DISCRETE SPACE-TIME AND INTEGRAL LORENTZ TRANSFORMATIONS

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Introduction. Modern physical theory, both classical and quantal, faces serious difficulties which arise from the divergence of certain integrals. Perhaps the best known of these "infinities" is the self-energy of the point electron. Most of the simpler devices used to eliminate the infinities, such as the introduction of a finite electron radius, are non-relativistic and must therefore be rejected. Relativistic theories[1] which do avoid some or all of the infinities are very complicated and often suffer from difficulty in physical interpretation.

The idea of introducing discreteness into space and time has occasionally been considered.[2] It seems likely that a physical theory based on a discrete space-time background will be free of the infinities which trouble contemporary quantum mechanics. The objection which is usually raised against such discrete schemes is that they are not invariant under the Lorentz group. The purpose of this investigation is to show that *there is a simple model of discrete space-time which, although not invariant under all Lorentz transformations, does admit a surprisingly large number of Lorentz transformations.* This group of transformations is, in fact, sufficiently large to make doubtful the validity of most physical objections raised against discrete space-times.

Apart from the physical speculations in the introduction, this paper is of a purely mathematical nature. We consider all events in Minkowski space-time whose four coordinates t, x, y, z are integers. (The velocity of light is taken as unity.) These events form a "cubic lattice"[3] in space-time. We first investigate the null lines which join lattice points, then the Lorentz transformations which leave the cubic lattice as a whole invariant. We shall call these *integral null lines* and *integral Lorentz transformations*, respectively. We also consider the time-like lines through lattice points which are mapped into lines parallel to the t-axis by an integral Lorentz transformation. These lines will be called *integral time lines.*

It may be noted that our model of discrete space-time involves a *fundamental length*[4] ϵ, namely, the least non-zero interval between lattice points. In the present investigation this fundamental distance has been chosen as the unit of length. In any physical theory based on our model, ϵ would probably be of the general order of magnitude of the classical electron radius (approximately 10^{-13} cm.).

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[1]G. Wentzel, *Rev. Mod. Phys.*, vol. 19 (1947), 1-18.

[2]V. Ambarzumian and D. Iwanenko, *Z. f. Phys.*, vol. 64 (1930), 563-567; L. Silberstein, "Discrete Space-Time," University of Toronto Studies, Physics Series (1936). For a short outline of the present paper, see *Phys. Rev.*, vol. 73 (1948), 414-415.

[3]"Hypercubic" would be the appropriate adjective—but we shall retain the shorter form.

[4]Cf. W. Heisenberg, *Ann. Phys.*, vol. 32 (1938), 20-33.

There are two attractive possibilities for making a first rough attempt at introducing physical theory on our discrete space-time background. The motion of a particle may be assumed to consist of a temporally ordered sequence of lattice points such that successive lattice points are joined by (*a*) integral null lines, or (*b*) integral time lines. In case (*a*), a particle always moves with an instantaneous velocity equal to the velocity of light, but it changes direction rapidly so that its average velocity can be quite low. This zigzag motion has a striking resemblance to some of the features of the Dirac electron.[5] Case (*b*) is rather similar to (*a*). The main difference is that the instantaneous velocity of a particle may now be zero; however it is interesting to note that the non-zero velocities associated with integral time lines are all very high and exceed 0.86 times the velocity of light (Sec. **8**).

Two of the results which we obtain are particularly striking. The first states that the spatial projections of integral null lines are dense (Sec. **4**). This means that particles, whose motion is of the type (*a*) above, can have instantaneous velocities in practically any direction of space. We shall also show that all integral null lines are equivalent in the sense that, given any two integral null lines, an integral Lorentz transformation can be found which maps one into the other (Sec. **7**).

The second result states that spatial projections of integral time lines are dense (Sec. **8**). This means that particles, whose motion is of the type (*b*) above, can have instantaneous velocities in practically any direction of space.

It is obvious that the cubic lattice which we are considering is invariant under all *translations* which map one lattice point into another. In this sense our discrete model of space-time is *homogeneous*. The two results stated above show that our model possesses also a large measure of *spatial isotropy*.

Of any physical theory based on our model of discrete space-time we require invariance under integral Lorentz transformations. The integral Lorentz transformations are independent of the fundamental length ϵ. Thus in the limit when ϵ tends to zero we expect the resulting equations of the physical theory to remain invariant under integral Lorentz transformations, although the background is now continuous Minkowski space-time. If the limiting equations are at all simple they are almost certain to be invariant under *all* Lorentz transformations, since it is difficult to visualize equations in continuous space-time which are invariant under as substantial a subgroup of Lorentz transformations as that considered here without these equations being completely Lorentz invariant. Thus it is reasonable to hope that equations based on our discrete space-time model might be found which, in the limit $\epsilon \to 0$, take the form of the equations of "continuous" relativistic physics, e.g. Maxwell's equations, Lorentz's equations of motion, and Dirac's equations for the electron. These equations of "continuous" physics would be a valid approximation for macroscopic phenomena and even for atomic and molecular

[5]E. Schrödinger, *Sitz. Ber. Preuss. Akad. Wiss.*, vol. 24 (1930), 418-428.

theory—but they would not be appropriate for the description of nuclear phenomena or the theory of elementary particles.

It is clear that we have merely chosen the simplest discrete model of space-time. Other regular point lattices in space-time might be considered and perhaps found more useful. In most essentials, however, these lattices would behave much the same as the cubic lattice studied here. For example, the Lorentz transformations which leave any such lattice invariant would all be associated with high velocities.

1. Gaussian Integers. In this section we collect some well-known definitions and theorems concerning Gaussian integers which will be used in the sequel.

A Gaussian integer is a complex number $a + ib$ whose real part a and imaginary part b are both integers. A real Gaussian integer is an ordinary integer. Gaussian integers can be added, subtracted and multiplied to yield other Gaussian integers; they form an *integral domain.* Here and in the following we shall refer to Gaussian integers simply as "integers." Sometimes, when we are dealing with ordinary integers, we shall add the adjective "real," but usually it will be clear from the context whether integers are real or complex (Gaussian).

The complex conjugate of $c = a + ib$ will be denoted by $\bar{c} = a - ib$; the absolute value of c by $|c| = +(a^2 + b^2)^{\frac{1}{2}}$.

A *unit* is an integer which divides all integers. There are exactly four units in the Gaussian integral domain: ± 1, and $\pm i$.

A *prime* p is an integer which is divisible only by the four units ± 1, $\pm i$, and by $\pm p$, $\pm ip$. Two integers are *relatively prime* if their only common factors are units. Similarly, a set of integers with units as their only common factors will be called *primitive*; thus a primitive vector is a vector whose components form a primitive set of integers.

One of the most important properties of the Gaussian integral domain is that it admits of *unique factorization* into primes.[6] By this is meant the following: An integer a can be written in the form

$$a = p_1 p_2 \ldots p_r, \tag{1.01}$$

where the p_i are primes other than units; if it can also be written in the form

$$a = q_1 q_2 \ldots q_s, \tag{1.02}$$

where the q_i are primes other than units, then $r = s$, and, for a suitable relabelling of the factors $q_1, \ldots, q_r$ we have

$$p_1 = u_1 q_1,\ p_2 = u_2 q_2, \ldots, p_r = u_r q_r, \tag{1.03}$$

where $u_1, u_2, \ldots, u_r$ are units.

We shall apply the term *real prime* to a prime, as defined above, which is real. This definition does not agree with the usual one for real integers in

[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers* (Oxford, 1938), 184, Theorem 215.

