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ON THE DIRECT SUM DECOMPOSITION OF AN ALGEBRA OF CONTINUOUS FUNCTIONS

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Let A and B be closed subalgebras of $C_r(K)$ whose direct sum is $C_r(K)$. According to a result of Stephen Fisher the set of common zeros of elements of B is a retract of K. We give examples to show that this result is not correct.

1. Introduction

Let K be a compact Hausdorff space and $C_r(K)$ the algebra of all continuous real-valued functions on K. Fisher [1] has proved that if $C_r(K)$ is the direct sum of proper closed subalgebras A and B with $1 \in A$, then $Z(B)$, the set of common zeros of elements of B, is non-empty and it is a retract of K. While the first assertion about $Z(B)$ is true, we give simple examples to show that the second assertion is false. We also show that the corollary in [1] is false, but examples therein are still valid.

2. Examples

EXAMPLE 1. Let $K = [-1, 1]$,

$$A_1 = \{f \in C_r[-1, 1] : f \text{ is constant on } [0, 1]\}$$

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and

$$B_1 = \{g \in C_r[-1, 1] : g \text{ is even}, g(1) = 0\} .$$

Then A_1, B_1 are proper closed subalgebras of $C_r[-1, 1]$, $1 \in A_1$ and $C_r[-1, 1] = A_1 \oplus B_1$. However, $Z(B_1) = \{-1, 1\}$ which is not a retract of $[-1, 1]$.

It is worth noting that while $Z(B_1)$ is not a retract of $[-1, 1]$, if we take $B_2 = \{g \in C_r[-1, 1] : g|_{[-1,0]} = 0\}$, then B_2 is a proper closed subalgebra of $C_r[-1, 1]$, $C_r[-1, 1] = A_1 \oplus B_2$ and $Z(B_2) = [-1, 0]$ is a retract of $[-1, 1]$. This raises the following question.

QUESTION. If a proper closed subalgebra A of $C_r(K)$ containing 1 is complemented by a proper closed subalgebra B , does there exist a proper closed subalgebra B' which complements A and for which $Z(B')$ is a retract of K ?

The following example shows that the corollary in [1] is also not correct.

EXAMPLE 2. Let $A = \{f \in C_r[-1, 1] : f \text{ is even}\}$ and $B = \{g \in C_r[-1, 1] : g|_{[-1,0]} = 0\}$. Then A and B are proper closed subalgebras of $C_r[-1, 1]$, $1 \in A$, and $C_r[-1, 1] = A \oplus B$. Here $Z(B) = [-1, 0]$ is a retract of $[-1, 1]$. Now if $p \in [-1, 0)$ and $W = \{x \in [-1, 1] : f(x) = f(p) \text{ for all } f \in A\}$, then $W = \{p, -p\}$ is not a retract of $[-1, 1]$, contrary to the corollary in [1].

REMARKS. (i) If $A \oplus B = C_r(K)$ and B is a closed ideal of $C_r(K)$, then it is well known [2] that $Z(B)$ is a retract of K . In fact, if Z is a closed subset of K , then the closed ideal I_Z of functions vanishing on Z is complemented in $C_r(K)$ by a subalgebra if and only if Z is a retract of K . Consequently Example 4 in [1] follows as the unit circle T is not a retract of the closed unit disc Δ .

(ii) If F is a closed subset of K and A_F is the algebra of all

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functions on K which are constant on F, then $A_F = I_F \oplus \mathbb{C}$. Hence such an algebra is complemented by a closed subalgebra if and only if I_F is complemented by a subalgebra. Therefore F must be a retract. This validates the last remark at the end of Example 4 in [1].

Finally we note that the theorem of Fisher [1, p. 220] would be correct if the phrase "and there is a retraction of K onto Z" were deleted. We further note that this modified result of Fisher would be valid for complex-valued functions if A and B were self-conjugate.

References

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