Canad. Math. Bull. Vol. 31 (1), 1988

COUNTEREXAMPLES CONCERNING SUPPORT THEOREMS FOR CONVEX SETS IN HILBERT SPACE

BY

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ABSTRACT. The Bishop-Phelps theorem guarantees the existence of support points and support functionals for a nonempty closed convex subset of a Banach space; equivalently, it guarantees the existence of subdifferentials and points of subdifferentiability of a proper lower semicontinuous convex function on a Banach space. In this note we show that most of these results cannot be extended to *pairs* of convex sets or functions, even in Hilbert space. For instance, two proper lower semicontinuous convex functions need not have a common point of subdifferentiability nor need they have a subdifferential in common. Negative answers are also obtained to certain questions concerning density of support points for the closed sum of two convex subsets of Hilbert space.

This note was motivated by a question arising in mathematical economics posed by Professor R. Vohra [18] concerning a particular form of the Bishop-Phelps theorem for the sum (assumed to be weak* closed) of two weak* closed convex subsets of a dual Banach space. It is easy to see that if a functional attains its supremum on the sum of two convex sets, then it attains its supremum on each of them, so we were led first to consider the question of the existence of such simultaneous support functionals. They need *not* exist; Edelstein and Thompson [10] have constructed a counterexample in c_0. What our first example shows is that this can be done in Hilbert space. This does not respond to Professor Vohra's question, however, since the sum of the two sets in our example is not closed. A related second example does, however, answer his question in the negative. These examples had their genesis in an earlier result [14] concerning common points of subdifferentiability of two convex functions on Hilbert space; this is reproduced in a third example. All of these examples reduce in one way or another to the following elementary lemma.

LEMMA. *If* $y \in l_2$ *is bounded above on* $B_\infty = \{x \in l_2: \|x\|_\infty \leqq 1\}$, *then* y *is an element of* l_1. *If, moreover,* y *attains its supremum on* B_∞, *then* y *is an finitely nonzero sequence.*

Received by the editors July 18, 1986, and, in revised form, December 8, 1986.

AMS Subject Classification (1980): 46B20, 46B22, 46C05, 52A05.

PROOF. Suppose that for some $M \geqq 0$ we have $\langle x, y\rangle \leqq M$ for all $x \in B_\infty$. For each $n > 0$ define $x(n) \in B_\infty$ by $x(n)_i = \operatorname{sgn} y_i$, $i = 1, 2, \ldots, n$, while $x(n)_i = 0$ if $i > n$. Then $\langle x(n), y\rangle = \sum_{i=1}^n |y_i| \leqq M$ for all n, which shows that $y \in l_1$. If there exists $x \in B_\infty$ such that $\langle z, y\rangle \leqq \langle x, y\rangle$ for all $z \in B_\infty$, define $x(n)$ as above to obtain $\sum_{i=1}^n |y_i| \leqq \sum x_i y_i$ for all n, hence $\|y\|_1 \leqq \sum x_i y_i \leqq \|x\|_\infty \|y\|_1 \leqq \|y\|_1$. This implies that $|x_i| = 1$ whenever $y_i \neq 0$; since $x_i \to 0$ as $i \to \infty$, we conclude that $\{y_i\}$ is finitely nonzero.

The example presented below exists in l_2, but it is more easily seen if we represent this space as the product space $l_2 \times \mathbf{R}$, where $\mathbf{R}$ is the real line. We pair this space with its dual (that is, with itself) by

$$\langle (x, r), (y, s) \rangle = \langle x, y\rangle - rs, \quad x, y \in l_2, \quad r, s \in \mathbf{R}.$$

EXAMPLE. 1. There exist two closed convex subsets C_1, C_2 of $l_2 \times \mathbf{R}$, each with nonempty interior, such that a functional $(y, s) \in l_2 \times \mathbf{R}$ is bounded above on both sets if (and only if) $\|y\|_1 \leqq s$, while *no* nonzero functional attains its supremum on both sets.

