfrak{a}$ is an integral ideal of K relatively prime to $\mathfrak{f}$. Since $1 \in \mathfrak{c}^{-1}$ and $(1+a) \in \mathfrak{c}'\mathfrak{c}^{-1} \subseteq \mathfrak{c}^{-1}$, we get $a \in \mathfrak{c}^{-1}$. Thus, we derive that

$$\begin{aligned} a\mathfrak{a}\mathfrak{c} &\subseteq \mathfrak{f}\mathfrak{c} \cap \mathfrak{a} \quad \text{by the facts that } a \in \mathfrak{f}\mathfrak{a}^{-1} \text{ and } a \in \mathfrak{c}^{-1} \\ &\subseteq \mathfrak{f} \cap \mathfrak{a} \\ &= \mathfrak{f}\mathfrak{a} \quad \text{because } \mathfrak{f} \text{ and } \mathfrak{a} \text{ are relatively prime,} \end{aligned}$$

from which it follows that $a \in \mathfrak{f}\mathfrak{c}^{-1}$. Using the fact that $\mathfrak{f} = N\mathcal{O}_K$ yields

$$\begin{aligned} \mathfrak{g}(1+a) &= \prod_{i=1}^{n}(1+a)^{\varphi_i} \in K^* \cap \prod_{i=1}^{n}(1+N(\mathfrak{c}^{-1}\mathcal{O}_L)^{\varphi_i}) \subseteq K^* \cap (1+N\mathcal{G}(\mathfrak{c})^{-1}\mathcal{O}_L) \\ &= 1+N\mathcal{G}(\mathfrak{c})^{-1}. \end{aligned} \tag{6.4}$$

Let

$$b_j' = \mathfrak{g}(1+a)^{-1}b_j \quad \text{and} \quad \boldsymbol{b}_j' = \Psi(b_j') \quad (1 \leqslant j \leqslant 2g). \tag{6.5}$$

We know that $\{\boldsymbol{b}_1', \ldots, \boldsymbol{b}_{2g}'\}$ is a $\mathbb{Z}$-basis of the lattice $\Psi(\mathcal{G}(\mathfrak{c}')^{-1})$ in $\mathbb{C}^g$ and

$$\boldsymbol{b}_j' = T\boldsymbol{b}_j \quad \text{with } T = \begin{bmatrix} (\mathfrak{g}(1+a)^{-1})^{\psi_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & (\mathfrak{g}(1+a)^{-1})^{\psi_g} \end{bmatrix}. \tag{6.6}$$

Furthermore, we get that

$$\begin{aligned} &\left[E_{\xi\mathcal{N}(\mathfrak{c}')}(\boldsymbol{b}_i', \boldsymbol{b}_j')\right]_{1\leqslant i,j\leqslant 2g} \\ &\quad = \left[\mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')b_i'\overline{b_j'})\right]_{1\leqslant i,j\leqslant 2g} \quad \text{by(5.1)} \\ &\quad = \left[\mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')\mathfrak{g}(1+a)^{-1}b_i\overline{\mathfrak{g}(1+a)^{-1}b_j})\right]_{1\leqslant i,j\leqslant 2g} \quad \text{by (6.5)} \\ &\quad = \left[\mathrm{Tr}_{K^*/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c}')\mathrm{N}_{K/\mathbb{Q}}(1+a)^{-1}b_i\overline{b_j})\right]_{1\leqslant i,j\leqslant 2g} \\ &\quad = \left[\mathrm{Tr}_{K/\mathbb{Q}}(\xi\mathcal{N}(\mathfrak{c})b_i\overline{b_j})\right]_{1\leqslant i,j\leqslant 2g} \\ &\qquad \text{by (6.3) and the fact that } \mathrm{N}_{K/\mathbb{Q}}(1+a) > 0 \\ &\quad = \left[E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{b}_i, \boldsymbol{b}_j)\right]_{1\leqslant i,j\leqslant 2g} \quad \text{by(5.1)} \\ &\quad = \begin{bmatrix} O_g & -I_g \\ I_g & O_g \end{bmatrix}. \end{aligned}$$

