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GEOMETRY OF SPACES OF VECTOR-VALUED HARMONIC FUNCTIONS

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ABSTRACT. It is shown that the space $h^p(D,X)$ has the Kadec-Klee property with respect to pointwise norm convergence in the Banach space X if and only if X has the Radon-Nikodym property and every point of the unit sphere of X is a denting point of the unit ball of X. In addition, it is shown that $h^p(D,X)$ is locally uniformly rotund if and only if X is locally uniformly rotund and has the Radon-Nikodym property.

1. **Introduction.** The space of real-valued harmonic functions on the open unit disc in the complex plane has long been of interest and, more recently, spaces of vector-valued harmonic functions have been the object of considerable study. In 1982, Bukhvalov and Danilevich [B-D] showed that a Banach space X has the Radon-Nikodym property if and only if every bounded X-valued harmonic function from the open unit disc, D, in the complex plane has almost everywhere radial values in X. They also showed that X has the Radon-Nikodym property if and only if every function in $h^p(D,X)$ (see Section 2 for the definition) has almost everywhere radial values in X for all p with $1 < p < \infty$. Subsequently, in 1992, Dowling and Lennard [D-L] considered Kadec-Klee and uniform Kadec-Klee-Huff properties of the space $h^p(D,X)$ for $1 < p < \infty$.

This paper is a continuation of the study of Kadec-Klee properties of the space $h^p(D,X)$ for $1 < p < \infty$. The topology on $h^p(D,X)$ with which Dowling and Lennard [D-L] worked exclusively is the so-called β topology, that is, the topology of norm uniform convergence on compact subsets of D. In Section 2 of this paper, some natural generalizations of the β topology, namely the weak-β topology on $h^p(D,X)$ and the weak*-β topology on $h^p(D,X^*)$, are introduced. Then, in Section 3, characterizations of the Kadec-Klee property with respect to these modes of convergence in $h^p(D,X)$ are given. In particular, it is shown that if X is a Banach space and if $1 < p < \infty$, then the following assertions are equivalent:

(1) If $\{f_n\}_{n=1}^{\infty}$ and f are in the unit sphere of $h^p(D,X)$ and satisfy $\{f_n(z)\}_{n=1}^{\infty}$ converges in norm in X to $f(z)$ for all z in the unit disc D, then $\{f_n\}_{n=1}^{\infty}$ converges in norm in $h^p(D,X)$ to f.

(1′) If $\{f_n\}_{n=1}^{\infty}$ and f are in the unit sphere of $h^p(D,X)$ and satisfy $\{f_n(z)\}_{n=1}^{\infty}$ converges weakly in X to $f(z)$ for all z in the unit disc D, then $\{f_n\}_{n=1}^{\infty}$ converges in norm in $h^p(D,X)$ to f.

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(2) The space X has the Radon-Nikodym property and every point of the unit sphere of X is a denting point of the unit ball of X.

The equivalence of (1) and (2) above solves a problem implicitly stated by Dowling and Lennard [D-L]. Section 3 concludes with the result that $h^p(D,X)$ is locally uniformly rotund if and only if the Banach space X is locally uniformly rotund and has the Radon-Nikodym property.

2. **Definitions and preliminaries.** Throughout this paper X will denote a real Banach space and D will denote the open unit disc in the complex plane. A function $f\colon D \longrightarrow X$ is said to be *harmonic* provided it is twice continuously differentiable and its Laplacian is zero. Hensgen [He] showed that $f\colon D \longrightarrow X$ is harmonic if and only if $x^*f\colon D \longrightarrow \mathbb{R}$ is harmonic for each x^* in X^*, the dual space of X. For $1 < p < \infty$ and a Banach space X, let

$$h^p(D,X) = \{f\colon D \longrightarrow X \mid f \text{ is harmonic and } \|f\|_p < \infty\},$$

where

$$\|f\|_p = \sup_{0 \le r < 1} \left[\int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi}\right]^{1/p}.$$

Let $\mathbb{T}$ denote the boundary of D; let $\mathcal{B}$ denote the Borel σ-algebra of subsets of $\mathbb{T}$; and let λ denote the normalized Haar measure on $\mathbb{T}$. For $1 < p < \infty$, the symbol $L^p(\mathbb{T},X)$ will denote the usual Lebesgue-Bochner function space $L^p(\mathbb{T},\mathcal{B},\lambda,X)$; see [D-U].