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which the criterion is the absence of *real* non-trivial factors; thus $2 = (1+i)(1-i)$ and $5 = (2+i)(2-i)$ are not real primes as we have defined the term. It is clear that any real integer p can be written in the form

$$p = q\,a\,\bar{a}, \tag{1.04}$$

where a, $\bar{a}$ are complex conjugate integers, and where q is the product of those real primes, each taken once, which are factors of p an odd number of times. This q is the real integer of least magnitude for which a decomposition of p in the form (1.04) is possible; apart from sign, q is uniquely determined by p. Since 2 is not a real prime, q must be odd.

Although we shall not require it in the sequel, we add the well-known theorem[7] that among the numbers 2, 3, 5, 7, . . . (which are usually called primes) those and only those of the form $4n + 3$, n being a real integer, are real primes.

An integer $a + ib$, where a and b are real, will be called *even* if a and b are either both even or both odd in the conventional sense; $a + ib$ will be called *odd* if one of a, b is odd and the other even. It is immediately obvious that for real integers our definitions of the terms even and odd coincide with the conventional meaning. The following facts are easily proved:

An even integer is divisible by the prime $1 + i$, an odd integer is not. (Note that the primes $1 - i$, $-1 + i$, $-1 - i$ differ from $1 + i$ only by the unit factors $-i$, i, -1, respectively.) The sum of two integers is even if the integers are both even or both odd; otherwise the sum is odd. The product of two integers is odd only if both factors are odd; otherwise it is even. These rules are easy to remember as they are all familiar from the conventional properties of even and odd real integers; in the case of Gaussian integers the conventional role of 2 is taken over by the prime $1 + i$ which is a repeated factor of 2:

$$2 = -i(1+i)^2. \tag{1.05}$$

We require some further theorems of which the first is a standard result:

If a and b are relatively prime, then there exist integers l and m such that

$$la - mb = 1. \tag{1.06}$$

Conversely (1.06) implies that a and b are relatively prime.

Equation (1.06) may be regarded as a diophantine equation for the unknown integers l and m. If l, m is a particular solution, then the general solution is $l + pb$, $m + pa$, where p is an arbitrary integer. Thus the general solution of (1.06), if relatively prime integers a and b are assigned, involves one discrete complex parameter p or two discrete real parameters. If (1.06) is now regarded as a diophantine equation for the four unknown integers a, b, l, m, then there is a *discrete sixfold infinity of solutions*, since the complex integers a and b can be chosen arbitrarily except for the restriction that they be relatively prime.

In (1.06), a and b are either both odd or else one of them is odd and the other even. If a and b are both odd, then one of l, m must be odd and the other even, so that $a + b + l + m$ is odd.

[7]Hardy and Wright, *Theory of Numbers*, 219, Theorem 252.

In the other case let us, for the sake of definiteness, take a even and b odd. Then one of two possibilities can arise: (i) l and m are both odd, so that $a + b + l + m$ is odd; (ii) l is even and m is odd, so that $a + b + l + m$ is even. Given a solution of (1.06) in which a is even and b, l, m are odd, then

$$(l + b)a - (m + a)b = 1, \tag{1.07}$$

and a,$(l + b)$ are even, b, $(m + a)$ are odd. We easily deduce the results:

If two relatively prime integers a and b are assigned, a being even and b odd, then there exists an even integer l and an odd integer m, satisfying (1.06).

Equation (1.06) has a discrete sixfold infinity of solutions in integers, such that $a + b + l + m$ is even.

2. Spinors and Tensors. We give here a short survey of the spinor calculus[8] in the form in which it will be applied to our problem.

In a complex plane (i.e. a plane with two complex coordinates), called the *spin space*, vectors and tensors are defined by their usual transformation properties. Thus

$$c'^{\alpha} = \lambda_{\beta}{}^{\alpha} c^{\beta}, \tag{2.01}$$

where the $\lambda_{\beta}{}^{\alpha}$ are constants, is the transformation equation of a *contravariant spinvector* c^{α}. Greek suffixes range over 1, 2 and the usual range convention and summation convention for repeated suffixes are assumed. The components of c^{α} are complex and we denote their complex conjugates by $c^{\dot\alpha}$. Then, obviously,

$$c'^{\dot\alpha} = \bar\lambda_{\beta}{}^{\alpha} c^{\dot\beta}, \tag{2.02}$$

where $\lambda_{\beta}{}^{\alpha}$ denotes the complex conjugate of $\lambda_{\beta}{}^{\alpha}$. Expressions such as $a^{\dot\alpha\beta}$, $a^{\alpha\dot\beta}$, $a^{\dot\alpha\beta\gamma}$, etc., are called *contravariant spintensors* or *spinors* if they have, respectively, the same transformation equations as $c^{\dot\alpha}c^{\beta}$, $c^{\alpha}c^{\dot\beta}$, $c^{\dot\alpha}c^{\beta}c^{\gamma}$, etc. If, in a spinor, dots are placed on undotted suffixes and the dots removed from dotted suffixes, then the resulting spinor denotes the complex conjugate of the original spinor; thus

$$a^{\alpha\dot\beta} = \overline{a^{\dot\alpha\beta}}, \quad a^{\dot\alpha\beta\gamma} = \overline{a^{\alpha\dot\beta\dot\gamma}}, \text{ etc.} \tag{2.03}$$

A spintensor $a^{\dot\alpha\beta}$ which has the symmetry property

$$a^{\dot\alpha\beta} = a^{\beta\dot\alpha}, \tag{2.04}$$

or, equivalently,

$$a^{\dot1 1} = \overline{a^{\dot1 1}}, \quad a^{\dot1 2} = \overline{a^{\dot2 1}}, \quad a^{\dot2 2} = \overline{a^{\dot2 2}}, \tag{2.05}$$

is said to be *Hermitian*.

Let us now consider *Minkowski space-time* which is a flat real 4-space with coordinates

$$(t, x, y, z) \equiv (x^0, x^1, x^2, x^3), \tag{2.06}$$

and with metric tensor

[8] O. Laporte and G. E. Uhlenbeck, *Phys. Rev.*, vol. 37 (1931), 1381; L. Infeld, *Phys. Zeitschrift*, vol. 33 (1932), 475. For an early use of a similar technique see also E. Goursat, *Ann. École Norm.* (3), vol. 6 (1889), 20, § 5.

(2.07) $$g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1,\ g_{rs} = 0 \text{ for } r \neq s.$$

Latin suffixes range over 0, 1, 2, 3 and the range and summation conventions are assumed. The *Lorentz transformations* are the linear transformations of the coordinates x^r which leave the components of the metric tensor g_{rs} invariant and which do not interchange past and future. We shall henceforth consider only transformations which leave the origin $x^r = 0$ fixed, i.e. homogeneous linear transformations. Then

(2.08) $$x'^r = L_s{}^r x^s$$

is a Lorentz transformation if

(2.09) $$g_{mn} L_r{}^m L_s{}^n = g_{rs}, \quad L_o{}^o > 0.$$

It immediately follows that the determinant of a Lorentz transformation is $+1$ or -1. Lorentz transformations with determinant $+1$ are called ***proper***.

We associate a real 4-vector A^r with a Hermitian spintensor $a^{\dot\alpha\beta}$ by the relations:

(2.10) $$\begin{aligned} a^{\dot11} &= A^0 + A^3, & A^0 &= \tfrac{1}{2}(a^{\dot11} + a^{\dot22}),\\ a^{\dot12} &= A^1 - iA^2, & A^1 &= \tfrac{1}{2}(a^{\dot12} + a^{\dot21}),\\ a^{\dot21} &= A^1 + iA^2, & A^2 &= \tfrac{1}{2}i(a^{\dot12} - a^{\dot21}),\\ a^{\dot22} &= A^0 - A^3, & A^3 &= \tfrac{1}{2}(a^{\dot11} - a^{\dot22}). \end{aligned}$$

We then have

(2.11) $$g_{mn} A^m A^n = a^{\dot11} a^{\dot22} - a^{\dot12} a^{\dot21} = \det(a^{\dot\alpha\beta}).$$

From this identity it follows that spin-transformations $\lambda_\beta{}^\alpha$, which leave the determinant of an arbitrary Hermitian spintensor $a^{\dot\alpha\beta}$ invariant, induce transformations $L_s{}^r$ of Minkowski space-time which leave $g_{mn} A^m A^n$ invariant, i.e. Lorentz transformations. Now

$$\det(a'^{\dot\alpha\beta}) = \det(a^{\dot\sigma\rho} \lambda_\sigma{}^\alpha \lambda_\rho{}^\beta) = \det(a^{\dot\alpha\beta}) \,|\det(\lambda_\beta{}^\alpha)|^2.$$

Thus we obtain the result: A spin-transformation $\lambda_\beta{}^\alpha$ induces a Lorentz transformation if and only if the absolute value of its determinant is unity, i.e.