Thus, $\{\boldsymbol{b}_1', \ldots, \boldsymbol{b}_{2g}'\}$ is a symplectic basis of $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}')^{-1}), E_{\xi\mathcal{N}(\mathfrak{c}')})$, and its associated CM-point $Z_{\mathfrak{c}'}^*$ is given by

$$\begin{aligned} Z_{\mathfrak{c}'}^* &= \begin{bmatrix}\boldsymbol{b}_{g+1}' & \cdots & \boldsymbol{b}_{2g}'\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{b}_1' & \cdots & \boldsymbol{b}_g'\end{bmatrix} \\ &= \begin{bmatrix}T\boldsymbol{b}_{g+1} & \cdots & T\boldsymbol{b}_{2g}\end{bmatrix}^{-1}\begin{bmatrix}T\boldsymbol{b}_1 & \cdots & T\boldsymbol{b}_g\end{bmatrix} \quad \text{by (6.6)} \\ &= Z_{\mathfrak{c}}^*. \end{aligned} \tag{6.7}$$

Let $\alpha = \begin{bmatrix}a_{ij}\end{bmatrix}$, $\alpha' = \begin{bmatrix}a_{ij}'\end{bmatrix} \in M_{2g}(\mathbb{Z})$ such that

$$\begin{bmatrix}\boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{2g}\end{bmatrix} = \begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g}\end{bmatrix}\alpha = \begin{bmatrix}\boldsymbol{b}_1' & \cdots & \boldsymbol{b}_{2g}'\end{bmatrix}\alpha'. \tag{6.8}$$

For each $1 \leqslant i \leqslant 2g$ we obtain that

$$\begin{aligned}\sum_{j=1}^{2g} a'_{ji} b_j &= \mathfrak{g}(1+a) \sum_{j=1}^{2g} a'_{ji} b'_j \quad \text{by (6.5)} \\ &= a_i \mathfrak{g}(1+a) \quad \text{by (6.8)} \\ &\in a_i(1 + N\mathcal{G}(\mathfrak{c})^{-1}) \quad \text{by (6.4)} \\ &\subseteq a_i + N\mathcal{G}(\mathfrak{c})^{-1} \quad \text{because } a_i \in \mathcal{O}_K \\ &= \sum_{j=1}^{2g} a_{ji} b_j + N \sum_{j=1}^{2g} \mathbb{Z} b_j \quad \text{by (6.8).}\end{aligned}$$

This yields $\alpha \equiv \alpha' \pmod{N \cdot M_{2g}(\mathbb{Z})}$, and hence

$$(1/N)\alpha \equiv (1/N)\alpha' \pmod{M_{2g}(\mathbb{Z})}. \tag{6.9}$$

Now, the result follows from (6.7), (6.9) and the property (S2) of $\{h_M(Z)\}_M$. □

7. Galois actions on $h_\mathfrak{f}(\mathcal{C})$

Finally, we shall show that if $h_\mathfrak{f}(\mathcal{C})$ is finite, then it lies in the ray class field $K_\mathfrak{f}$ and satisfies the natural transformation formula under the Artin reciprocity map for $\mathfrak{f}$.

Let $r\colon K^* \to M_{2g}(\mathbb{Q})$ be the regular representation with respect to the ordered basis $\{a_1, \ldots, a_{2g}\}$ of K^* over $\mathbb{Q}$ given by

$$a \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} = r(a) \begin{bmatrix} a_1 \\ \vdots \\ a_{2g} \end{bmatrix} \quad (a \in K^*). \tag{7.1}$$

Then it can be extended to the map $r\colon (K^*)_\mathbb{A} \to M_{2g}(\mathbb{Q}_\mathbb{A})$ of adele rings.

LEMMA 7.1 (Shimura's reciprocity law). *Let f be an element of $\mathcal{F}$ that is finite at $Z_\mathfrak{c}^*$.*

(i) *The special value $f(Z_\mathfrak{c}^*)$ lies in K_{ab}.*

(ii) *For every $s \in K_\mathbb{A}^\times$ we have $r(\mathfrak{g}(s)) \in G_{\mathbb{A}+}$ and*

$$f(Z_\mathfrak{c}^*)^{[s,K]} = f^{\tau(r(\mathfrak{g}(s)^{-1}))}(Z_\mathfrak{c}^*).$$

Proof. See [13, lemma 9.5 and theorem 9.6]. □

THEOREM 7.2. *If $h_\mathfrak{f}(\mathcal{C})$ is finite, then it belongs to $K_\mathfrak{f}$. Furthermore, it satisfies*

$$h_\mathfrak{f}(\mathcal{C})^{\sigma_\mathfrak{f}(\mathcal{D})} = h_\mathfrak{f}(\mathcal{C}\mathcal{D}) \quad \textit{for every } \mathcal{D} \in \mathrm{Cl}(\mathfrak{f}),$$

where $\sigma_\mathfrak{f}$ is the Artin reciprocity map for $\mathfrak{f}$.

