21

Euclidean approach for bosons

One can distinguish two basic approaches to quantum field theory. In the more traditional approach, one views the underlying physical Hilbert space equipped with the self-adjoint generator of the dynamics – Hamiltonian or Liouvillean – as the basic object. There also exists a different philosophy, whose starting point is paths (trajectories). The physical space and the physical Hamiltonian or Liouvillean are treated as derived objects (if they can be defined at all).

The second approach is often viewed as more modern and useful by physicists active in quantum field theory. Also from the mathematical point of view, the method of paths has turned out to be in many cases more efficient than the operator-theoretic approach. This chapter is devoted to a brief description of a certain version of this method, called often the *Euclidean approach*.

Let us first explain the origin of the word *Euclidean* in the name of this approach. Originally the Euclidean approach amounted to replacing the real time variable t by the imaginary $\mathrm{i}s$, an operation called the *Wick rotation*. Under this transformation, the Minkowski space $\mathbb{R}^{1,d}$ becomes the Euclidean space $\mathbb{R}^{1+d}$. After the Wick rotation, the unitary group generated by the Hamiltonian $\mathrm{e}^{\mathrm{i}tH}$ becomes the self-adjoint group of contractions e^{-sH}. One can then study e^{-sH} from the point of view of the so-called *path space*. In particular, it is sometimes easier to construct or study interacting models of quantum field theory on the Euclidean space than on the Minkowski space.

In the literature the term "Euclidean approach" seems to have acquired a wider meaning, going beyond quantum field theory on a Euclidean space. It sometimes denotes a method for obtaining a unitary group $\mathrm{e}^{\mathrm{i}tH}$ by first constructing the self-adjoint semi-group e^{-sH} for $s \geq 0$. In some cases one can try to represent the integral kernel of e^{-sH} by a measure on the so-called *path space*. This allows us to use methods of measure theory, which are sometimes quite powerful. In particular, one can treat very singular perturbations with little effort, provided they fit into the framework – essentially, they need to be representable as multiplication operators.

This approach also works in ordinary quantum mechanics. For example, it can be used to construct Schrödinger Hamiltonians $H = -\frac{1}{2}\Delta_x + V(x)$ on $L^2(\mathbb{R}^d)$, where V is a real potential. In the absence of the potential, $\mathrm{e}^{\frac{t}{2}\Delta_x}$ is simply the well-known *heat semi-group*. Its distribution kernel $K_0(t, x, y)$ can be interpreted as the probability that a Brownian path starting from y arrives at x at time t. The perturbed heat kernel $K(t, x, y)$ can now be explicitly expressed in terms

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of $K_0(t,x,y)$ and the integral of the potential along Brownian paths by the so-called *Feynman–Kac formula*. We will briefly describe this construction in Sect. 21.1.

In this chapter we describe the Euclidean method for bosons in an abstract framework. We describe the construction of a class of interacting Hamiltonians starting from free ones, using the Feynman–Kac(–Nelson) formula.

In the usual version of the Euclidean approach one assumes that the generator of the physical dynamics, called the *Hamiltonian*, is bounded from below. Physically, this corresponds to the zero temperature, which is typical for most applications of quantum field theory. There also exists a version of the Euclidean approach for bosonic quantum fields at positive temperatures. Its aim is to construct an interacting KMS state and a dynamics at inverse temperature β. The dynamics is now generated by a self-adjoint operator L, the *Liouvillean*, which is not bounded from below or from above. This leads to some additional technical difficulties. However, the system can be described in a way similar to zero-temperature path spaces. There is an important difference: as a consequence of the KMS condition, the path space is now β-periodic. Thus, the Euclidean space is replaced with a cylinder of circumference β.

One of the interesting features of the Euclidean approach is the use of various non-trivial tools from functional analysis. One of them is the concept of *local Hermitian semi-groups* (see Thm. 2.69). They are indispensable in the positive temperature case. They are also sometimes useful at zero temperature, which happens if the perturbation is unbounded and destroys the positivity of the generator.

To motivate the reader, let us briefly discuss *Gaussian Markov path spaces*, which are usually the starting point for applications of the Euclidean approach. Let $\mathcal{Z}$ be a Hilbert space equipped with a conjugation τ. As we have seen in Subsect. 9.3.5, in such a case the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$ can be unitarily identified with $L^2(Q,\mathrm{d}\mu)$ for some probability space $(Q,\mathfrak{S},\mu)$. In the Euclidean approach we study operators on $L^2(Q,\mathrm{d}\mu)$ using the space of paths, that is, functions from $\mathbb{R}$ with values in Q.