(2.12) $$|\det(\lambda_\beta{}^\alpha)| = 1.$$

It is easily seen that the two spin-transformations $\lambda_\beta{}^\alpha$ and $\lambda_\beta{}^\alpha e^{i\theta}$ (θ any real number) induce the same Lorentz transformation. It follows that we may limit the spin-transformations to those with determinant $+1$, without reducing the set of Lorentz transformations which are induced by them. It can also be shown[9] that every proper Lorentz transformation can be obtained from a spin-transformation. We summarize our conclusions as follows:

A proper Lorentz transformation determines a spin-transformation, which satisfies (2.12), uniquely to within an arbitrary *phase factor* $e^{i\theta}$.

Every spin-transformation $\lambda_\beta{}^\alpha$, satisfying

(2.13) $$\det(\lambda_\beta{}^\alpha) = 1,$$

induces a proper Lorentz transformation. To every proper Lorentz

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[9] O. Veblen and J. von Neumann, "Geometry of Complex Domains," Institute for Advanced Study mimeographed notes (Princeton, 1936).

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transformation there correspond exactly two spin-transformations which satisfy (2.13) and which differ in sign only.

Spin-transformations which satisfy (2.13) leave invariant the components of the real skew-symmetric spintensor $\epsilon^{\alpha\beta}$, defined by:

(2.14) $$\epsilon^{11} = \epsilon^{22} = 0, \quad \epsilon^{12} = -\epsilon^{21} = 1, \quad \epsilon^{\dot\alpha\dot\beta} = \epsilon^{\alpha\beta}.$$

This spintensor may be used to lower the suffixes of other spinors, and thus to introduce *covariant spinors* c_α, $a_{\dot\alpha\beta}$, etc. as follows:

(2.15) $$c^\alpha = \epsilon^{\alpha\beta} c_\beta, \qquad c^1 = c_2, \quad c^2 = -c_1;$$

(2.16) $$a^{\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta} a_{\dot\gamma\delta},$$

$$a^{\dot 11} = a_{\dot 22}, \quad a^{\dot 12} = -a_{\dot 21}, \quad a^{\dot 21} = -a_{\dot 12}, \quad a^{\dot 22} = a_{\dot 11}.$$

In particular, we find that

(2.17) $$\epsilon_{11} = \epsilon_{22} = 0, \quad \epsilon_{12} = -\epsilon_{21} = 1.$$

If the vector A^r is associated with the Hermitian spintensor $a^{\dot\alpha\beta}$ by the relations (2.10), and the vector B^r associated similarly with the spintensor $b^{\dot\alpha\beta}$, we deduce easily that

(2.18) $$A^m B_m = g_{mn} A^m B^n = \tfrac{1}{2} a^{\dot\alpha\beta} b_{\dot\alpha\beta}.$$

3. The Cubic Lattice, Integral Null Vectors, Integral Spinvectors. Consider the points in Minkowski space whose coordinates t, x, y, z are all real integers. The set of these points will be called *the cubic lattice*.

The coordinates of a point P of the lattice may be regarded as the components of the vector OP which joins the origin O to P. Such a vector will be called an *integral vector* since its components are integers. For most purposes it suffices to restrict ourselves to *primitive integral vectors*, whose components have no common factor, as all other integral vectors are multiples of these.

As in the previous section, we can associate with an integral vector t, x, y, z a Hermitian spintensor $a^{\dot\alpha\beta}$ by the relations

(3.01) $$a^{\dot 11} = t + z, \quad a^{\dot 12} = x - iy, \qquad a^{\dot 21} = x + iy, \quad a^{\dot 22} = t - z,$$

and

(3.02) $$t = \tfrac{1}{2}(a^{\dot 11} + a^{\dot 22}), \quad x = \tfrac{1}{2}(a^{\dot 12} + a^{\dot 21}), \quad y = \tfrac{1}{2}i(a^{\dot 12} - a^{\dot 21}), \quad z = \tfrac{1}{2}(a^{\dot 11} - a^{\dot 22}).$$

It immediately follows from (3.01) that the components of $a^{\dot\alpha\beta}$ are Gaussian integers. We also see from (3.01) that any common factor of t, x, y, z must be a factor of all $a^{\dot\alpha\beta}$. Equation (3.02) shows that any factor common to the $a^{\dot\alpha\beta}$, other than a factor of 2, must be a factor of t, x, y, z. In particular, if the vector t, x, y, z is to be primitive, any common factor of $a^{\dot\alpha\beta}$ must be a factor of 2.

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We shall now study *integral null vectors*, whose components satisfy the equation

$$t^2 - x^2 - y^2 - z^2 = 0. \tag{3.03}$$

By (2.11), this implies

$$a^{\dot{1}1}\, a^{\dot{2}2} = a^{\dot{1}2}\, a^{\dot{2}1}. \tag{3.04}$$

Making use of the unique factorization theorem for Gaussian integers, $a^{\dot{1}1}$ must split into two factors, of which one is a factor of $a^{\dot{1}2}$, the other being a factor of $a^{\dot{2}1}$. Since $a^{\dot{1}1}$ is real, those factors can be written in the form $mc^{\dot{1}}$ and nc^1, where m and n are real and relatively prime, and where $c^{\dot{1}} = \overline{c^1}$. Similarly, $a^{\dot{2}2}$ splits into factors $rc^{\dot{2}}$ and sc^2, $rc^{\dot{2}}$ being a factor of $a^{\dot{2}1}$ and sc^2 a factor of $a^{\dot{1}2}$, where r and s are real and relatively prime, and where $c^{\dot{2}} = \overline{c^2}$. Thus we have

$$a^{\dot{1}1} = mn\, c^{\dot{1}}c^1, \quad a^{\dot{1}2} = ms\, c^{\dot{1}}c^2,$$
$$a^{\dot{2}1} = rn\, c^{\dot{2}}c^1, \quad a^{\dot{2}2} = rs\, c^{\dot{2}}c^2.$$

Since $a^{\dot{1}2} = \overline{a^{\dot{2}1}}$, we have $ms = rn$. It follows that $m = r$, $n = s$, or $m = -r$, $n = -s$. Then

$$a^{\dot{1}1} = \quad mn\, c^{\dot{1}}c^1, \quad a^{\dot{1}2} = \pm\, mn\, c^{\dot{1}}c^2,$$
$$a^{\dot{2}1} = \pm\, mn\, c^{\dot{2}}c^1, \quad a^{\dot{2}2} = \quad mn\, c^{\dot{2}}c^2.$$

The factor ± 1 in the second and third of these expressions can be removed by absorbing it in c^1 or in c^2. Doing this and writing p for mn, we have

$$a^{\dot{\alpha}\beta} = pc^{\dot{\alpha}}c^{\beta}, \tag{3.05}$$

where p is a real integer. Decomposing p in the form (1.04), i.e. $p = qa\bar{a}$, we can absorb the complex integer a in both c^1 and c^2, thus reducing (3.05) to the form

$$a^{\dot{\alpha}\beta} = qc^{\dot{\alpha}}c^{\beta}, \tag{3.06}$$

where, as is easily seen, q is the product of those real primes, each taken once, which are contained an odd number of times in the greatest common factor of t, x, y, z.