Proof. Since $h_\mathfrak{f}(\mathcal{C})$ belongs to K_{ab} by lemma 7.1(i), there is a sufficiently large positive integer M so that $N|M$ and $h_\mathfrak{f}(\mathcal{C}) \in K_\mathfrak{m}$ with $\mathfrak{m} = M\mathcal{O}_K$. Take an integral

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ideal $\mathfrak{d}$ in $\mathcal{D}$ relatively prime to $\mathfrak{m}$ by using the surjectivity of the natural map $\mathrm{Cl}(\mathfrak{m}) \to \mathrm{Cl}(\mathfrak{f})$. Let $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{2g}\}$ be a symplectic basis of the principally polarized abelian variety $(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}), E_{\xi\mathcal{N}(\mathfrak{c}\mathfrak{d})})$, and let $d_1, \ldots, d_{2g}$ be elements of $\mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}$ such that $\boldsymbol{d}_j = \Psi(d_j)$ $(1 \leqslant j \leqslant 2g)$. Since $\mathcal{G}(\mathfrak{c})^{-1} \subseteq \mathcal{G}(\mathfrak{c}\mathfrak{d})^{-1}$, we get

$$\begin{bmatrix}\boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g}\end{bmatrix} = \begin{bmatrix}\boldsymbol{d}_1 & \cdots & \boldsymbol{d}_{2g}\end{bmatrix}\delta \quad \text{for some } \delta \in M_{2g}(\mathbb{Z}) \cap \mathrm{GL}_{2g}(\mathbb{Q}). \tag{7.2}$$

We then have that

$$\begin{aligned}\begin{bmatrix}O_g & -I_g\\ I_g & O_g\end{bmatrix} &= \left[E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{b}_i, \boldsymbol{b}_j)\right]_{1\leqslant i,j\leqslant 2g}\\ &= \delta^{\mathrm{T}}\left[E_{\xi\mathcal{N}(\mathfrak{c})}(\boldsymbol{d}_i, \boldsymbol{d}_j)\right]_{1\leqslant i,j\leqslant 2g}\delta \quad \text{by (7.2)}\\ &= \delta^{\mathrm{T}}\left[\mathcal{N}(\mathfrak{c})\mathcal{N}(\mathfrak{c}\mathfrak{d})^{-1}E_{\xi\mathcal{N}(\mathfrak{c}\mathfrak{d})}(\boldsymbol{d}_i, \boldsymbol{d}_j)\right]_{1\leqslant i,j\leqslant 2g}\delta\\ &= \mathcal{N}(\mathfrak{d})^{-1}\delta^{\mathrm{T}}\begin{bmatrix}O_g & -I_g\\ I_g & O_g\end{bmatrix}\delta.\end{aligned}$$

This claims that

$$\delta \in M_{2g}(\mathbb{Z}) \cap G_+ \text{ with } \nu(\delta) = \mathcal{N}(\mathfrak{d}). \tag{7.3}$$

Furthermore, if we let $Z^*_{\mathfrak{c}\mathfrak{d}}$ be the CM-point associated with $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_{2g}\}$, then we obtain

$$Z^*_{\mathfrak{c}\mathfrak{d}} = (\delta^{-1})^{\mathrm{T}}(Z^*_{\mathfrak{c}}) \tag{7.4}$$

in a similar way to the argument in the proof of proposition 6.1.

Let $s = (s_p)_p$ be an idele of K such that

$$\left.\begin{aligned} s_p &= 1 && \text{if } p|M,\\ s_p(\mathcal{O}_K)_p &= \mathfrak{d}_p && \text{if } p\nmid M.\end{aligned}\right\} \tag{7.5}$$

If we set $\tilde{\mathcal{D}}$ to be the ray class in $\mathrm{Cl}(\mathfrak{m})$ containing $\mathfrak{d}$, then by (7.5) we attain

$$[s, K]|_{K_{\mathfrak{m}}} = \sigma_{\mathfrak{m}}(\tilde{\mathcal{D}}), \tag{7.6}$$

$$\mathfrak{g}(s)_p^{-1}(\mathcal{O}_{K^*})_p = \mathcal{G}(\mathfrak{d})_p^{-1} \quad \text{for all rational primes } p. \tag{7.7}$$