A typical situation where Euclidean methods apply arises when we consider a real (commuting with τ) self-adjoint operator $a \geq 0$ on $\mathcal{Z}$. Recall that the semi-group $\mathrm{e}^{-t\mathrm{d}\Gamma(a)}$ is then positivity improving as an operator on $L^2(Q,\mathrm{d}\mu)$. We will see that for such operators the expectation value $(F|\mathrm{e}^{-t\mathrm{d}\Gamma(a)}G)$ can be written in terms of a measure on the set of *paths*. Field operators for real (τ-invariant) arguments can be interpreted as multiplication operators on $L^2(Q,\mathrm{d}\mu)$. Therefore, operators of the form $P(\phi)$, where P is a polynomial based on $\mathcal{Z}^\tau$, the real subspace of $\mathcal{Z}$, can be interpreted as multiplication operators in the Q-space representation. The Euclidean approach gives a powerful tool to study operators of the form $\mathrm{d}\Gamma(a)+P(\phi)$.

Throughout the chapter, we will use the terminology of abstract measure theory discussed in Chap. 5. Recall, in particular, that if $\mathfrak{T}_i$, $i\in I$, is a family of subsets of a set Q, we denote by $\bigvee_{i\in I}\mathfrak{T}_i$ the σ-algebra generated by $\bigcup_{i\in J}\mathfrak{T}_i$.

Throughout the chapter, we will use t as the generic variable in $\mathbb{R}$ denoting time.

21.1 A simple example: Brownian motion

In this section we illustrate the Euclidean approach by recalling the well-known representation of the heat semi-group e^{-tH_0}, $t \geq 0$, for $H_0 = -\frac{1}{2}\Delta$ on $L^2(\mathbb{R}^d)$, using Brownian motion.

From Subsect. 4.1.8 we obtain that the distribution kernel of e^{-tH_0} is

$$\mathrm{e}^{-tH_0}(x,y) = (2\pi t)^{-d/2}\mathrm{e}^{-(x-y)^2/2t}. \tag{21.1}$$

Consider the real Hilbert space $\mathcal{X} = L^2([0,\infty[,\mathbb{R}^d) \simeq L^2([0,\infty[,\mathbb{R}) \otimes \mathbb{R}^d$ and the Gaussian measure on $\mathcal{X}$ with covariance $\mathbb{1}$. Let ϕ denote the generic variable in $\mathcal{X}$. The associated Gaussian $\mathbf{L}^2$ space $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}\phi^2}\mathrm{d}\phi)$ can be realized as $L^2(Q,\mathfrak{S},\mathrm{d}\mu)$. Following Remark 5.66, we still denote by ϕ the generic variable on Q. For a Borel subset $I \subset \mathbb{R}$, the function $\mathbb{1}_I \otimes \mathbb{1}$ is a projection in $\mathcal{X}$. The corresponding conditional expectation of a measurable function F on Q will be denoted $E_I[F]$. In particular $E_\emptyset[F] = \int F(\phi)\mathrm{d}\mu(\phi)$.

Definition 21.1 *The* Brownian motion in $\mathbb{R}^d$ *is the family* $\{B_t\}_{t\geq 0}$ *of* $\mathbb{R}^d$*-valued measurable functions on* Q *defined by*

$$\xi \cdot B_t(\phi) := \langle \phi | \mathbb{1}_{[0,t]} \otimes \xi \rangle,\ \ \xi \in \mathbb{R}^d,\ \ t \geq 0.$$

The Wiener process in $\mathbb{R}^d$ *is*

$$X_t(x,\phi) := x + B_t(\phi),\ \ t \geq 0,\ \ x \in \mathbb{R}^d.$$

We will often drop ϕ from $B_t(\phi)$ and $X_t(x,\phi)$.

The following lemma expresses the Markov property of the Wiener process:

Lemma 21.2 *For* $t_1, t_2 \geq 0$ *and almost all (a.a.)* $x \in \mathbb{R}^d$

$$\mathrm{E}_{[0,t_1]}\Big[f\big(X_{t_2+t_1}(x)\big)\Big] = \int f\Big(X_{t_2}\big(X_{t_1}(x),\phi\big)\Big)\mathrm{d}\mu(\phi),$$

for all bounded measurable functions $f : \mathbb{R}^d \to \mathbb{C}$.