Let us now consider *primitive integral null vectors*. Since the square of a real integer leaves a remainder of 1 or 0 on division by 4, according as the integer is odd or even, it is easily seen from (3.03) that, of the components of a primitive integral null vector, t and one of x, y, z must be odd, while the two remaining components (two of x, y, z) must be even.

For a primitive integral null vector, q in (3.06) must be ± 1, and we arrive at the following result:

Each primitive integral null vector determines a spinvector c^{α} with integral components c^1, c^2, such that

$$a^{\dot{\alpha}\beta} = \pm\, c^{\dot{\alpha}}c^{\beta}, \tag{3.07}$$

where the upper or lower sign must be taken throughout. Since $c^{\dot{1}}c^1$ and $c^{\dot{2}}c^2$ are both positive, we see from (3.02) that t is positive or negative according as the plus or minus sign is chosen in (3.07). For primitive integral null vectors pointing into the future we have $t > 0$, and thus

$$a^{\dot{\alpha}\beta} = c^{\dot{\alpha}}c^{\beta}. \tag{3.08}$$

Some non-primitive null vectors pointing into the future can also be represented in the form (3.08). Whether this is possible or whether the representation takes the more complicated form (3.06), with $q > 1$, depends only on the properties of the greatest common factor of the components of the null vector.

From (3.08) and (3.02) it is seen that a spinvector c^α with integral components determines an integral null vector (t, x, y, z) if and only if *c^1, c^2 are both odd or both even.* Such a spinvector will be called an ***integral spinvector***. Note that, even if c^1, c^2 are integers, c^α is not an integral spinvector if $c^1 + c^2$ is odd.

The sum and difference of integral spinvectors are again integral spinvectors; the product of an integer and an integral spinvector is an integral spinvector. Thus the integral spinvectors form a two-dimensional complex vector space with coefficients in the ring of complex integers. It is easy to see that the independent integral spinvectors

$$\mathbf{e}_{(1)} \equiv (1 + i\,, 0), \quad \mathbf{e}_{(2)} \equiv (1\,, 1) \tag{3.09}$$

form a basis; this means that any integral spinvector $\mathbf{c}$ can be written in the form

$$\mathbf{c} = a\mathbf{e}_{(1)} + b\mathbf{e}_{(2)}\,, \tag{3.10}$$

where a and b are integers, and that conversely any spinvector of this form is integral.

The following theorem can be derived:

The null vector associated, by (3.08), (3.02), *with an integral spinvector c^α is integral and primitive if and only if one or other of the following two conditions is satisfied:*

I *c^1, c^2 are both odd and relatively prime.*

(3.11) II $c^\alpha = (1 + i)d^\alpha$,
where d^1, d^2 are relatively prime and one of them is even, the other odd.

In the first case t is odd, z is even, and one of x, y is odd, the other even; in the second case t, z are odd and x, y are even. By (3.08), $\pm c^\alpha$ and $\pm\, ic^\alpha$ determine the same null vector.

The criterion which we have just stated solves our basic problem of determining all primitive integral null vectors.

If we drop the requirement that c^1, c^2 be integers, we may enquire to what extent the spinvector c^α is determined by a null vector in space-time. By (3.01), a null vector determines a unique Hermitian spintensor $a^{\dot\alpha\beta}$. Let

$$a^{\dot\alpha\beta} = c^{\dot\alpha}\, c^\beta = c'^{\dot\alpha}\, c'^\beta\,, \tag{3.12}$$

or, equivalently,

$$c^{\dot 1}c^1 = c'^{\dot 1}c'^1 \quad , \quad c^{\dot 2}c^2 = c'^{\dot 2}c'^2\,,$$
$$c^{\dot 1}c^2 = c'^{\dot 1}c'^2 \quad , \quad c^{\dot 2}c^1 = c'^{\dot 2}c'^1\,.$$

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The first two of these equations show that $c'^1 = c^1e^{i\theta}$, $c'^2 = c^2e^{i\phi}$ (θ, ϕ real), and the last two equations imply $\theta = \phi$. Hence

(3.13) $$c'^\alpha = c^\alpha e^{i\theta}.$$

Thus a null vector t, x, y, z, determines a spinvector c^α uniquely to within an arbitrary phase factor $e^{i\theta}$.

4. Integral Null Vectors are Spatially Dense. In (3.08) let us write

(4.01) $$c^1 = (1 + i)\,(p + iq) \quad , \quad c^2 = (1 + i)r,$$

where p, q, r are real integers. By (3.02) the spinvector c^α determines a null vector with components

(4.02) $$t = p^2 + q^2 + r^2,$$

(4.03) $$x = 2pr, \quad y = 2qr, \quad z = p^2 + q^2 - r^2.$$

Equations (4.02), (4.03) determine a discrete three parameter set of integral null vectors which are not necessarily primitive. We shall now show that the spatial projections of these null vectors, i.e. the directions defined by (4.03), are everywhere dense.

Consider an arbitrary direction D_0 in space, and let l_0, m_0, n_0 be its direction cosines. We can then define real numbers a_0, b_0, c_0 by the equations

(4.04) $$l_0 = 2a_0c_0, \quad m_0 = 2b_0c_0, \quad n_0 = a_0^2 + b_0^2 - c_0^2.$$

We obtain, by virtue of $l_0^2 + m_0^2 + n_0^2 = 1$,

(4.05) $$a_0 = l_0[2(1 - n_0)]^{-\frac{1}{2}}, \quad b_0 = m_0[2(1 - n_0)]^{-\frac{1}{2}}, \quad c_0 = [\tfrac{1}{2}(1 - n_0)]^{\frac{1}{2}}.$$

It is obvious that a_0, b_0, c_0 can be approximated by *rational* fractions a, b, c such that l, m, n, defined by[10]

(4.06) $$l = 2ac, \quad m = 2bc, \quad n = a^2 + b^2 - c^2,$$

are arbitrarily close to l_0, m_0, n_0, respectively. Thus the direction D, with direction ratios l, m, n, makes an arbitrarily small angle with D_0. Let the integer d be the least common denominator of the rational fractions a, b, c. Then p, q, r, defined by

(4.07) $$p = ad, \quad q = bd, \quad r = cd,$$

are real integers. If we substitute these integers into (4.02) and (4.03) we obtain an integral null line whose spatial component (x, y, z) is immediately seen to have the direction D. Since D approximates D_0, our assertion is proved.

Having shown that a subset of all integral null vectors is spatially dense, it follows, *a fortiori*, that the same is true for the set of all integral null vectors. Since every integral null vector is codirectional with a primitive integral null vector, the *set of all primitive integral null vectors is spatially dense.*[11]

[10] $l^2 + m^2 + n^2$ is not necessarily 1.

[11] By saying that a set of vectors is spatially dense, we mean, more precisely, that the *directions* of the spatial projections of the vectors in the set are dense. This remark applies also to Sec. 8.

5. Integral Lorentz Transformations. A Lorentz transformation

$$x'^r = L_s{}^r x^s \tag{5.01}$$

is *integral* if it maps into itself, i.e. leaves invariant as a whole, the cubic lattice which consists of all points with integral coordinates x^r.

Consider the integral vector $x^s \equiv \delta_m{}^s$, where m is 0, 1, 2, or 3, and where $\delta_m{}^s$ is 1 if $s = m$ and 0 if $s \neq m$. The transformation (5.01) maps this vector into $x'^r \equiv L_m{}^r$. If (5.01) is an integral Lorentz transformation then x'^r must be an integral vector; thus the components $L_m{}^r$ must be real integers. It is obvious that then x'^r is always integral whenever x^r is.