The following is a list of some properties concerning the spaces $L^p(\mathbb{T},X)$ and $h^p(D,X)$ that will be used in the sequel.

(2.1) The mapping $I\colon L^p(\mathbb{T},X) \longrightarrow h^p(D,X)$, given by

$$I(f)(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t) f(e^{it}) \frac{dt}{2\pi}$$

for each $0 \le r < 1$ and $0 \le \theta \le 2\pi$, defines an isometric embedding, where $\{P_r\}_{0 \le r < 1}$ is the Poisson kernel given, for $0 \le r < 1$ and $0 \le u \le 2\pi$, by

$$P_r(u) = \frac{1 - r^2}{1 - 2r\cos u + r^2}.$$

(2.2) A Banach space X has the Radon-Nikodym property if and only if the mapping I, given in (2.1), is surjective; see [B-D].

(2.3) If $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $L^q(\mathbb{T},X)^*$ is isometrically isomorphic to $h^p(D,X^*)$; see [Do]. In the sequel, a reference to the weak* topology on $h^p(D,X^*)$ will mean the isomorphic image in $h^p(D,X^*)$ of the weak* topology on $L^q(\mathbb{T},X)^*$.

(2.4) Let $j\colon X \longrightarrow X^{**}$ be the natural inclusion mapping. Then, for $1 < p < \infty$, the mapping $J\colon h^p(D,X) \longrightarrow h^p(D,X^{**})$, defined by $J(f)(z) = j\big(f(z)\big)$ for z in D, is an isometric embedding.

Dowling and Lennard [D-L] introduced the notion of β convergence; a sequence $\{f_n\}_{n=1}^{\infty}$ in $h^p(D,X)$ is said to *converge in the β topology* to f in $h^p(D,X)$ provided

$$\lim_n (\sup\{\|f_n(z) - f(z)\| : z \in K\}) = 0$$

for all compact subsets K of D. As in the scalar-valued case (that is, the case $X = \mathbb{R}$), it can be shown that a bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $h^p(D,X)$ converges in the β topology to f in $h^p(D,X)$ if and only if $\{f_n(z)\}_{n=1}^{\infty}$ converges in norm in X to $f(z)$ for all z in D. Consequently, it is natural to define weak-β convergence as follows: A sequence $\{f_n\}_{n=1}^{\infty}$ in $h^p(D,X)$ is said to *converge in the* weak-β *topology* to f in $h^p(D,X)$ provided $\{f_n(z)\}_{n=1}^{\infty}$ converges weakly in X to $f(z)$ for all z in D. The notion of weak*-β convergence of a sequence in $h^p(D,X^*)$ is defined in an analogous manner.

The symbol B_X will denote the closed unit ball of X and S_X will denote the unit sphere of X. Let τ be a topological vector space topology on X which is weaker than the norm topology on X. Then X is said to have the *Kadec-Klee property with respect to τ convergence* provided that whenever $\{x_n\}_{n=1}^{\infty}$ and x are in S_X and satisfy $\{x_n\}_{n=1}^{\infty}$ converges in the τ topology to x, it follows that $\{x_n\}_{n=1}^{\infty}$ converges in norm to x.

A point x in B_X is called an *extreme point* of B_X provided x is not the midpoint of any non-trivial line segment lying in B_X. A Banach space X is said to be *strictly convex* provided every x in S_X is an extreme point of B_X. The following is a list of various well-known types of extreme points; the list is given in order of increasing strength.

(i) A point x in B_X is called a *strongly extreme point* of B_X provided that whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in X with $\lim_n \|x+x_n\| = 1$ and $\lim_n \|x-x_n\| = 1$, it follows that $\lim_n \|x_n\| = 0$. A Banach space X is said to be *midpoint locally uniformly rotund* provided every x in S_X is a strongly extreme point of B_X.