Proof We first prove the lemma for $f(x) = \mathrm{e}^{\mathrm{i}\xi\cdot x}$, $\xi \in \mathbb{R}^d$. Indeed, for such a function both sides equal $\mathrm{e}^{-\xi^2 t_2/2}\mathrm{e}^{\mathrm{i}X_{t_1}(x)\cdot\xi}$. By Fourier transformation, this proves the lemma for $f \in C_\mathrm{c}^\infty(\mathbb{R}^d)$. By the usual argument, the identity extends to all bounded measurable functions f. □

Proposition 21.3 *Let* $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. *Then*

$$\mathrm{e}^{-tH_0}f(x) = \int f\big(X_t(x)\big)\mathrm{d}\mu, \quad t \geq 0, \quad \textit{for a.a. } x \in \mathbb{R}^d.$$

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Proof Let $f \in C_{\rm c}^\infty(\mathbb{R}^d)$ and $\hat{f}$ be its Fourier transform. Then

$$\begin{aligned}\int f\big(X_t(x)\big)\mathrm{d}\mu &= (2\pi)^{-d}\int \hat{f}(\xi)\mathrm{e}^{\mathrm{i}\xi\cdot x}\int \mathrm{e}^{\mathrm{i}\xi\cdot B_t}\mathrm{d}\mu\mathrm{d}\xi\\ &= (2\pi)^{-d}\int \hat{f}(\xi)\mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{e}^{-t\xi^2/2}\mathrm{d}\xi = \mathrm{e}^{-tH_0}f(x).\end{aligned}$$

If $f \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, we choose a sequence $f_n \in C_{\rm c}^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in L^2, $f_n \to f$ a.e. and $\sup_n \|f_n\|_\infty < \infty$. From (21.1) we obtain that $\mathrm{e}^{-tH_0}f_n(x) \to \mathrm{e}^{-tH_0}f(x)$ for a.a. x. The convergence of the r.h.s. to $\int f\big(X_t(x)\big)\mathrm{d}\mu$ follows from the dominated convergence. □

We end this section by proving the celebrated *Feynman–Kac formula* in a simple situation. We denote by $C_{\rm b}(\mathbb{R}^d)$ the space of bounded continuous functions on $\mathbb{R}^d$.

Theorem 21.4 *Let $V \in C_{\rm b}(\mathbb{R}^d)$ be a real potential, $f \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $t \geq 0$. Then, for all $x \in \mathbb{R}^d$, $\mathrm{e}^{-\int_0^t V\left(X_s(x)\right)\mathrm{d}s}f\big(X_t(x)\big)$ is a bounded measurable function on Q and*

$$\mathrm{e}^{-t(H_0+V)}f(x) = \int \mathrm{e}^{-\int_0^t V(X_s(x))\mathrm{d}s}f(X_t(x))\mathrm{d}\mu,\ \textit{for a.a. } x \in \mathbb{R}^d. \tag{21.2}$$

Lemma 21.5 *Let $g_1,\dots,g_{n-1} \in L^\infty(\mathbb{R}^d)$, $h \in L^2(\mathbb{R}^d)$. Let $s_1,\dots,s_n > 0$ and $t_i = t_{i-1}+s_i$, $t_1 = s_1$. Then*

$$\begin{aligned}&\mathrm{e}^{-s_1H_0}g_1\mathrm{e}^{-s_2H_0}\cdots g_{n-1}\mathrm{e}^{-s_nH_0}h(x)\\ &\quad= \int\prod_{i=1}^{n} g_i\big(X_{t_i}(x)\big)h\big(X_{t_n}(x)\big)\mathrm{d}\mu.\end{aligned} \tag{21.3}$$