PROOF. We define

$$C_1 = \{ (x, r) \in l_2 \times \mathbf{R}\colon r \geqq 0 \quad \text{and} \quad \|x\|_\infty \leqq r + 1\} \quad \text{and}$$

$$C_2 = \{ (x, r) \in l_2 \times \mathbf{R}\colon \|x\|_\infty \leqq 1 \quad \text{and} \quad r \geqq \langle x, a\rangle \},$$

where $a = (2^{-i}) \in l_1 \subset l_2$. Since $x \to \|x\|_\infty$ is convex and continuous on l_2, it is immediate that both sets are closed, convex and have nonempty interior. Suppose, first, that (y, s) defines a functional which is bounded above on C_1. Since $(\alpha z, \|\alpha z\|_\infty - 1) \in C_1$ whenever $z \in l_2$ and $\alpha > 0$, there is a constant $M > 0$ such that

$$\langle \alpha z, y\rangle - s(\|\alpha z\|_\infty - 1) \leqq M$$

for all such α and z. Dividing by α and letting the latter tend to $+\infty$ shows that $\langle z, y\rangle \leqq s\|z\|_\infty$ for all $z \in l_2$. It follows readily (take $z = x(n)$ as in the proof of the above lemma) that $s \geqq \|y\|_1$. This proves the "only if" assertion. Suppose now, that $(y, s) \neq (0, 0)$ attains its supremum on C_1 at the point (x, r). This means that $r \geqq 0$, $\|x\|_\infty \leqq r + 1$ and

$$(*) \qquad \langle z, y\rangle - ts \leqq \langle x, y\rangle - rs$$

whenever $(z, t) \in C_1$. If $z \in l_2$ with $\|z\|_\infty \leqq r + 1$, then $(z, r) \in C_1$, so from (*) we have $\langle z, y\rangle \leqq \langle x, y\rangle$. This says that the functional on l_2 represented by y attains its supremum on $(r + 1)B_\infty$ at the point x, and hence by the Lemma, y must be a finitely nonzero sequence. Note that since $s \geqq \|y\|_1$ and $(y, s) \neq (0, 0)$, we must have $s > 0$.

Suppose, next, that the functional represented by (y, s) also attains its

https://doi.org/10.4153/CMB-1988-019-1 Published online by Cambridge University Press

supremum on the set C_2, at a point (w, u), say. This means that $\|w\|_\infty \leqq 1$ and $u \geqq \langle w, a \rangle$ and that for all $(z, t) \in C_2$ we have

$$(**) \qquad \langle z, y \rangle - st \leqq \langle w, y \rangle - su.$$

Now, for any $z \in l_2$ with $\|z\|_\infty \leqq 1$ we have $(z, \langle z, a \rangle) \in C_2$ and hence it follows from (**) that

$$\begin{aligned} \langle z, y - sa \rangle &= \langle z, y \rangle - s\langle z, a \rangle \leqq \langle w, y \rangle - su \\ &= \langle w, y - sa \rangle + (\langle w, a \rangle - u)s \leqq \langle w, y - sa \rangle. \end{aligned}$$

This inequality shows that the functional represented by $y - sa$ attains its supremum on B_∞, hence by the Lemma is a finitely nonzero sequence. But since $s > 0$ and $a_i > 0$ for all i, it is impossible for both y and $y - sa$ to be finitely nonzero, so no functional attains its supremum on both sets.

It remains to prove that any element of the cone $K = \{(y, s) \in l_2 : s \geqq \|y\|_1\}$ is bounded above on both sets; this will show that there are many such functionals. First, if $(y, s) \in K$ and $(x, r) \in C_1$, then

$$\langle x, y \rangle - rs \leqq \langle x, y \rangle - r\|y\|_1 \leqq (\|x\|_\infty - r)\|y\|_1 \leqq \|y\|_1,$$

so (y, s) is bounded above on C_1. Moreover, if $(x, r) \in C_2$, then

$$\langle x, y \rangle - rs \leqq \langle x, y \rangle - s\langle x, a \rangle = \langle x, y - sa \rangle \leqq \|y - sa\|_1,$$

so (y, s) is bounded above on C_2 as well.

As a corollary to the result above, we see that the sum $C = C_1 + C_2$ is actually open, because it has empty boundary: Since C_1 has nonempty interior, then so does C, hence any boundary point x of C would be a support point, and any decomposition $x = u + v$ would yield simultaneous support points u, v of C_1, C_2 respectively.