(ii) A point x in B_X is called a *denting point* of B_X provided x is not an element of the closed convex hull of $\{y \in B_X : \|y - x\| > \varepsilon\}$ for each $\varepsilon > 0$. A Banach space X is said to have *property* (G) provided every x in S_X is a denting point of B_X. A dual space X^* is said to have *property* (G^*) provided every x^* in S_{X^*} is a weak* denting point of B_{X^*}, where a weak* denting point is defined as above for a denting point but with the closed convex hull replaced by the weak* closed convex hull.

(iii) A point x in B_X is called a *locally uniformly rotund point* of B_X provided that whenever $\{x_n\}_{n=1}^{\infty}$ is a sequence in B_X with $\lim_n \|x + x_n\| = 2$, it follows that $\lim_n \|x - x_n\| = 0$. A Banach space X is said to be *locally uniformly rotund* provided every x in S_X is a locally rotund point of B_X.

3. **Kadec-Klee properties of $h^p(D,X)$.** The main results in this section are characterizations of various Kadec-Klee properties of the space $h^p(D,X)$ for $1 < p < \infty$. The first proposition is due to Dowling and Lennard [D-L]; it is stated here for both the sake of completeness and subsequent reference.

PROPOSITION 3.1. *Let $1 < p < \infty$ and let X be a Banach space. If $h^p(D,X)$ has the Kadec-Klee property with respect to β convergence, then X has the Radon-Nikodym property and is strictly convex.*

THEOREM 3.2. *Let $1 < p < \infty$ and let X be a Banach space. Then $h^p(D,X^*)$ has the Kadec-Klee property with respect to weak*-β convergence if and only if X^* has property (G^*).*

PROOF. Suppose $h^p(D,X^*)$ has the Kadec-Klee property with respect to weak*-β convergence. It follows immediately from the definition that $h^p(D,X^*)$ has the Kadec-Klee property with respect to β-convergence, and so, by Proposition 3.1, the space X^* is strictly convex. By considering the constant X^*-valued harmonic functions, it is easily seen that X^* has the Kadec-Klee property with respect to weak* convergence. Hu and Lin [H-L1] proved that X^* is strictly convex and has the Kadec-Klee property with respect to weak* convergence if and only if X^* has property (G^*). Consequently, X^* has property (G^*) whenever $h^p(D,X^*)$ has the Kadec-Klee property with respect to weak*-β convergence.

Conversely, suppose X^* has property (G^*). Note that the weak* topology (see (2.3)) and the weak*-β topology on $B_{h^p(D,X^*)}$ are equivalent since they are comparable compact Hausdorff topologies. Thus $h^p(D,X^*)$ has the Kadec-Klee property with respect to weak*-β convergence if and only if $h^p(D,X^*)$ has the Kadec-Klee property with respect to weak* convergence. Since X^* has property (G^*), it follows that X^* has the Radon-Nikodym property (see [H-L1]), and hence, by (2.3), the space $h^p(D,X^*)$ is isometrically isomorphic to $L^p(\mathbb{T},X^*)$. So $h^p(D,X^*)$ has the Kadec-Klee property with respect to weak* convergence whenever $L^p(\mathbb{T},X^*)$ has the Kadec-Klee property with respect to weak* convergence. However, Hu and Lin [H-L3] proved that $L^p(\mathbb{T},X^*)$ has property (G^*), and hence the Kadec-Klee property with respect to weak* convergence, whenever X^* has property (G^*). This completes the proof.

LEMMA 3.3. *If x is in B_X and x is not a denting point of B_X, then there exist a sequence $\{f_n\}_{n=1}^\infty$ in $B_{L^p(\mathbb{T},X)}$ and $\varepsilon > 0$ such that $\|f_n - f\|_p > \varepsilon$ for each n in $\mathbb{N}$, where $f = x\chi_{\mathbb{T}}$, and $\{f_n\}_{n=1}^\infty$ converges in the σ-topology to f, where $\sigma = \sigma\big(L^p(\mathbb{T},X), L^q(\mathbb{T},X^*)\big)$ and $1 < p$, $q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