Proof We prove (21.3) for $n = 2$; the general case follows easily by induction. We have

$$\begin{aligned}&\mathrm{e}^{-s_1H_0}g\mathrm{e}^{-s_2H_0}h(x)\\ &\quad= \int g\big(X_{s_1}(x)\big)\mathrm{e}^{-s_2H_0}h\big(X_{s_1}(x)\big)\mathrm{d}\mu\\ &\quad= \int g\big(X_{s_1}(x,\phi_1)\big)\int h\Big(X_{s_2}\big(X_{s_1}(x,\phi_2)\phi_1\big)\Big)\mathrm{d}\mu(\phi_2)\mathrm{d}\mu(\phi_1)\\ &\quad= \int g\big(X_{s_1}(x)\big)E_{[0,s_1]}\Big[h\big(X_{s_1+s_2}(x)\big)\Big]\mathrm{d}\mu\\ &\quad= \int g\big(X_{s_1}(x)\big)h\big(X_{s_1+s_2}(x)\big)\mathrm{d}\mu,\end{aligned}$$

by Lemma 21.2. □

Lemma 21.6 *For $V \in C_{\rm b}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$ the map*

$$[0,+\infty[\ni t \mapsto V\big(X_t(x)\big) \in L^2(Q)$$

is continuous.

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Proof For $t \geq 0$, $\delta > 0$, we have

$$\begin{aligned}&\int \left|V\big(X_{t+\delta}(x)\big) - V\big(X_t(x)\big)\right|^2 \mathrm{d}\mu \\ &= \int V^2\big(X_{t+\delta}(x)\big)\mathrm{d}\mu + \int V^2\big(X_t(x)\big)\mathrm{d}\mu - 2\int V\big(X_t(x)\big)V\big(X_{t+\delta}(x)\big)\mathrm{d}\mu \\ &= \mathrm{e}^{-(t+\delta)H_0}V^2(x) + \mathrm{e}^{-tH_0}V^2(x) - 2\mathrm{e}^{-\delta H_0}V\mathrm{e}^{-tH_0}V(x),\end{aligned}$$

where in the last line we use (21.3). From (21.1) we see that e^{-tH_0} is a semi-group of contractions on $C_{\mathrm{b}}(\mathbb{R}^d)$. Moreover it is easy to see that, for $G \in C_{\mathrm{b}}(\mathbb{R}^d)$ and all $x \in \mathbb{R}^d$, the map

$$[0, +\infty[\ni t \mapsto \mathrm{e}^{-tH_0}G(x) \in \mathbb{R}$$

is continuous. This proves the right continuity at all $t \geq 0$. The proof of the left continuity at all $t > 0$ is similar. □

Proof of Thm. 21.4. By Lemma 21.6, $\int_0^t V\big(X_s(x)\big)\mathrm{d}s$ is a bounded measurable function on Q. Hence, the integrand in the r.h.s. of (21.2) is bounded measurable on Q.

Let $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. By Trotter's product formula (see Thm. 2.75) we have

$$\mathrm{e}^{-t(H_0+V)}f = \lim_{n\to\infty}\big(\mathrm{e}^{-(t/n)H_0}\mathrm{e}^{-(t/n)V}\big)^n f, \quad \text{in } L^2(\mathbb{R}^d),$$

and after extracting a subsequence we can assume that

$$\mathrm{e}^{-t(H_0+V)}f(x) = \lim_{n\to\infty}\big(\mathrm{e}^{-(t/n)H_0}\mathrm{e}^{-(t/n)V}\big)^n f(x), \quad \text{for a.a. } x.$$

Applying (21.3) to $h = \mathrm{e}^{-(t/n)V}f$, $g_j = \mathrm{e}^{-(t/n)V}$ for $1 \leq j \leq n-1$, we get

$$\mathrm{e}^{-t(H_0+V)}f(x) = \int \mathrm{e}^{-F_n(x)}f\big(X_t(x)\big)\mathrm{d}\mu,$$

for $F_n(x) = \frac{t}{n}\sum_{j=1}^{n} V\big(X_{tj/n}(x)\big)$. Set $F(x) = \int_0^t V\big(X_s(x)\big)\mathrm{d}s$. We claim that

$$\mathrm{e}^{-F_n(x)} \to \mathrm{e}^{-F(x)} \ \text{ in } L^2(Q), \quad \text{for a.a. } x, \tag{21.4}$$

which will complete the proof of the theorem. Since $|\mathrm{e}^{-F_n}|$, $|\mathrm{e}^{-F}| \leq \mathrm{e}^{t\|V\|_\infty}$, it suffices to prove that $F_n(x) \to F(x)$ in $L^2(Q)$ for a.a. x. Since F_n is a Riemann sum for the integral defining F, this follows from Lemma 21.6. □

21.2 Euclidean approach at zero temperature

Most of this section is devoted to a description of the Euclidean approach at zero temperature in an abstract setting. We start with the definition of an abstract version of *Markov path spaces*. We will restrict ourselves to path spaces with a finite measure, which is sufficient for most applications to quantum field theory.

