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A CURE FOR THE TELEPHONE DISEASE

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The following problem due to A. Boyd, has enjoyed a certain popularity in recent months with several mathematicians. A different solution to the one given here has been given independently by R. T. Bumby and J. Spencer.(2)

The Problem. There are n ladies, and each one of them knows an item of scandal which is not known to any of the others. They communicate by telephone, and whenever two ladies make a call, they pass on to each other, as much scandal as they know at that time. How many calls are needed before all the ladies know all the scandal?

If $f(n)$ is the minimum number of calls needed, then it is easy to verify that $f(1)=0, f(2)=1, f(3)=3, f(4)=4$. It is also easy to see that $f(n+1)\leq f(n)+2$, for the $(n+1)$-th lady first calls one of the others and someone calls her back after the remaining n ladies have communicated all the scandal to each other. It follows that $f(n)\leq 2n-4$ $(n\geq 4)$. We will prove that

$$f(n) = 2n-4 \qquad (n \geq 4). \tag{1}$$

We shall represent the n ladies by the set of vertices, V, of a multigraph. A sequence of calls

$$c(1), c(2), \ldots, c(t) \tag{2}$$

between them can be represented by the edges of the multigraph labelled according to the order in which the calls are made.

The interchange rule. Suppose (2) is a given sequence of calls, and suppose that the a calls $c(i), c(i+1), \ldots, c(i+a-1)$ are vertex disjoint from the succeeding b calls $c(i+a), \ldots, c(i+a+b-1)$. Then we can interchange the order of these two blocks of a and b calls, i.e. if we make the same calls as in (2) but in the order

$$c(1), \ldots, c(i-1)c(i+a), \ldots, c(i+a+b-1),$$
$$c(i), \ldots, c(i+a-1), c(i+a+b), \ldots, c(t)$$

then the total information conveyed is exactly the same as for the sequence (2). If $c'(1), \ldots, c'(t)$ is a rearrangement of the sequence (2) obtained by a number of interchanges of adjacent blocks of vertex disjoint calls of the kind just described, we say that c' is an *equivalent* calling system and write $c' \sim c$.

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(2) Since this paper was written we have received another solution from R. Tijdeman. His paper will appear in Nieuw Archief voor Wiskunde.

Let (2) be a given sequence of calls. A vertex x of the graph will be called an F-point if the corresponding lady knows everything after the t calls have been made. Obviously, if $c' \sim c$, then the sequence of calls $c'(1), \ldots, c'(t)$ gives the same F-points as c. In order that there be any F-points at all, the graph G, with vertex set V and edge set (2), must be connected. Consequently, we have

LEMMA 1. *There are no F-points after n-2 calls.*

In order to prove (1) it is enough to prove

LEMMA 2. *After $n+k-4$ calls there are at most k F-points.*

Proof. We shall actually prove the following stronger assertion
$P(k)$: *If $c(1), \ldots, c(n+k-4)$ is a sequence of $n+k-4$ calls, then there are at most k F-points. Further, if there are k F-points, then there is an equivalent calling sequence $c' \sim c$ in which the last k calls*

$$c'(n-3), c'(n-2), \ldots, c'(n+k-4)$$

are all between F-points.

The first part of $P(k)$ follows from Lemma 1 if $k=0$, 1, or 2 and for these values of k the second part of $P(k)$ is satisfied vacuously. We now assume $k>2$ and use induction on k.

Suppose there are $k+1$ F-points after the $n+k-4$ calls. Since the last call $c(n+k-4)$ can produce at most two F-points, it follows from the induction hypothesis that there must be $k-1$ F-points $\{x_1, \ldots, x_{k-1}\}$ after the first $n+k-5$ calls and the last call $c(n+k-4)$ is between two additional F-points $\{x_k, x_{k+1}\}$. By the second part of $P(k-1)$, we can assume that the last $k-1$ calls of the sequence $c(1), \ldots, c(n+k-5)$ are between the F-points $\{x_1, \ldots, x_{k-1}\}$. By the interchange rule, the last call $c(n+k-4)$ could be made before $c(n-3), \ldots, c(n+k-5)$. It follows that after the $n-3$ calls

$$c(1), c(2), \ldots, c(n-4), c(n+k-4)$$

there would be two F-points $\{x_k, x_{k+1}\}$ contrary to Lemma 1. This shows that there can be at most k F-points.

To complete the proof we must show that the second part of the inductive statement $P(k)$ holds.