It then follows from (7.1)–(7.7) that for every rational prime p, the entries of each of the vectors

$$r(\mathfrak{g}(s)^{-1})_p\begin{bmatrix}b_1\\ \vdots\\ b_{2g}\end{bmatrix} \quad \text{and} \quad (\delta^{-1})^{\mathrm{T}}\begin{bmatrix}b_1\\ \vdots\\ b_{2g}\end{bmatrix}$$

form a basis of $\mathcal{G}(\mathfrak{c}\mathfrak{d})_p^{-1} = \mathcal{G}(\mathfrak{c})^{-1}\mathcal{G}(\mathfrak{d})_p^{-1}$. So, there exists a matrix $u = (u_p)_p \in \prod_p \mathrm{GL}_{2g}(\mathbb{Z}_p)$ satisfying

$$r(\mathfrak{g}(s)^{-1}) = u(\delta^{-1})^{\mathrm{T}}. \tag{7.8}$$

Since δ^{T} and

$$\begin{bmatrix}I_g & O_g\\ O_g & \mathcal{N}(\delta)I_g\end{bmatrix}$$

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can be viewed as elements of $\mathrm{GSp}_{2g}(Z/M\mathbb{Z})$ by (7.3), there exists a matrix $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ such that

$$\delta^{\mathrm{T}} \equiv \begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix} \gamma \pmod{M \cdot M_{2g}(\mathbb{Z})} \tag{7.9}$$

owing to the surjectivity of the reduction $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/M\mathbb{Z})$. Since

$$r(\mathfrak{g}(s)^{-1})_p = I_{2g} \quad \text{for all } p|M$$

by (7.5), we get $u_p = \delta^{\mathrm{T}}$ for all $p|M$ by (7.8). Hence, we deduce using (7.9) that

$$u_p\gamma^{-1} \equiv \begin{bmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{bmatrix} \pmod{M \cdot M_{2g}(\mathbb{Z}_p)} \quad \text{for all rational primes } p. \tag{7.10}$$

On the other hand, we have by (5.4) and (7.2) that

$$\begin{aligned} \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{2g} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g} \end{bmatrix} \alpha \\ &= (\begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_{2g} \end{bmatrix} \delta^{-1})(\delta\alpha) \\ &= \begin{bmatrix} \boldsymbol{d}_1 & \cdots & \boldsymbol{d}_{2g} \end{bmatrix} (\delta\alpha). \end{aligned} \tag{7.11}$$

Letting

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

we induce the following:

$$\begin{aligned} h_{\mathfrak{f}}(\mathcal{C})^{\sigma_{\mathfrak{m}}(\tilde{\mathcal{D}})} &= h_{\mathfrak{f}}(\mathcal{C})^{[s,K]} \quad \text{by (7.6)} \\ &= h_{(1/N)\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z_{\mathfrak{c}}^*)^{[s,K]} \quad \text{by definition 5.4} \\ &= h_{(1/N)\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z)^{\tau(r(\mathfrak{g}(s)^{-1}))}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by lemma 7.1(ii)} \\ &= h_{(1/N)\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z)^{\tau(u(\delta^{-1})^{\mathrm{T}})}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (7.8)} \\ &= h_{(1/N)\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z)^{\tau(u\gamma^{-1})\tau(\gamma)\tau((\delta^{-1})^{\mathrm{T}})}|_{Z=Z_{\mathfrak{c}}^*} \\ &= h_{(1/N)\left[\begin{smallmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{smallmatrix}\right]\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z)^{\tau(\gamma)\tau((\delta^{-1})^{\mathrm{T}})}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (7.10) and (S3)} \\ &= h_{(1/N)\gamma^{\mathrm{T}}\left[\begin{smallmatrix} I_g & O_g \\ O_g & \mathcal{N}(\delta)I_g \end{smallmatrix}\right]\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z)^{\tau((\delta^{-1})^{\mathrm{T}})}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (S3)} \\ &= h_{(1/N)\delta\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}(Z)^{\tau((\delta^{-1})^{\mathrm{T}})}|_{Z=Z_{\mathfrak{c}}^*} \quad \text{by (7.9) and (S2)} \\ &= h_{(1/N)\delta\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]}((\delta^{-1})^{\mathrm{T}}(Z_{\mathfrak{c}}^*)) \quad \text{due to the fact that } \delta \in G_+ \text{ and by (A1)} \\ &= h_{\mathfrak{f}}(\mathcal{C}\mathcal{D}) \quad \text{by (7.4), (7.11) and definition 5.4.} \end{aligned}$$