Since the determinant of a Lorentz transformation is ± 1, the components $(L^{-1})_s{}^r$ of the inverse Lorentz transformation will be real integers if $L_s{}^r$ are real integers. Thus a Lorentz transformation with integral components $L_s{}^r$ maps the set of all integral vectors into the set of all integral vectors, and not into a proper subset of the latter. The following conclusion is immediate:

A Lorentz transformation $L_s{}^r$ is integral if and only if all its components $L_s{}^r$ are real integers.

Thus, by (2.09), our problem of determining all integral Lorentz transformations reduces to the solution of 10 quadratic diophantine equations in 16 unknown integers. This rather formidable mathematical problem can be approached indirectly by considering integral null vectors, spinvectors and spin-transformations, as will be shown in this section and the next.

We shall now prove the following theorem: A necessary and sufficient condition for a Lorentz transformation to be integral is that the Lorentz transformation, as well as its inverse, map primitive integral null vectors into integral null vectors. The necessity of the condition is trivial; we shall therefore consider only its sufficiency.

Consider the four independent primitive integral null vectors

$$\begin{aligned}
\mathbf{N}_{(0)} &\equiv (1, -1, \ 0, \ 0),\\
\mathbf{N}_{(1)} &\equiv (1, \ 1, \ 0, \ 0),\\
\mathbf{N}_{(2)} &\equiv (1, \ 0, \ 1, \ 0),\\
\mathbf{N}_{(3)} &\equiv (1, \ 0, \ 0, \ 1).
\end{aligned} \tag{5.02}$$

We have

$$\begin{aligned}
\mathbf{E}_{(0)} &\equiv (1, \ 0, \ 0, \ 0) = \tfrac{1}{2}(\mathbf{N}_{(0)} + \mathbf{N}_{(1)}),\\
\mathbf{E}_{(1)} &\equiv (0, \ 1, \ 0, \ 0) = \tfrac{1}{2}(-\mathbf{N}_{(0)} + \mathbf{N}_{(1)}),\\
\mathbf{E}_{(2)} &\equiv (0, \ 0, \ 1, \ 0) = \mathbf{N}_{(2)} - \tfrac{1}{2}(\mathbf{N}_{(0)} + \mathbf{N}_{(1)}),\\
\mathbf{E}_{(3)} &\equiv (0, \ 0, \ 0, \ 1) = \mathbf{N}_{(3)} - \tfrac{1}{2}(\mathbf{N}_{(0)} + \mathbf{N}_{(1)}).
\end{aligned} \tag{5.03}$$

A Lorentz transformation, satisfying the hypothesis of our theorem, maps the vectors $\mathbf{N}_{(r)}$ into integral null vectors

$$\mathbf{N}'_{(r)} \equiv (t_{(r)}, x_{(r)}, y_{(r)}, z_{(r)}), \quad r = 0, 1, 2, 3, \tag{5.04}$$

and it maps the vectors $\mathbf{E}_{(r)}$ into

$$\begin{aligned}
\mathbf{E}'_{(0)} &= \tfrac{1}{2}(\mathbf{N}'_{(0)} + \mathbf{N}'_{(1)}),\\
\mathbf{E}'_{(1)} &= \tfrac{1}{2}(-\mathbf{N}'_{(0)} + \mathbf{N}'_{(1)}),\\
\mathbf{E}'_{(2)} &= \mathbf{N}'_{(2)} - \tfrac{1}{2}(\mathbf{N}'_{(0)} + \mathbf{N}'_{(1)}),\\
\mathbf{E}'_{(3)} &= \mathbf{N}'_{(3)} - \tfrac{1}{2}(\mathbf{N}'_{(0)} + \mathbf{N}'_{(1)}).
\end{aligned} \tag{5.05}$$

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If $\mathbf{N}'_{(r)} = \mathbf{M}'_{(r)}d$, where d is an integer and $\mathbf{M}'_{(r)}$ an integral null vector, then, by hypothesis, the inverse of the Lorentz transformation considered here maps $\mathbf{M}'_{(r)}$ into an integral null vector $\mathbf{M}_{(r)}$, and therefore maps $\mathbf{N}'_{(r)}$ into $\mathbf{N}_{(r)} = \mathbf{M}_{(r)}d$. But, by (5.02), the vectors $\mathbf{N}_{(r)}$ are all primitive. It follows that d must be a unit. Hence the integral null vectors $\mathbf{N}'_{(r)}$ must be primitive. Then by Sec. **3**, $t_{(r)}$ and one of $x_{(r)}$, $y_{(r)}$, $z_{(r)}$ must be odd, the other two being even.

Since the scalar product of two vectors is an invariant, we have

$$g_{mn}\,\mathbf{N}'_{(0)}{}^{m}\,\mathbf{N}'_{(1)}{}^{n} = g_{mn}\,\mathbf{N}_{(0)}{}^{m}\,\mathbf{N}_{(1)}{}^{n},$$

or

$$t_{(0)}\,t_{(1)} - x_{(0)}\,x_{(1)} - y_{(0)}\,y_{(1)} - z_{(0)}\,z_{(1)} = 2. \tag{5.06}$$

Thus the left-hand side of this equation is even. Combining this fact with the last statement of the preceding paragraph, we see that $t_{(0)}\,t_{(1)}$ and one of the three products $x_{(0)}\,x_{(1)}$, $y_{(0)}\,y_{(1)}$, $z_{(0)}\,z_{(1)}$ must be odd. In order to be definite, let us take $x_{(0)}\,x_{(1)}$ odd. Then $t_{(0)}$, $t_{(1)}$, $x_{(0)}$, $x_{(1)}$ are odd, and $y_{(0)}$, $y_{(1)}$, $z_{(0)}$, $z_{(1)}$ are even. It follows that $t_{(1)} \pm t_{(0)}$, $x_{(1)} \pm x_{(0)}$, $y_{(1)} \pm y_{(0)}$, $z_{(1)} \pm z_{(0)}$ are all even integers. Hence $\mathbf{E}'_{(r)}$, defined in (5.05), are integral vectors.

Applying the Lorentz transformation (5.01) to the vectors $\mathbf{E}_{(r)}$, given by (5.03), we obtain

$$\mathbf{E}'_{(r)}{}^{s} = L_r{}^{s}\,. \tag{5.07}$$

Thus $L_r{}^s$ are integers and the sufficiency of our condition is demonstrated.

6. Integral Spin-Transformations. It seems legitimate to deduce from the preceding theorem that a spin-transformation is associated with an integral Lorentz transformation if and only if both the spin-transformation and its inverse map integral spinvectors into integral spinvectors. It must be pointed out, though, that this statement is not *a priori* obvious and that we must proceed with caution. The reason is that a spinvector is not uniquely determined by a primitive null vector in space-time, but is determined only to within an arbitrary phase factor. However, we shall show now that the statement made above is true if the spin-transformation is taken with a suitable phase factor.

If a spin-transformation and its inverse map integral spinvectors into integral spinvectors, then the corresponding Lorentz transformation is integral since it and its inverse map primitive integral null vectors into integral null vectors. It is therefore sufficient to show that, given an integral Lorentz transformation, a spin-transformation can be found which represents it and which, as well as its inverse, maps integral spinvectors into integral spinvectors.

Let $L_s{}^r$ be an arbitrary integral Lorentz transformation. Since $L_s{}^r$ are integers, it is clear, by (5.01), that the greatest common factor of the components of an integral vector x^r is a common factor of the components of the transform x'^r of x^r under the integral Lorentz transformation. Since x^r can also be obtained from x'^r by the integral Lorentz transformation

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$(L^{-1})_s{}^r$, it follows that the x^r and the x'^r have the same greatest common factor. In particular, we have that, if an integral null vector can be represented in the form (3.08), the transform of this null vector under any integral Lorentz transformation can again be so represented. The following is easily deduced:

If $\lambda_\beta{}^\alpha$ is a spin-transformation which represents an integral Lorentz transformation $L_s{}^r$ then $\lambda_\beta{}^\alpha$ maps any integral spinvector into a spinvector which differs from an integral spinvector by at most a phase factor.