Given a Markov path space there is a canonical construction of a positivity improving semi-group $\{P(t)\}_{t\in[0,\infty[}$ possessing a unique ground state. Its generator is sometimes called the *Hamiltonian*. It acts on the so-called *physical Hilbert space*. A converse construction is also possible: every contractive positivity improving semi-group with a ground state can be dilated to a Markov path space.

The concept of a Markov path space is closely related to *unitary dilations of contractive semi-groups*. Indeed, each Markov path space involves a unitary group $\{U_t\}_{t\in\mathbb{R}}$ of measure preserving transformations of the underlying space which is a dilation of the physical semi-group $\{P(t)\}_{t\in[0,\infty[}$.

The most important class of examples of Markov path spaces are *Gaussian Markov path spaces*, which can be used to describe free bosonic quantum field theories in a Euclidean setting. They can be viewed as the real-wave quantization of a dilation of a contractive semi-group.

21.2.1 Markov path spaces

Definition 21.7 *A* generalized path space $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ *consists of*

(1) *a complete probability space* $(Q, \mathfrak{S}, \mu)$*;*
(2) *a distinguished sub-*σ*-algebra* $\mathfrak{S}_0$ *of* $\mathfrak{S}$*;*
(3) *a one-parameter group* $\mathbb{R} \ni t \mapsto U_t$ *of measure preserving* $*$*-automorphisms of* $L^\infty(Q, \mathfrak{S}, \mu)$*, strongly continuous for the* σ*-weak topology;*
(4) *a measure preserving* $*$*-automorphism* R *of* $L^\infty(Q, \mathfrak{S}, \mu)$ *such that* $RU_t = U_{-t}R$*,* $R^2 = \mathbb{1}$*.*

Moreover, one assumes that

$$\mathfrak{S} = \bigvee_{t\in\mathbb{R}} U_t \mathfrak{S}_0. \tag{21.5}$$

In what follows, $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ is a generalized path space. By Prop. 5.33 (2)(iii) and (2)(iv), U_t extends to a strongly continuous group of isometries of $L^p(Q, \mathfrak{S}, \mu)$, and R extends to an isometry of $L^p(Q, \mathfrak{S}, \mu)$, for $1 \leq p < \infty$.

Definition 21.8 *We set* $\mathfrak{S}_t := U_t\mathfrak{S}_0$*,* $\mathfrak{S}_I := \bigvee_{t\in I} \mathfrak{S}_t$*, for* $I \subset \mathbb{R}$*, and denote by* E_I *the conditional expectation w.r.t.* $\mathfrak{S}_I$*.*

Definition 21.9 *The generalized path space* $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ *is a* Markov path space *if it satisfies*

(1) *the* reflection property*:* $RE_0 = E_0$*,*
(2) *the* Markov property*:* $E_{[0,+\infty[}E_{]-\infty,0]} = E_0$*.*

21.2.2 Reconstruction theorem

Let $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ be a Markov path space.

Definition 21.10 *The* physical Hilbert space associated with $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ *is*

$$\mathcal{H} := L^2(Q, \mathfrak{S}_0, \mu).$$

The function $1 \in \mathcal{H}$ *will be denoted by* Ω. *The Abelian* $*$*-algebra* $\mathfrak{A} := L^\infty(Q, \mathfrak{S}_0, \mu)$ *acting on* $\mathcal{H}$ *is called the* algebra of time-zero fields.

Theorem 21.11 (1) $P(t) := E_0 U_t E_0,\ t \geq 0$, *is a strongly continuous semi-group of self-adjoint contractions on* $\mathcal{H}$ *preserving* Ω.

(2) $P(t)$ *is doubly Markovian.*

(3) $P(t)$ *is a contraction semi-group on* $L^p(Q, \mathfrak{S}_0, \mu)$ *for* $1 \leq p \leq \infty$. *It is strongly continuous for* $1 \leq p < \infty$.