In particular, suppose that $\mathfrak{d} = d\mathcal{O}_K$ for some $d \in \mathcal{O}_K$ such that $d \equiv 1 \pmod{\mathfrak{f}}$. Then $\mathcal{D}$ is the identity class of $\mathrm{Cl}(\mathfrak{f})$, and so the above observation implies that $\sigma_{\mathfrak{m}}(\tilde{\mathcal{D}})$ leaves $h_{\mathfrak{f}}(\mathcal{C})$ fixed. Therefore, we conclude that $h_{\mathfrak{f}}(\mathcal{C})$ lies in $K_{\mathfrak{f}}$. □

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Corollary 7.3. *Let H be a subgroup of* $\mathrm{Cl}(\mathfrak{f})$ *defined by*

$$H = \langle \mathcal{D} \in \mathrm{Cl}(\mathfrak{f}) \mid \mathcal{D} \text{ contains an integral ideal } \mathfrak{d} \text{ of } K \text{ for which } \mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \text{ for some } d \in \mathcal{O}_K \text{ such that } \mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}\rangle,$$

and let $K_\mathfrak{f}^H$ be the fixed field of H. If $h_\mathfrak{f}(\mathcal{C})$ is finite, then it belongs to $K_\mathfrak{f}^H$.

Proof. Let $\mathcal{C}_0$ be the identity class of $\mathrm{Cl}(\mathfrak{f})$. Since $h_\mathfrak{f}(\mathcal{C}_0) \in K_\mathfrak{f}$ by theorem 7.2, $K(h_\mathfrak{f}(\mathcal{C}_0))$ is a Galois extension of K as a subfield of $K_\mathfrak{f}$. Furthermore, since

$$h_\mathfrak{f}(\mathcal{C}_0)^{\sigma_\mathfrak{f}(\mathcal{C})} = h_\mathfrak{f}(\mathcal{C}_0\mathcal{C}) = h_\mathfrak{f}(\mathcal{C})$$

by theorem 7.2, $K(h_\mathfrak{f}(\mathcal{C}_0))$ contains $h_\mathfrak{f}(\mathcal{C})$. Thus, it suffices to show that $h_\mathfrak{f}(\mathcal{C}_0)$ belongs to $K_\mathfrak{f}^H$.

To this end, let $\mathcal{D}$ be an element of $\mathrm{Cl}(\mathfrak{f})$ containing an integral ideal $\mathfrak{d}$ of K for which

$$\mathcal{G}(\mathfrak{d}) = \mathfrak{g}(d)\mathcal{O}_{K^*} \quad \text{for some } d \in \mathcal{O}_K \text{ such that } \mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}.$$

Now that

$$(\mathbb{C}^g/\Psi(\mathcal{G}(\mathfrak{d})^{-1}), E_{\xi\mathcal{N}(\mathfrak{d})}) = (\mathbb{C}^g/\Psi(\mathfrak{g}(d)^{-1}\mathcal{O}_{K^*}), E_{\xi\mathcal{N}(d\mathcal{O}_K)}),$$

we obtain

$$h_\mathfrak{f}(\mathcal{C}_0)^{\sigma_\mathfrak{f}(\mathcal{D})} = h_\mathfrak{f}(\mathcal{D}) = h_\mathfrak{f}([d\mathcal{O}_K]),$$

where $[\mathfrak{a}]$ is the ray class containing $\mathfrak{a}$ for a fractional ideal $\mathfrak{a}$ of K. Moreover, since $\mathfrak{g}(d) \equiv 1 \pmod{N\mathcal{O}_{K^*}}$, we obtain

$$h_\mathfrak{f}([d\mathcal{O}_K]) = h_\mathfrak{f}([\mathcal{O}_K]) = h_\mathfrak{f}(\mathcal{C}_0)$$

analogously to the proof of proposition 6.3. This proves that $h_\mathfrak{f}(\mathcal{C}_0)$ belongs to $K_\mathfrak{f}^H$. □

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