PROOF. Since x is not a denting point of B_X, there exists $\varepsilon > 0$ such that x is in $\overline{\mathrm{co}}\big(B_X \setminus B(x,\varepsilon)\big)$, where $B(x,\varepsilon) = \{y \in X : \|y - x\| \le \varepsilon\}$. Therefore, for each n in $\mathbb{N}$, there exist m_n in $\mathbb{N}$ with $m_n \ge 2$ and a finite set $\{x_n^k\}_{k=1}^{m_n}$ in $B_X \setminus B(x,\varepsilon)$ such that

$$\Big\| \frac{1}{m_n} \sum_{k=1}^{m_n} x_n^k - x \Big\| < \frac{1}{n}.$$

For n in $\mathbb{N}$, let $M_n = m_1 m_2 \cdots m_n$ and define, for $1 \le j \le M_n$,

$$E_n^j = \Big\{ e^{it} : \frac{2\pi(j-1)}{M_n} \le t < \frac{2\pi j}{M_n} \Big\}.$$

Now, let $\Pi_n = \{E_n^j : 1 \le j \le M_n\}$ for each n in $\mathbb{N}$. Clearly, each Π_n is a partition of $\mathbb{T}$ and the σ-algebra generated by $\bigcup_{n\ge 1} \Pi_n$ is $\mathcal{B}$. For each n in $\mathbb{N}$, define

$$f_n = \sum_{j=1}^{M_n} x_n^{r_n(j)} \chi_{E_n^j},$$

where $1 \le r_n(j) \le m_n$ and $r_n(j) \equiv j \pmod{m_n}$. Note that $\{f_n\}_{n=1}^\infty$ is a sequence in the closed unit ball of $L^p(\mathbb{T},X)$. For each t in $\mathbb{T}$, it is easily seen that $\|f_n(t) - f(t)\| > \varepsilon$ and

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hence $\|f_n - f\|_p > \varepsilon$ for each n in $\mathbb{N}$. Now for each k in $\mathbb{N}$ and x^* in X^*, it is the case that, for each E in $\prod_k$ and $n > k$,

$$\Big|\int_{\mathbb{T}} \langle x^*\chi_E, f_n - f\rangle \, d\lambda\Big| = \Big|\int_E x^*\big(f_n(t) - f(t)\big)\, d\lambda(t)\Big| \le \|x^*\| \frac{1}{n}\lambda(E).$$

Since $L^q(\mathbb{T}, X^*)$ is the closed linear span of $\{x^*\chi_E : E \in \bigcup_{k\ge 1}\prod_k \text{ and } x^* \in X^*\}$, the calculation above shows that $\{f_n\}_{n=1}^{\infty}$ converges in the σ topology to f. This completes the proof.

REMARK. Using arguments similar to those in the proof of Lemma 3.3, it can be shown that if (Ω, Σ, μ) is an atom-free measure space and f is a $\sigma\big(L^p(\Omega, X), L^q(\Omega, X^*)\big)$ point of sequential continuity of $B_{L^p(\Omega,X)}$, then f is a denting point of $B_{L^p(\Omega,X)}$; this strengthens a result of Hu and Lin [H-L2] which concludes that such an f is a strongly extreme point of $B_{L^p(\Omega,X)}$.

THEOREM 3.4. *Let $1 < p < \infty$ and let X be a Banach space. Then $h^p(D, X)$ has the Kadec-Klee property with respect to weak-β convergence if and only if X has the Radon-Nikodym property and property (G).*

PROOF. Suppose $h^p(D, X)$ has the Kadec-Klee property with respect to weak-β convergence. It follows immediately from the definition that $h^p(D, X)$ has the Kadec-Klee property with respect to β convergence, and so, by Proposition 3.1, the space X has the Radon-Nikodym property. To obtain a contradiction, suppose X fails to have property (G). Then there exists a point x in S_X which is not a denting point of B_X. By Lemma 3.3, there exist a sequence $\{f_n\}_{n=1}^{\infty}$ in $B_{L^p(\mathbb{T},X)}$ and $\varepsilon > 0$ such that $\|f_n - f\|_p > \varepsilon$ for each n in $\mathbb{N}$, where $f = x\chi_{\mathbb{T}}$, and $\{f_n\}_{n=1}^{\infty}$ converges in the σ topology to f. Let $I: L^p(\mathbb{T}, X) \longrightarrow h^p(D, X)$ be the isometric embedding defined in (2.1). For $z = re^{i\theta}$ in D and x^* in X^*,

$$x^*\big(I(f_n)(z)\big) = x^*\Big(\int_0^{2\pi} P_r(\theta - t) f_n(e^{it})\frac{dt}{2\pi}\Big) = \int_0^{2\pi} \langle P_r(\theta - t)x^*, f_n(e^{it})\rangle \frac{dt}{2\pi}$$

and

$$x^*\big(I(f)(z)\big) = \int_0^{2\pi} \langle P_r(\theta - t)x^*, f(e^{it})\rangle \frac{dt}{2\pi}.$$