Therefore, if we introduce the two spinvectors

(6.01) $$\mathbf{e}_{(1)} \equiv (1+i, 0), \quad \mathbf{e}_{(2)} \equiv (1, 1),$$

we must have

(6.02) $$\lambda_\beta{}^\alpha e_{(1)}{}^\beta = e^{-i\phi} e'_{(1)}{}^\alpha, \qquad \lambda_\beta{}^\alpha e_{(2)}{}^\beta = e^{-i\theta} e'_{(2)}{}^\alpha,$$

where $\mathbf{e}'_{(1)}$, $\mathbf{e}'_{(2)}$, are integral spinvectors. Since $\lambda_\beta{}^\alpha$ is determined by $L_s{}^r$ only to within an arbitrary phase factor, we can choose this phase factor so that, in (6.02), $\phi = 0$. We can then write:

(6.03) $$(\lambda^{-1})_\beta{}^\alpha e'_{(1)}{}^\beta = e_{(1)}{}^\alpha, \qquad (\lambda^{-1})_\beta{}^\alpha e'_{(2)}{}^\beta = e^{i\theta} e_{(2)}{}^\alpha,$$

where $(\lambda^{-1})_\beta{}^\alpha$ is the inverse spin-transformation which exists, by (2.12), and which represents the integral Lorentz transformation $(L^{-1})_s{}^r$.

From (6.03) we obtain, on addition,

(6.04) $$(\lambda^{-1})_\beta{}^\alpha (e'_{(1)}{}^\beta + e'_{(2)}{}^\beta) = e_{(1)}{}^\alpha + e^{i\theta} e_{(2)}{}^\alpha.$$

Since $e'_{(1)}{}^\beta + e'_{(2)}{}^\beta$ is integral, $e_{(1)}{}^\alpha + e^{i\theta} e_{(2)}{}^\alpha$ must be of the form $e^{i\psi}(p, q)$, p and q being integers and ψ real. Thus, by (6.01), we have

(6.05) $$1 + i + e^{i\theta} = e^{i\psi} p,$$

(6.06) $$e^{i\theta} = e^{i\psi} q.$$

From (6.06) it follows that $|q| = 1$ and hence that q and $u = 1/q$ are units. Then (6.05) can be written

(6.07) $$1 + i = e^{i\theta}(up - 1).$$

Taking absolute values, we find that $|up - 1| = 2^{\frac{1}{2}}$ and thus $up - 1$ must be one of $1 + i$, $1 - i$, $-1 + i$, or $-1 - i$, since these are the only integers of absolute value $2^{\frac{1}{2}}$. In either of these cases $e^{-i\theta}$ is a unit, by (6.07). Then $e''_{(2)}{}^\beta = e^{-i\theta} e'_{(2)}{}^\beta$ is an integer and we can rewrite (6.03) as follows:

(6.08) $$\lambda_\beta{}^\alpha e_{(1)}{}^\beta = e''_{(1)}{}^\alpha, \qquad \lambda_\beta{}^\alpha e_{(2)}{}^\beta = e''_{(2)}{}^\alpha,$$

where $e''_{(1)}{}^\alpha = e'_{(1)}{}^\alpha$. Since an arbitrary integral spinvector can be written in the form (3.10), we immediately see that $\lambda_\beta{}^\alpha$ maps integral spinvectors into integral spinvectors. Similarly, corresponding to $(L^{-1})_s{}^r$, there must exist a spin-transformation $\mu_\beta{}^\alpha$ which maps integral spinvectors into integral spinvectors. But $\mu_\beta{}^\alpha$ can differ from $(\lambda^{-1})_\beta{}^\alpha$ by at most a phase factor e^{ix} and, since $(\lambda^{-1})_\beta{}^\alpha$ maps $\mathbf{e}''_{(2)}$ into $\mathbf{e}_{(2)}$, $\mu_\beta{}^\alpha$ maps $\mathbf{e}''_{(2)}$ into an integral spinvector $e^{ix}\,\mathbf{e}_{(2)}$. It follows that e^{ix} is a unit and that therefore $(\lambda^{-1})_\beta{}^\alpha$ maps integral spinvectors into integral spinvectors. This establishes our assertion.

Let us denote by v the determinant of $\lambda_\beta{}^\alpha$; we know, by (2.12), that $|v| = 1$. Writing (6.08) in the form

(6.09) $$\lambda_\beta{}^\alpha e_{(\gamma)}{}^\beta = e''_{(\gamma)}{}^\alpha,$$

and taking determinants on both sides, we have

(6.10) $$v(1+i) = \det(e''_{(\gamma)}{}^{\alpha}),$$

by (6.01). The right-hand side of (6.10) is obviously an integer. By the same argument as that applied to (6.07), it follows that v is a unit, i.e. $v = 1$, $v = i$, or $v = -1$, $v = -i$. In the latter two cases we can absorb the phase factor i in $\lambda_\beta{}^\alpha$, thus reducing these cases to the first two.

It is now clear that every proper integral Lorentz transformation is represented by two spin-transformations, differing in sign only, which satisfy the condition

(6.11) $$\det(\lambda_\beta{}^\alpha) = 1 \text{ or } i,$$

and which are such that both the spin-transformation and its inverse map the two integral spinvectors $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$, given by (6.01), into integral spinvectors. Conversely, the conditions just imposed on a spin-transformation are sufficient to insure that the spin-transformation corresponds to an integral Lorentz transformation.

Spin-transformations which satisfy the above conditions will be called *integral spin-transformations*. We shall now obtain the conditions on integral spin-transformations in a more explicit form.

If $\det(\lambda_\beta{}^\alpha) = 1$, (6.11), we have

(6.12) $$(\lambda^{-1})_1{}^1 = \lambda_2{}^2,\ (\lambda^{-1})_2{}^1 = -\lambda_2{}^1,\ (\lambda^{-1})_1{}^2 = -\lambda_1{}^2,\ (\lambda^{-1})_2{}^2 = \lambda_1{}^1.$$

On transforming $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$ by $\lambda_\beta{}^\alpha$ and by $(\lambda^{-1})_\beta{}^\alpha$ we obtain the following four spinvectors:

(6.13) $$\begin{gathered}((1+i)\lambda_1{}^1,\ (1+i)\lambda_1{}^2),\ (\lambda_1{}^1+\lambda_2{}^1,\ \lambda_1{}^2+\lambda_2{}^2),\\ ((1+i)\lambda_2{}^2,\ -(1+i)\lambda_1{}^2),\ (\lambda_2{}^2-\lambda_2{}^1,\ -\lambda_1{}^2+\lambda_1{}^1).\end{gathered}$$

If $\det(\lambda_\beta{}^\alpha) = i$, (6.11), we obtain by the same procedure four spinvectors which differ from those in (6.13) by unit factors only. Thus, in either case, each of the spinvectors (6.13) must be integral, i.e. the two components must be integers, both odd or both even. We easily deduce the following result:

A spin-transformation $\lambda_\beta{}^\alpha$ is integral if and only if one of the following four conditions is satisfied:

I $\lambda_\beta{}^\alpha$ *are integers such that*

(6.14) $$\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2 = 1,$$

and such that $\lambda_1{}^1+\lambda_2{}^1+\lambda_1{}^2+\lambda_2{}^2$ *is even.*

(6.15) II $$\lambda_\beta{}^\alpha = \mu_\beta{}^\alpha/(1+i),$$

where $\mu_\beta{}^\alpha$ *are odd integers such that*

(6.16) $$\mu_1{}^1\mu_2{}^2 - \mu_2{}^1\mu_1{}^2 = 2i.$$

III $\lambda_\beta{}^\alpha$ *are integers such that*

(6.17) $$\lambda_1{}^1\lambda_2{}^2 - \lambda_2{}^1\lambda_1{}^2 = i,$$

and such that $\lambda_1{}^1+\lambda_2{}^1+\lambda_1{}^2+\lambda_2{}^2$ *is even.*

$$\text{IV} \qquad \lambda_\beta{}^\alpha = \mu_\beta{}^\alpha/(1+i),$$

where $\mu_\beta{}^\alpha$ are odd integers such that

$$\mu_1{}^1\,\mu_2{}^2 - \mu_2{}^1\,\mu_1{}^2 = -2. \tag{6.18}$$

In cases II and IV, (6.16) and (6.18) are, by (6.15), equivalent respectively to (6.14) and (6.17). In these cases the condition that the sum of the $\lambda_\beta{}^\alpha$ be an even integer need not be stated separately since it follows from the other requirements, as can be seen by examining the possible remainders of $\mu_\beta{}^\alpha$ on division by 2.