The cone K of elements of $l_2 \times \mathbf{R}$ which are bounded above on both of the sets C_1 and C_2 in Example 1 is the epigraph of the l_1 norm on l_2. Thus, while it is "big" in a certain sense (since l_1 is dense in l_2), it clearly has empty interior. (Otherwise, the l_1 norm would be continuous, hence equivalent to the l_2 norm.) In fact, if the set of functionals which are bounded above on two closed convex subsets of l_2 has nonempty interior, then there are many functionals which simultaneously support both sets; this is a consequence of the fact that reflexive Banach spaces have the Radon-Nikodym property, together with the following proposition. (For information about spaces with the Radon-Nikodym property and strongly exposed points see either of the monographs [4] or [9].)

PROPOSITION. *Suppose that the Banach space E has the Radon-Nikodym property, that C_1, C_2 are nonempty closed convex subsets of E and that the cone*

$K = \{f \in E^*: \sup f(C_i) < \infty, i = 1, 2\}$ *has nonempty interior* int K. *Then there is a dense subset of functionals in K which simultaneously support both* C_1 *and* C_2.

PROOF. If C_1 and C_2 are bounded, then this is trivial: For $i = 1, 2$, the set of functionals which strongly expose some point of C_i is a dense G_δ subset of E^* [4, p. 55], so the intersection of these two sets is also a dense G_δ. We prove the general result by reducing it to the bounded case, as follows: Suppose that f_0 is an interior point of K, let $\alpha = \sup f_0(C_1) - 1$ and define the slice S_1 of C_1 by

$$S_1 = \{x \in C_1: f_0(x) \geqq \alpha\}.$$

Obviously, S_1 is closed, nonempty and convex. To see that it is bounded, it suffices to prove that it is weakly bounded, that is, that $\sup f(S_1) < \infty$ for every $f \in E^*$. Since f_0 is an interior point of K there exists $\epsilon > 0$ such that $f_0 + \epsilon f \in K$. If $x \in B_1$, then

$$\alpha + \epsilon f(x) \leqq f_0(x) + \epsilon f(x) \leqq \sup(f_0 + \epsilon f)(S_1) \leqq \sup(f_0 + \epsilon f)(C_1) < \infty,$$

so $\sup f(S_1) < \infty$. Since S_1 is bounded, there is a dense G_δ subset of E^* consisting of functionals which strongly expose points of S_1. Those which are in a sufficiently small neighborhood of f_0 actually strongly expose a point of C_1. To see this, first choose a point x_0 of C_1 such that

$$f_0(x_0) > \sup f_0(C_1) - 1/2 \equiv \alpha + 1/2$$

and use the boundedness of S_1 to choose $M > 0$ such that $\|x_0 - x\| \leqq M$ for all $x \in S_1$. Choose f such that $\|f - f_0\| < 1/2M$ and f strongly exposes S_1 at x, say. It will strongly expose C_1 at x provided $f_0(x) > \alpha$, so suppose that $f_0(x) = \alpha$. Now, $f_0(x_0 - x) > 1/2$, so $x \neq x_0$ and therefore $f(x) > f(x_0)$. Thus,

$$\begin{aligned}\|f_0 - f\| &\geqq (f_0 - f)[(x_0 - x)/\|x_0 - x\|] \\ &> [1/2 + f(x) - f(x_0)]/\|x_0 - x\| > 1/2M,\end{aligned}$$

a contradiction.

By carrying out this same construction for C_2 we obtain a neighborhood of f_0 which contains a dense G_δ set of functionals which simultaneously support both C_1 and C_2.

What if the Banach space E does not have the Radon-Nikodym property? Is it possible to find two *bounded* closed convex nonempty sets with no common support functionals? As noted earlier, Edelstein and Thompson [10] have shown that this is possible in c_0. Both of their sets have nonempty interior, one of them being the unit ball and the other an isomorph of the unit ball. Their proof uses the same fact which we have exploited: Those functionals which attain their supremum on the unit ball of c_0 are finitely nonzero elements of l_1. Moreover, the inverse of the adjoint of the associated isomorphism carries such elements into those having infinitely many nonzero terms. Further investigations stemming from the Edelstein-Thompson paper have been carried out by J. Borwein [2], Cobzas [6, 7, 8], Edelstein [11] and Fonf [12].