Since $\{f_n\}_{n=1}^{\infty}$ converges in the σ topology to f, it follows from the calculation above that $\{I(f_n)\}_{n=1}^{\infty}$ converges in the weak-β topology to $I(f)$. Now, since $\|\cdot\|_p$ is a weak-β lower semicontinuous function on $h^p(D, X)$, since $\{f_n\}_{n=1}^{\infty}$ is in $B_{L^p(\mathbb{T},X)}$, and since $\|I(f)\|_p = \|f\|_p = 1$, it follows that $\lim_n \|I(f_n)\|_p = \|I(f)\|_p = 1$. So the hypothesis that $h^p(D, X)$ has the Kadec-Klee property with respect to weak-β convergence yields that $\lim_n \|I(f_n) - I(f)\|_p = 0$ and hence $\lim_n \|f_n - f\|_p = 0$. This contradicts the fact that $\|f_n - f\|_p > \varepsilon$, for each n in $\mathbb{N}$, and hence this portion of the proof is complete.

Conversely, suppose X has the Radon-Nikodym property and property (G). Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of norm-one functions in $h^p(D, X)$ which converges in the weak-

β topology to the norm-one function f in $h^p(D,X)$. Then, for each z in D, it follows that $\{f_n(z)\}_{n=1}^{\infty}$ converges weakly in X to $f(z)$, and hence $\{j(f_n(z))\}_{n=1}^{\infty}$ converges in the weak* topology in X^{**} to $j(f(z))$, where $j: X \longrightarrow X^{**}$ is the natural inclusion mapping. Thus $\{J(f_n)(z)\}_{n=1}^{\infty}$ converges in the weak* topology in X^{**} to $J(f)(z)$ for each z in D, where J is the mapping defined in (2.4); that is, $\{J(f_n)\}_{n=1}^{\infty}$ converges in the weak*-β topology in $h^p(D,X^{**})$ to $J(f)$. As was noted in the proof of Theorem 3.2, this is equivalent to saying $\{J(f_n)\}_{n=1}^{\infty}$ converges in the weak* topology (see (2.3)) in $h^p(D,X^{**})$ to $J(f)$. Since J is an isometry, it follows that $J(f_n)$, for each n in $\mathbb{N}$, and $J(f)$ are norm-one functions in $h^p(D,X^{**})$. But $h^p(D,X^{**})$ is isometrically isomorphic to $L^q(\mathbb{T},X^*)^*$, where $\frac{1}{p}+\frac{1}{q}=1$, and thus $\{J(f_n)\}_{n=1}^{\infty}$ may be considered as a sequence of norm-one functions in $L^q(\mathbb{T},X^*)^*$ which converges in the weak* topology in $L^q(\mathbb{T},X^*)^*$ to the norm-one function $J(f)$. Now, if $J(f)$ were a weak* denting point of $B_{L^q(\mathbb{T},X^*)^*}$, then it would follow that $\lim_n \|J(f_n)-J(f)\|_p = 0$ and so $\lim_n \|f_n - f\|_p = 0$, which would complete the proof. To see that this is the case, note that, since X has the Radon-Nikodym property, the space $h^p(D,X)$ is isometrically isomorphic to $L^p(\mathbb{T},X)$ by (2.2) and, since X has property (G), the space $L^p(\mathbb{T},X)$ also has property (G) by a result of Lin and Lin [L-L]. Hence, by considering f as an element of $L^p(\mathbb{T},X)$, it is the case that f is a denting point of $B_{L^p(\mathbb{T},X)}$. Notice that the natural embedding of $L^p(\mathbb{T},X)$ into $L^q(\mathbb{T},X^*)^*$ sends f to $J(f)$. Finally, Hu and Lin [H-L3] proved that $J(f)$ is a weak* denting point of $B_{L^q(\mathbb{T},X^*)^*}$ whenever f is a denting point of $B_{L^p(\mathbb{T},X)}$. This completes the proof.