The integral spin-transformations of types III and IV can be replaced by spin-transformations of determinant $+1$ if the phase factor $e^{-i\pi/4} = 2^{\frac{1}{2}}/(1+i)$ is introduced. This procedure has the disadvantage of introducing the irrationality $2^{\frac{1}{2}}$, but it has the advantage that the resulting spin-transformations together with those of types I and II form a group.

From the discussion of the diophantine equation (6.14) in Sec. **1** we see that there is a discrete sixfold infinity of integral spin-transformations of the type I. Similarly, it can be shown for each of the types II, III, and IV, that there is a discrete sixfold infinity of integral spin-transformations. Since there is a 2 - 1 correspondence between integral spin-transformations and proper integral Lorentz transformations, we have:

The group of proper integral Lorentz transformations is a discrete, sixfold infinite set.

We have not hesitated to "count" the order of infinity of the integral Lorentz group because this emphasizes the large number of integral Lorentz transformations. However, since we are dealing with an enumerably infinite discrete group of transformations without infinitesimal elements, the statement that the group is sixfold infinite has no invariant significance and must not be taken too literally. A different parametrization of the elements of the group may easily result in an order of infinity other than six.

7. Equivalence of Primitive Integral Null Vectors. We shall now prove that, *given two primitive integral null vectors, an integral Lorentz transformation can be found which maps the one into the other.* Thus all primitive integral null vectors are equivalent in the sense that no single such vector possesses an invariant property which is not shared by all others.

In Sec. **3** we saw that if the vector (t, x, y, z) is a primitive integral null vector, then t is odd and one of x, y, z is odd, the remaining two components being even. A primitive integral null vector with y or x odd is mapped into a vector with z odd by the proper integral Lorentz transformation which cyclicly permutes the $x-$, $y-$, and $z-$ axes once or twice. It follows that it is sufficient to prove the italicized statement for the case where the two assigned primitive integral null vectors have odd $z-$ components. Then, by Sec. **3**, the two null vectors are represented by spinvectors of the form (3.11):

$$c^1 = (1+i)d^1, \quad c^2 = (1+i)d^2, \tag{7.01}$$

where d^1 and d^2 are relatively prime integers of which one is even and one

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odd. It is obviously sufficient to show that there always exists an integral spin-transformation mapping an integral spinvector of the type considered into the spinvector $\mathbf{e}_{(1)} \equiv (1 + i, 0)$.

Consider the spin-transformation

$$\lambda_\beta{}^\alpha c^\beta = e_{(1)}{}^\alpha. \tag{7.02}$$

By (7.01), we can write this

$$\lambda_1{}^1 d^1 + \lambda_2{}^1 d^2 = 1, \tag{7.03}$$

$$\lambda_1{}^2 d^1 + \lambda_2{}^2 d^2 = 0. \tag{7.04}$$

The last equation is satisfied if we put

$$\lambda_1{}^2 = - d^2, \quad \lambda_2{}^2 = d^1. \tag{7.05}$$

Then equation (7.03) is identical with the condition (6.14). Since, by (7.05), $\lambda_1{}^2$ and $\lambda_2{}^2$ are relatively prime integers and $\lambda_1{}^2 + \lambda_2{}^2$ is odd, we can, by Sec. 1, find integers $\lambda_1{}^1$ and $\lambda_2{}^1$ satisfying (7.03), or equivalently (6.14), and such that $\lambda_1{}^1 + \lambda_2{}^1$ is odd. It follows that $\lambda_1{}^1 + \lambda_2{}^1 + \lambda_1{}^2 + \lambda_2{}^2$ is even. Thus conditions I (Sec. 6) for integral spin-transformations are satisfied by $\lambda_\beta{}^\alpha$ and our proof is complete.

8. Integral Time Lines are Spatially Dense. *Integral time lines* are the transforms of the t-axis ($x = y = z = 0$) under integral Lorentz transformations. A primitive integral vector having the direction of an integral time line will be called a *primitive integral time vector*; it is the transform under an integral Lorentz transformation of the vector

$$\mathbf{E}_{(0)} \equiv (1, 0, 0, 0). \tag{8.01}$$

Thus far integral Lorentz transformations have been regarded as mappings of space-time into itself, which map the points of the cubic lattice into other lattice points. However, an integral Lorentz transformation can also be regarded as a change to a new coordinate system, such that the points of the cubic lattice have again integral coordinates with respect to the new coordinate axes. Such a coordinate system will be called an *integral Lorentz frame.* Integral time lines are merely the t-axes of integral Lorentz frames.

Before we investigate the main theorem of this section we shall consider briefly the velocities associated with integral time lines. By "velocity" is meant the velocity of a particle whose world line coincides with the integral time line, or, equivalently, the velocity of a particle at rest in the corresponding integral Lorentz frame.

The components t, x, y, z of a primitive integral time vector satisfy the diophantine equation

$$t^2 - x^2 - y^2 - z^2 = 1. \tag{8.02}$$

The velocity v associated with this integral time vector is given by

$$v = \left(\frac{x^2}{t^2} + \frac{y^2}{t^2} + \frac{z^2}{t^2}\right)^{\frac{1}{2}}. \tag{8.03}$$

By (8.02), this reduces to

$$(8.04)\qquad v = \left(1 - \frac{1}{t^2}\right)^{\frac{1}{2}} = \frac{1}{t}(t^2 - 1)^{\frac{1}{2}}.$$

Since t must be an integer we see that the only possible velocities are, for $t = 1, 2, 3, 4, \ldots,$

$$(8.05)\qquad v = 0,\ \ \tfrac{1}{2}\,3^{\frac{1}{2}},\ \ \tfrac{2}{3}\,2^{\frac{1}{2}},\ \ \tfrac{1}{4}\,(15)^{\frac{1}{2}},\ldots.$$

Remembering that we have chosen the velocity of light c equal to unity, we see that *the velocities* (*other than zero*) *associated with integral time lines are very high, the smallest velocity being* $\frac{1}{2}3^{\frac{1}{2}} = 0.866$ *times the velocity of light.*

An example of an integral time line, associated with the minimum non-zero velocity $\frac{1}{2}3^{\frac{1}{2}}$, is given by the transform of the t-axis under the integral Lorentz transformation

$$(8.06)\qquad L_s{}^r \equiv \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

We now proceed to show that integral time lines are spatially dense. The vector $\mathbf{E}_{(0)}$ (8.01) is associated with the Hermitian spintensor $a_{(0)}{}^{\dot\alpha\beta}$, given by

$$(8.07)\qquad a_{(0)}{}^{\dot 1 1} = a_{(0)}{}^{\dot 2 2} = 1,\quad a_{(0)}{}^{\dot 1 2} = a_{(0)}{}^{\dot 2 1} = 0.$$