(4) *Let* $A_i \in L^\infty(Q, \mathfrak{S}_0, \mu)$, $i = 1, \dots, n$ *and* $t_1 \leq \dots \leq t_n$. *Then*

$$\big(\Omega | A_1 P(t_1 - t_2) A_2 \cdots P(t_{n-1} - t_n) A_n \Omega\big) = \int_Q \prod_{i=1}^n U_{t_i}(A_i) \mathrm{d}\mu.$$

Proof $P(t)$ is clearly a contraction. It is self-adjoint:

$$P(t)^* = E_0 U_{-t} E_0 = E_0 U_{-t} R E_0 = E_0 R U_t E_0 = E_0 U_t E_0 = P(t).$$

Let us prove the semi-group property. Note that $U_t E_0 U_{-t} = E_t$. The Markov property implies, for $t, s \geq 0$,

$$E_{-t} E_0 E_s = E_{-t} E_{]-\infty,0]} E_{[0,+\infty[} E_s = E_{-t} E_s.$$

This yields

$$\begin{aligned} P(t)P(s) &= E_0 U_t E_0 U_s E_0 = U_t E_{-t} E_0 E_s U_{-s} \\ &= U_t E_{-t} E_s U_{-s} = E_0 U_t U_s E_0 = P(t+s). \end{aligned}$$

Finally, since $t \mapsto U_t$ is strongly continuous, so is $t \mapsto P(t)$.

U_t, E_0 are clearly positivity preserving. Hence so is $P(t)$. U_t, E_0 preserve 1. Hence so does $P(t)$. This proves (2). (3) follows from (2) by Prop. 5.24. We leave (4) to the reader. □

Definition 21.12 *The unique positive self-adjoint operator* H *on* $\mathcal{H}$ *such that* $P(t) = \mathrm{e}^{-tH}$ *is called the* Hamiltonian.

Clearly, $H\Omega = 0$.

Remark 21.13 *Often instead of Markov path spaces one uses more general* OS-positive path spaces, *named after Osterwalder and Schrader, where Def. 21.9 is replaced by the condition that* $E_{[0,+\infty[} R E_{[0,+\infty[} \geq 0$. *The OS-positivity condition is one of the Osterwalder–Schrader axioms; see Osterwalder–Schrader (1973, 1975). They are Euclidean analogs of the Gårding–Wightman axioms.*

In space dimensions 2 or higher it is believed that sharp time interacting fields do not exist, hence the Markov property cannot be used. Results similar to those

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in this chapter can be established in the framework of OS-positive path spaces, with similar proofs.

21.2.3 Gaussian path spaces I

Let $\mathcal{X}$ be a real Hilbert space with a self-adjoint operator $\epsilon > 0$. (All the constructions of this subsection have their complex counterparts; we assume the reality to simplify the exposition and in view of the application in the next subsection.) Consider the real Hilbert space

$$L^2(\mathbb{R}, \mathcal{X}) \simeq L^2(\mathbb{R}, \mathbb{R}) \otimes \mathcal{X} \tag{21.6}$$

and the positive self-adjoint operator

$$C = (D_t^2 + \epsilon^2)^{-1}$$

on (21.6). Introduce the real Hilbert space $\mathcal{Q} := C^{-\frac{1}{2}} L^2(\mathbb{R}, \mathcal{X})$. Its dual $\mathcal{Q}^{\#}$ can be identified with $C^{\frac{1}{2}} L^2(\mathbb{R}, \mathcal{X})$. Note that the operator C is orthogonal from $\mathcal{Q}$ to $\mathcal{Q}^{\#}$.

Definition 21.14 *For $t \in \mathbb{R}$ let us define the map*

$$j_t : (2\epsilon)^{\frac{1}{2}} \mathcal{X} \ni g \mapsto \delta_t \otimes g \in \mathcal{Q}. \tag{21.7}$$

Lemma 21.15 *We have*

$$(j_{t_1} g_1 | j_{t_2} g_2)_{\mathcal{Q}} = \Big(g_1 | \frac{\mathrm{e}^{-|t_1 - t_2|\epsilon}}{2\epsilon} g_2 \Big)_{\mathcal{X}}.$$

In particular j_t is isometric.