The next goal is to characterize the Banach spaces X such that $h^p(D,X)$ has the Kadec-Klee property with respect to β convergence. To accomplish this, the following notion will be used: For $1 < p < \infty$ and X a Banach space, a point x in S_X is said to be a *strong h^p-point* provided that whenever $\{f_n\}_{n=1}^{\infty}$ is a sequence in $h^p(D,X)$ with $\{f_n(0)\}_{n=1}^{\infty}$ converging in norm in X to x and $\lim_n \|f_n\|_p = 1$, it follows that $\lim_n \|f_n - x\|_p = 0$, where the symbol x in $\|f_n - x\|_p$ is to be interpreted as the constant function in $h^p(D,X)$ with value x. It is easy to show that if x in S_X is a strong h^p-point, then x is a strongly extreme point of B_X.

PROPOSITION 3.5. *Let $1 < p < \infty$ and let X be a Banach space. If $h^p(D,X)$ has the Kadec-Klee property with respect to β convergence, then every point in the unit sphere of X is a strong h^p-point.*

PROOF. Let x be a point in S_X. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence in $h^p(D,X)$ with $\{f_n(0)\}_{n=1}^{\infty}$ converging in norm in X to x and $\lim_n \|f_n\|_p = 1$. For each n in $\mathbb{N}$, define $g_n: D \longrightarrow X$ by $g_n(z) = f_n(z^n)$ for each z in D and define $g: D \longrightarrow X$ by $g(z) = x$ for each z in D. Clearly, each g_n and g are elements of $h^p(D,X)$ with $\|g_n\|_p = \|f_n\|_p$ (by a computation similar to that given below), for each n in $\mathbb{N}$, and $\|g\|_p = 1$. Also, since $\{g_n\}_{n=1}^{\infty}$ is a bounded sequence in $h^p(D,X)$, it follows that $\{g_n(z)\}_{n=1}^{\infty}$ converges in norm in X to $g(z)$ for each z in D. Hence, since $h^p(D,X)$ has the Kadec-Klee property with respect to

β convergence, it follows that $\lim_n \|g_n - g\|_p = 0$. However,

$$\begin{aligned}
\|g_n - g\|_p^p &= \sup_{0\le r<1} \int_0^{2\pi} \|g_n(re^{i\theta}) - x\|^p \frac{d\theta}{2\pi} \\
&= \sup_{0\le r<1} \int_0^{2\pi} \|f_n(r^n e^{in\theta}) - x\|^p \frac{d\theta}{2\pi} \\
&= \sup_{0\le r<1} \int_0^{2\pi} \|f_n(r^n e^{i\theta}) - x\|^p \frac{d\theta}{2\pi} \\
&= \sup_{0\le r<1} \int_0^{2\pi} \|f_n(re^{i\theta}) - x\|^p \frac{d\theta}{2\pi} \\
&= \|f_n - x\|_p^p.
\end{aligned}$$

Thus $\lim_n \|f_n - x\|_p = 0$. This shows that x is a strong h^p-point and the proof is complete.

Let K be a closed bounded convex subset of a Banach space X. A point x in K is called a *very strong extreme point* of K provided that for every sequence $\{x_n\}_{n=1}^\infty$ of K-valued, Bochner integrable functions on $[0,1]$ satisfying $\lim_n \| \int_0^1 x_n(t)\,dt - x\| = 0$, it follows that $\lim_n \int_0^1 \|x_n(t) - x\|\,dt = 0$. Lin, Lin and Troyanski [L-L-T] showed that x in K is a very strong extreme point of K if and only if x is a denting point of K.