Suppose there are k F-points after the $n+k-4$ calls

$$c(1), c(2), \ldots, c(n+k-4).$$

Consider the disconnected graph G_0 with vertex set V and edge set $E_0 = \{c(1), \ldots, c(n-2)\}$. Suppose G_0 has an isolated vertex x. There are at least $k-1$ F-points $x_i \neq x (1 \leq i < k)$ and each of these is connected to x by a path from the edge set $E_1=\{c(n-1), \ldots, c(n+k-4)\}$. This implies that the points x, x_i $(1 \le i < k)$ are in a single component of the graph on V with edge set E_1. This is impossible since $|E_1|+1<k$. Thus G_0 has no isolated vertex and each component of this graph has at least one edge. By the interchange rule, the first $n-3$ calls can be equivalently re-ordered so that the $(n-3)$-rd call is in a different component of G_0 to $c(n-2)$. Therefore, we may assume that $c(n-3)$ and $c(n-2)$ are disjoint.

Now suppose that the last k calls of the given sequence are not all between F-points. Then there is p, $1 \le p \le k$, such that the last $p-1$ calls $c(n+k-p-2), \ldots, c(n+k-4)$ are all between F-points but the preceding call, $c(n+k-p-3)$, is adjacent to at most one F-point. In fact, we can assume that $p<k$. For, if $p=k$ we can, by the last paragraph, consider instead the equivalent calling sequence obtained by interchanging $c(n-3)$ and $c(n-2)$.

If $c(n+k-p-3)$ is not adjacent to any F-point, then by the interchange rule, this call could be made last and then there would be k F-points after only $n+k-5$ calls

$$c(1), \ldots, c(n+k-p-4), c(n+k-p-2), \ldots, c(n+k-4).$$

This contradicts the induction hypothesis and so we can assume that $c(n+k-p-3)$ is adjacent to exactly one F-point.

Consider the graph $\bar{G}$ on V having the p edges $c(n+k-p-3), \ldots, c(n+k-4)$ and let C be the component of this graph containing the edge $c(n+k-p-3)$. Let $\bar{c}(1)=c(n+k-p-3)$, $\bar{c}(2), \ldots, \bar{c}(r)$ be the edges of C in the order in which these calls are made and let $\bar{\bar{c}}(1), \bar{\bar{c}}(2), \ldots, \bar{\bar{c}}(p-r)$ be the remaining edges of $\bar{G}$ in order. By the interchange rule, $\bar{\bar{c}}(1)$ can be made before any of the calls in C and similarly for $\bar{\bar{c}}(2), \ldots, \bar{\bar{c}}(p-r)$. Thus the original calling sequence is equivalent to the sequence of calls

$$(3) \qquad c(1), c(2), \ldots, c(n+k-p-4), \bar{\bar{c}}(1), \ldots, \bar{\bar{c}}(p-r), \bar{c}(1), \ldots, \bar{c}(r).$$

Since $\bar{c}(1)$ is adjacent to only one F-point, the component C contains at most r F-points (C has r edges and at most $r+1$ points). It follows that after the first $n+k-r-4$ calls in the sequence (3), there are at least $k-r$ F-points. Therefore, by the induction hypothesis there must be exactly $k-r$ such F-points (and the component C contains exactly r F-points) and there is an equivalent re-ordering of these $n+k-r-4$ calls so that the last $k-r$ are between the $k-r$ F-points not in C. In this way we obtain an equivalent calling sequence, say

$$(4) \qquad c_1(1), \ldots, c_1(n+k-r-4), \bar{c}(1), \ldots, \bar{c}(r).$$

Since the $k-r$ calls $c_1(n-3), \ldots, c_1(n+k-r-4)$ are vertex disjoint from $\bar{c}(1), \ldots, \bar{c}(r)$ (they are between F-points not in C) it follows, again by the interchange rule, that an equivalent sequence is

$$(5) \qquad c_1(1), \ldots, c_1(n-4), \bar{c}(1), \ldots, \bar{c}(r), c_1(n-3), \ldots, c_1(n+k-r-4).$$

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The first $n-4+r$ calls in the sequence (5) give rise to the r F-points in C. Therefore, by the induction hypothesis, these calls can be rearranged so that the last r calls are between F-points. After re-ordering the first $n+r-4$ calls of (5) in this way we obtain an equivalent calling system $c' \sim c$ in which the last k calls are all between F-points. This completes the proof of Lemma 2.

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