We now return to Professor Vohra's original question: Suppose that C_1, C_2 are closed convex sets and that $C = C_1 + C_2$ is closed. Given z in the boundary of C with $z = z^1 + z^2$, $z^i \in C_i$, and $\delta > 0$, does there exist a support point of C of the form $x^1 + x^2$, $x^i \in C_i$, such that $\|x^i - z^i\| < \delta$? (Actually, he also assumes that the C_i are weak* closed convex subsets of a dual space which contains a weak* closed convex cone K for which $K + C_i \subset C_i \subset K$; we will address this below.) The example which follows gives a strong negative answer to this question.

EXAMPLE 2. There exists a weakly compact convex subset B_1 of l_2 which shows, taking $C_1 = C_2 = B_1$, that the answer to the above question is negative.

PROOF. Let $B_1 = \{x \in l_2 \colon \sum |x_i| \leqq 1\}$; this is a closed, convex and bounded subset of l_2, hence is weakly compact. Suppose that $0 \neq y \in l_2$ attains its supremum on B_1 at $x \in B_1$. This imposes the following restrictions on x and y: Assume, without loss of generality, that $\|y\|_\infty = 1$. If we define u by $u_i = x_i \operatorname{sgn}(x_i y_i)$, then $u \in B_1$ and

$$(*) \qquad \sum |x_i y_i| = \langle u, y\rangle \leqq \langle x, y\rangle = \sum x_i y_i \leqq \|y\|_\infty \cdot \sum |x_i| \leqq 1.$$

The first inequality in (*) shows that $x_i y_i \geqq 0$ for all i. Since $y_i \to 0$, we must have $|y_i| = 1$ for some i, so if $\mathbf{e}^i$ denotes the i-th basis vector for l_2, then $u = (\operatorname{sgn} y_i)\mathbf{e}^i \in B_1$ and $\langle u, y\rangle = 1$; combining this with the second inequality in (*) shows that $\langle x, y\rangle = 1$. If $x_k \neq 0$ for some k, and if $|y_k| < 1$, then $x_k y_k < |x_k|$ and therefore $\langle x, y\rangle = \sum x_i y_i < \sum |x_i| \leqq 1$; that is, $x_k \neq 0$ implies that $|y_k| = 1$.

Write $0 = \mathbf{e}^1 - \mathbf{e}^1 \in B_1 + B_1$, let $0 < \delta < 1$, and suppose that there exists a support point $u + v \in B_1 + B_1$ such that $\|u - \mathbf{e}^1\| < \delta$ and $\|v - (-\mathbf{e}^1)\| < \delta$. Let $y \neq 0$ support $B_1 + B_1$ at $u + v$; then it supports B_1 at both u and v. Assume that $\|y\|_\infty = 1$. Since $|u_1 - 1| < 1$ and $|v_1 + 1| < 1$, we must have $u_1 > 0$ and $v_1 < 0$. Since $u_1 \neq 0$ (for instance), what we have shown above implies that $|y_1| = 1$. Moreover, we must also have $u_1 y_1 > 0$ and $v_1 y_1 > 0$, an impossibility.

This example can be placed in the context of Professor Vohra's original question by letting $K \subset l_2 \times \mathbf{R}$ be the cone generated by $B_1 \times \{1\}$ and letting $C_1 = C_2 = K$. The details go as follows:

We will pair $l_2 \times \mathbf{R}$ with itself as before, and put an l_2 norm on the product. Suppose that $(y, s) \neq (0, 0)$ attains its supremum on K at $(x, r) \neq (0, 0)$. The latter implies that $r > 0$ and it is straightforward to verify that y attains its supremum on B_1 at $r^{-1}x \in B_1$ and that $y \neq 0$. Suppose, now, that $0 < \delta < 1/2$ and write $(0, 1) = (\mathbf{e}^1, 1) + (-\mathbf{e}^1, 1) \in K + K = K$. Suppose, further, that there were a support point $(u, r) + (v, t)$ of $K + K$ with

$$\|(u, r) - (\mathbf{e}^1, 1)\| < \delta \text{ and } \|(v, t) - (-\mathbf{e}^1, 1)\| < \delta.$$