PROPOSITION 3.6. *Let $1 < p < \infty$ and let X be a Banach space. If x in S_X is a strong h^p-point, then x is a very strong extreme point of B_X (that is, x is a denting point of B_X).*

PROOF. It is clear that in the definition of a very strong extreme point that $[0,1]$ with Lebesgue measure can be replaced by $\mathbb{T}$ with its normalized Haar measure. Let x in S_X be a strong h^p-point. Suppose that $\{x_n\}_{n=1}^\infty$ is a sequence of B_X-valued, Bochner integrable functions on $\mathbb{T}$ with

$$\lim_n \left\| \int_0^{2\pi} x_n(e^{it}) \frac{dt}{2\pi} - x \right\| = 0.$$

For each n in $\mathbb{N}$, define $f_n : D \longrightarrow X$ by

$$f_n(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t)\, x_n(e^{it}) \frac{dt}{2\pi}$$

for each $0 \le r < 1$ and $0 \le \theta \le 2\pi$. Clearly, f_n is harmonic and, since x_n is B_X-valued, it follows that $\|f_n(z)\| \le 1$ for all n in $\mathbb{N}$ and all z in D. Thus each f_n is an element of $h^p(D,X)$ and $\|f_n\|_p \le 1$. Note that, for all n in $\mathbb{N}$,

$$f_n(0) = \int_0^{2\pi} x_n(e^{it}) \frac{dt}{2\pi}.$$

By the supposition concerning $\{x_n\}_{n=1}^\infty$ above, it follows that $\{f_n(0)\}_{n=1}^\infty$ converges in norm in X to x and so $\lim_n \|f_n(0)\| = 1$. Since f is harmonic, it follows that $\|f_n(0)\| \le \|f_n\|_p \le 1$ and hence $\lim_n \|f_n\|_p = 1$. Now, since x is a strong h^p-point, it follows that $\lim_n \|f_n - x\|_p = 0$. But $\|f_n - x\|_p = \|x_n - x\|_p$ and so

$$\int_0^{2\pi} \|x_n(e^{it}) - x\| \frac{dt}{2\pi} = \|x_n - x\|_1 \le \|x_n - x\|_p = \|f_n - x\|_p,$$

from which it follows that

$$\lim_n \int_0^{2\pi} \|x_n(e^{it}) - x\| \frac{dt}{2\pi} = 0.$$

This shows that x is a very strong extreme point of B_X and so the proof is complete.

THEOREM 3.7. *Let $1 < p < \infty$ and let X be a Banach space. Then the following assertions are equivalent:*

(1) The space $h^p(D,X)$ has the Kadec-Klee property with respect to weak-β convergence;

(2) The space $h^p(D,X)$ has the Kadec-Klee property with respect to β convergence;

(3) The space X has the Radon-Nikodym property and property (G).

PROOF. Assertion (1) implies (2) by definition; and (1) and (3) are equivalent by Theorem 3.4. It remains to show (2) implies (3). So, suppose that $h^p(D,X)$ has the Kadec-Klee property with respect to β convergence. Then X has the Radon-Nikodym property by Proposition 3.1. Now, every point in S_X is a strong h^p-point by Proposition 3.5 and hence every point in S_X is a denting point of B_X by Proposition 3.6. Thus X has property (G) and the proof is complete.

REMARK. Let (Ω, Σ, μ) be a measure space and let p be such that $1 \leq p < \infty$. Let τ be a topological vector space topology on a Banach space X that is weaker than the norm topology and is such that the norm on X is a τ lower semicontinuous function. It is interesting to compare the results of this section with the following result noted by Besbes, Dilworth, Dowling and Lennard [B-D-D-L]: The space X has the Kadec-Klee property with respect to τ convergence if and only if whenever $\{f_n\}_{n=1}^{\infty}$ and f are in $S_{L^p(\Omega,X)}$ and satisfy $\{f_n(\omega)\}_{n=1}^{\infty}$ converges in the τ topology to $f(\omega)$ for μ-almost all ω in Ω, it follows that $\lim_n \|f_n - f\|_{L^p(\Omega,X)} = 0$. Thus, in the $S_{L^p(\Omega,X)}$ setting, norm convergence of $\{f_n(\omega)\}_{n=1}^{\infty}$ to $f(\omega)$ for μ-almost all ω in Ω always yields norm convergence of $\{f_n\}_{n=1}^{\infty}$ to f in $L^p(\Omega, X)$, whereas weak convergence of $\{f_n(\omega)\}_{n=1}^{\infty}$ to $f(\omega)$ for μ-almost all ω in Ω yields norm convergence of $\{f_n\}_{n=1}^{\infty}$ to f in $L^p(\Omega, X)$ precisely whenever X has the Kadec-Klee property with respect to weak convergence. That is, mimicking the language used in this paper, it can be said that $L^p(\Omega, X)$ always has the Kadec-Klee property with respect to almost everywhere convergence, whereas $L^p(\Omega, X)$ has the Kadec-Klee property with respect to weak almost everywhere convergence if and only if X has the Kadec-Klee property with respect to weak convergence.