The integral spin-transformation $\lambda_\beta{}^\alpha$ of type I (Sec. 6) maps $a_{(0)}{}^{\dot\alpha\beta}$ into the spintensor $a^{\dot\alpha\beta}$, given by

$$(8.08)\qquad \begin{aligned} a^{\dot 1 1} &= \bar\lambda_1{}^1\,\lambda_1{}^1 + \bar\lambda_2{}^1\,\lambda_2{}^1, & a^{\dot 1 2} &= \bar\lambda_1{}^1\,\lambda_1{}^2 + \bar\lambda_2{}^1\,\lambda_2{}^2, \\ a^{\dot 2 1} &= \bar\lambda_1{}^2\,\lambda_1{}^1 + \bar\lambda_2{}^2\,\lambda_2{}^1, & a^{\dot 2 2} &= \bar\lambda_1{}^2\,\lambda_1{}^2 + \bar\lambda_2{}^2\,\lambda_2{}^2, \end{aligned}$$

and $a^{\dot\alpha\beta}$ is in turn associated with the primitive integral time vector whose components t, x, y, z are given by

$$(8.09)\qquad \begin{aligned} t &= \tfrac{1}{2}(\bar\lambda_1{}^1\,\lambda_1{}^1 + \bar\lambda_2{}^1\,\lambda_2{}^1 + \bar\lambda_1{}^2\,\lambda_1{}^2 + \bar\lambda_2{}^2\,\lambda_2{}^2), \\ x + iy &= \bar\lambda_1{}^2\,\lambda_1{}^1 + \bar\lambda_2{}^2\,\lambda_2{}^1, \\ z &= \tfrac{1}{2}(\bar\lambda_1{}^1\,\lambda_1{}^1 + \bar\lambda_2{}^1\,\lambda_2{}^1 - \bar\lambda_1{}^2\,\lambda_1{}^2 - \bar\lambda_2{}^2\,\lambda_2{}^2). \end{aligned}$$

We shall now show that the spatial projections of the integral time vectors (8.09) are dense, or, equivalently, that the expression

$$(8.10)\qquad \frac{x}{z} + i\frac{y}{z} = \frac{2(\bar\lambda_1{}^2\,\lambda_1{}^1 + \bar\lambda_2{}^2\,\lambda_2{}^1)}{\bar\lambda_1{}^1\,\lambda_1{}^1 + \bar\lambda_2{}^1\,\lambda_2{}^1 - \bar\lambda_1{}^2\,\lambda_1{}^2 - \bar\lambda_2{}^2\,\lambda_2{}^2}$$

can be made to approximate to an arbitrary degree any preassigned complex number β, which we may assume to be non-zero.

Given an arbitrary non-zero complex number β, we define α by the equation

$$(8.11)\qquad \beta = \frac{2\alpha}{\bar\alpha\alpha - 1}.$$

We may take α to be

$$(8.12)\qquad \alpha = [1 + (1 + \bar\beta\beta)^{\frac{1}{2}}]/\bar\beta,$$

so that $|\alpha| > 1$, and therefore $\bar\alpha\alpha - 1 > 0$.

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It is obvious that, given a small positive ϵ, we can find complex integers λ_1^1, and λ_1^2, which are relatively prime, such that

$$(8.13) \qquad \left| \alpha - \frac{\lambda_1^1}{\lambda_1^2} \right| < \frac{\epsilon}{2}, \quad |\lambda_1^2| > \frac{2}{\epsilon},$$

and such that λ_1^1 is even. Then, by Sec. **1**, non-zero integers λ_2^2, λ_2^1, can be found, satisfying

$$(8.14) \qquad \lambda_1^1 \lambda_2^2 - \lambda_2^1 \lambda_1^2 = 1,$$

and such that $\lambda_1^1 + \lambda_2^1 + \lambda_1^2 + \lambda_2^2$ is even. Then λ_β^α are the components of an integral spin-transformation of type I (Sec. **6**). From (8.14) we obtain

$$(8.15) \qquad \left| \frac{\lambda_1^1}{\lambda_1^2} - \frac{\lambda_2^1}{\lambda_2^2} \right| = \left| \frac{1}{\lambda_1^2 \lambda_2^2} \right| < \frac{\epsilon}{2},$$

by the second inequality of (8.13). Combining (8.13) with (8.15), we have

$$(8.16) \qquad \left| \alpha - \frac{\lambda_2^1}{\lambda_2^2} \right| < \epsilon .$$

Thus both λ_1^1/λ_1^2 and λ_2^1/λ_2^2 approximate α. Substituting in (8.11), we see that the number β is approximated by the two fractions

$$(8.17) \qquad \frac{2\bar{\lambda}_1^2 \lambda_1^1}{\bar{\lambda}_1^1 \lambda_1^1 - \bar{\lambda}_1^2 \lambda_1^2} \text{ and } \frac{2\bar{\lambda}_2^2 \lambda_2^1}{\bar{\lambda}_2^1 \lambda_2^1 - \bar{\lambda}_2^2 \lambda_2^2},$$

the two denominators being positive, since $\bar{\alpha}\alpha - 1 > 0$. It follows that β is approximated by the fraction which is obtained by adding the numerators and denominators of the fractions (8.17), i.e. β is approximated by (8.10). This completes the proof that we can find integral time lines whose spatial projections approximate any preassigned direction in space.

Since every integral time line is codirectional with a primitive integral time vector, we deduce the following theorem:

The set of all primitive integral time vectors is spatially dense.

We add, without proof, the statement of a more general theorem which can be verified by arguments more complicated than, but quite similar to those just given above.

Consider an integral Lorentz frame and any integral vector. *The transforms of the integral vector under all integral Lorentz transformations form a set which is spatially dense.*

The preceding theorem is a special case of this. So also is the theorem of Sec. **4**, once the equivalence of primitive null vectors (Sec. **7**) is established.

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Appendix

Professor H. S. M. Coxeter was kind enough to show me some independent work of his which is essentially equivalent to our problem of finding all integral Lorentz transformations. He considers a lattice in *hyperbolic 3-space* consisting of the points of our cubic lattice which lie on the unit "sphere"

$$t^2 - x^2 - y^2 - z^2 = 1. \tag{A}$$

The congruent transformations of hyperbolic space which leave this lattice invariant as a whole are exactly our integral Lorentz transformations.

Coxeter chooses as his basic operation the reflection in 4-space which consists of adding the quantity $t - x - y - z$ to each of the four coordinates t, x, y, z of a point. In our notation this transformation is given by

$$L_s{}^r = \begin{pmatrix} 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}. \tag{B}$$

This is easily seen to be an integral Lorentz transformation. Combining iteration of this transformation with the trivial operations of permuting the spatial coordinates x, y, z and of changing the signs of any of the coordinates t, x, y, z, all integral Lorentz transformations (including reflections) are obtained.

This procedure may simplify slightly some of the proofs in this paper. For example, to show that primitive integral null vectors are equivalent, take such a vector (t, x, y, z) and by changing signs make certain that t, x, y, z are all positive or zero. Then so long as $t > 1$ at least two of x, y, z must be non-zero since (t, x, y, z) is assumed primitive. Hence we have

$$t = (x^2 + y^2 + z^2)^{\frac{1}{2}} < x + y + z \leqslant \{3(x^2 + y^2 + z^2)\}^{\frac{1}{2}} < 2t.$$

It follows that $-t < t - x - y - z < 0$. Thus performing (B) and changing signs again, we obtain an integral null vector whose t-component has been decreased. Repeating this process it is clear that we must finally arrive at one of the forms $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, or $(1, 0, 0, 1)$. Permuting the spatial coordinates we can reduce the given primitive integral null vector to the standard form

$$(1, 1, 0, 0). \tag{C}$$

This establishes the theorem of Sec. **7**.

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