We have $|r - 1| < \delta$, so $r > 0$ and $r^{-1}u \in B_1$. Since $\|r^{-1}u\|_1 \leqq 1$ it follows that $\|r^{-1}u\| \leqq 1$ and hence $\|u\| \leqq r$. Consequently,

$$\begin{aligned} \|r^{-1}u - \mathbf{e}^1\| &\leqq \|r^{-1}u - u\| + \|u - \mathbf{e}^1\| \\ &\leqq |r^{-1} - 1| \cdot \|u\| + \delta \leqq |1 - r| + \delta < 2\delta < 1. \end{aligned}$$

Similar assertions are true for $t^{-1}v$ and $-\mathbf{e}^1$. Let (y, s) be any functional which supports K at the sum $(u, r) + (v, t)$; then it necessarily supports K at both points, and $y \neq 0$. Without loss of generality, we can assume that $\|y\|_\infty = 1$. By the observations made above, y supports B_1 at $r^{-1}u$ and $t^{-1}v$. The argument given for Example 2 shows that this is impossible.

Suppose that E is a Banach space and that F is a partially ordered Banach space. A function φ with values in F and domain a convex subset dom φ of E is defined to be "convex" in exact analogy with the real-valued case, by simply interpreting the convexity inequality in terms of the partial ordering on F. One can adjoin the element ∞ to F, require that it satisfy certain obvious properties, and set φ equal to ∞ outside of dom φ. Thus, the latter set (now called the *effective domain* of φ) is the set of x where $\varphi(x) \neq \infty$. The *subdifferential set* $\partial\varphi(x)$ at a point x in dom φ is defined to be the set of all continuous linear maps $T:E \to F$ satisfying

$$Ty - Tx \leqq \varphi(y) - \varphi(x), \; y \in E.$$

The existence of such operators has been investigated by a number of authors [1, 3, 13, 15, 16, 17, 19]. These results all involve continuity assumptions on φ (as well as additional hypotheses on the order structure of F). In the case of an extended *real-valued* convex function, Brøndsted and Rockafellar [5] extended the Bishop-Phelps theorem to prove that if φ is lower semicontinuous, then there is a dense set of points x in dom φ for which $\partial\varphi(x)$ is nonempty. Since there is a natural definition of semicontinuity in the vector-valued case, it is reasonable to hope that this result can be generalized. In the example which follows we show that this is impossible, even in the simplest case when F is two-dimensional. Note that in this case we can represent our convex function φ by a pair of extended real-valued functions having the same effective domain, and $\partial\varphi(x)$ will be nonempty if and only if each of these functions has a nonempty subdifferential at the same point x.

EXAMPLE 3. There exist two proper lower semicontinuous convex functions f_1 and f_2 on l_2 with common effective domain l_1 such that no point of l_1 is a point of subdifferentiability for both functions.

PROOF. Let $a = (2^{-n}) \in l_1 \subset l_2$ as in Example 1, define $f_1(x) = \sum |x_n|$ and let $f_2(x) = f_1(x - a)$, $x \in l_2$. As the supremum of a sequence $\sum_{n=1}^{N} |x_n|$ of continuous convex functions f_1 (and hence f_2) is lower semicontinuous and clearly $\operatorname{dom} f_1 = \operatorname{dom} f_2 = l_1$. A subdifferential of f_1 at a point x is any element $y \in l_2$ satisfying $\langle x, y \rangle = \sum |x_n|$ and $|y_n| \leqq 1$ for all n. As in the proof of Lemma 1, this implies that $|y_n| = 1$ whenever $x_n \neq 0$ and hence $x_n = 0$ for all but finitely many n. Any point with the latter property is clearly in $\operatorname{dom} \partial f_1 = \{x \in l_1 \colon \partial f_1(x) \text{ is nonempty}\}$, so this set consists precisely of the finitely nonzero sequences. It is easily seen that $\operatorname{dom} \partial f_2 = \operatorname{dom} \partial f_1 + a$, so these two sets have no points in common.

The two sets in Example 1 were obtained by first taking the epigraphs of the conjugate convex functions f_1^* and f_2^* to f_1 and f_2 in Example 3, then modifying the epigraph of f_1^* so as to avoid "vertical" support functionals. The usual duality theory shows that f_1^* and f_2^* have no subdifferentials in common.

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