The final result of this paper gives a very pleasing characterization of local uniform in rotundity in $h^p(D,X)$.

THEOREM 3.8. *Let $1 < p < \infty$ and let X be a Banach space. Then $h^p(D,X)$ is locally uniformly rotund if and only if X is locally uniformly rotund and has the Radon-Nikodym property.*

PROOF. Suppose $h^p(D,X)$ is locally uniformly rotund. Then it follows at once that X is locally uniformly rotund. To show that X has the Radon-Nikodym property, it suffices to show that the mapping $I\colon L^p(\mathbb{T}, X) \longrightarrow h^p(D,X)$, defined in (2.1), is surjective.

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Toward this end, let f be in $h^p(D,X)$ with $\|f\|_p = 1$. Then $J(f)$ is a norm-one function in $h^p(D,X^{**})$. It is known (see [H-L3]) that $B_{L^p(\mathbb{T},X)}$ is weak* dense in $B_{L^q(\mathbb{T},X^*)^*}$, where $\frac{1}{p}+\frac{1}{q} = 1$, and so, since $B_{L^q(\mathbb{T},X^*)^*}$ can be identified with $B_{h^p(D,X^{**})}$ by (2.3), there exists a net $\{f_\lambda\}_{\lambda\in\Lambda}$ in $B_{L^p(\mathbb{T},X)}$ such that $\{J(I(f_\lambda))\}_{\lambda\in\Lambda}$ converges in the weak* topology in $B_{L^q(\mathbb{T},X^*)^*}$ to $J(f)$. Thus $\{\|J(I(f_\lambda)) + J(f)\|_p\}_{\lambda\in\Lambda}$ converges to 2 and so $\{\|I(f_\lambda)+f\|_p\}_{\lambda\in\Lambda}$ converges to 2. Now, since $h^p(D,X)$ is locally uniformly rotund, it follows that $\{I(f_\lambda)\}_{\lambda\in\Lambda}$ converges in norm to f. Hence f is an element of $I(L^p(\mathbb{T},X))$, since $I(L^p(\mathbb{T},X))$ is closed in $h^p(D,X)$. This shows that I is surjective.

Conversely, suppose X is locally uniformly rotund and has the Radon-Nikodym property. Since X is locally uniformly rotund, the space $L^p(\mathbb{T},X)$ is locally uniformly rotund by a result of Smith and Turett [S-T] and, since X has the Radon-Nikodym property, it follows from (2.2) that $h^p(D,X)$ is isometrically isomorphic to $L^p(\mathbb{T},X)$. Hence $h^p(D,X)$ is locally uniformly rotund and the proof is complete.

REMARK. For $1 < p < \infty$, the space $h^p(D,\mathbb{R})$ is uniformly convex and therefore is locally uniformly rotund. Rainwater [R] showed that if c_0 is equipped with Day's norm $|||\cdot|||$, then $(c_0, |||\cdot|||)$ is locally uniformly rotund. However, $h^p(D,(c_0,|||\cdot|||))$ is not locally uniformly rotund, by Theorem 3.8, since c_0 fails to have the Radon-Nikodym property. In fact, more can be said. No equivalent norm on $h^p(D,c_0)$ is locally uniformly rotund since $h^p(D,c_0)$ contains an isomorphic copy of ℓ^∞ and ℓ^∞ has no equivalent locally uniformly rotund norm by a result of Lindenstrauss [L]. It is unknown, at the moment, whether X has the Radon-Nikodym property whenever $h^p(D,X)$ has an equivalent locally uniformly rotund norm. An affirmative answer to this question would be an improvement of the main result of Daher [Da] which is that X has the Radon-Nikodym property whenever $h^p(D,X)$ is separable.

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