Proof We use the identity

$$\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tk} \frac{2\epsilon}{k^2 + \epsilon^2} \mathrm{d}k = 2\pi \mathrm{e}^{-|t|\epsilon}, \quad t \in \mathbb{R}, \tag{21.8}$$

which follows from Fourier transform and functional calculus. □

Definition 21.16 *For $t \in \mathbb{R}$ we set $\mathcal{Q}_t := j_t (2\epsilon)^{\frac{1}{2}} \mathcal{X}$. Let e_t denote the orthogonal projection onto $\mathcal{Q}_t$.*

For $I \subset \mathbb{R}$ we set $\mathcal{Q}_I := \big(\sum_{t \in I} \mathcal{Q}_t\big)^{\mathrm{cl}}$. The orthogonal projection onto $\mathcal{Q}_I$ will be denoted e_I.

Note that $e_t = j_t j_t^{\#}$.

For explicit formulas, in the following proposition we prefer to use the space $\mathcal{Q}^{\#}$ rather than $\mathcal{Q}$, by transporting operators with the help of the operator C.

Definition 21.17 *We write e^t, resp. e^I for $C e_t C^{-1}$, resp. $C e_I C^{-1}$.*

Definition 21.18 *We define*

$$(rf)(s) := f(-s), \quad (u_t f)(s) = f(s - t), \quad f \in \mathcal{Q}, \quad s, t \in \mathbb{R}.$$

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Definition 21.19 $\mathrm{e}^{-t\epsilon}$, *defined originally on $\mathcal{X}$, determines in an obvious way a contractive semi-group on $(2\epsilon)^{\frac{1}{2}}\mathcal{X}$, which will be denoted by the same symbol. We set $p(t) := j_0 \mathrm{e}^{-t\epsilon} j_0^{\#}$, which is a contractive semi-group on $\mathcal{Q}_0$.*

Proposition 21.20 (1) *Let t, $t_1 < t_2$, $f \in \mathcal{Q}^{\#}$. We have*

$$\begin{aligned}
e^t f(s) &:= \mathrm{e}^{-\epsilon|t-s|} f(t),\\
e^{[t,\infty[} f(s) &= 1\!\!1_{[t,\infty[}(s) f(s) + \mathrm{e}^{-|s-t|\epsilon} 1\!\!1_{]-\infty,t[}(s) f(t),\\
e^{]-\infty,t]} f(s) &= 1\!\!1_{]-\infty,t]}(s) f(s) + \mathrm{e}^{-|s-t|\epsilon} 1\!\!1_{]t,\infty[}(s) f(t),\\
e^{[t_1,t_2]} f(s) &= 1\!\!1_{[t_1,t_2]}(s) f(s) + \mathrm{e}^{-|s-t_1|\epsilon} 1\!\!1_{]-\infty,t_1[}(s) f(t_1) + \mathrm{e}^{-|s-t_2|\epsilon} 1\!\!1_{]t_2,\infty[}(s) f(t_2).
\end{aligned}$$

(2) *$C_{\mathrm{c}}^{\infty}(]t_1, t_2[, \mathrm{Dom}\,\epsilon)$ is dense in $\mathcal{Q}_{]t_1,t_2[}$.*
(3) *$\mathbb{R} \ni t \mapsto u_t$ is an orthogonal C_0-group on $\mathcal{Q}$.*
(4) *r is an orthogonal operator satisfying $ru_t = u_{-t}r$ and $r^2 = 1\!\!1$.*
(5) *$\sum_{t\in\mathbb{R}} u_t \mathcal{Q}_0$ is dense in $\mathcal{Q}$.*
(6) *$re_0 = e_0$.*
(7) *$e_{[0,\infty[} e_{]-\infty,0]} = e_0$.*
(8) *$e_0 u_t e_0 = p\big(|t|\big)$.*

Remark 21.21 *Let $[0,\infty[\ni t \mapsto p(t)$ be a contractive C_0-semi-group on a Hilbert space $\mathcal{Q}_0$. We say that $(\mathcal{Q}, u_t, e_0)$ is a* unitary dilation *of $\{p(t)\}_{t\in[0,\infty[}$ if $\mathcal{Q}$ is a Hilbert space, e_0 is an orthogonal projection from $\mathcal{Q}$ onto $\mathcal{Q}_0$, $\{u_t\}_{t\in\mathbb{R}}$ is a unitary C_0-group on $\mathcal{Q}$ and $p(t) = e_0 u_t e_0$, $t \geq 0$. We say that the dilation $(\mathcal{Q}, u_t, e_0)$ is* minimal *if $\sum_{t\in\mathbb{R}} u_t \mathcal{Q}_0$ is dense in $\mathcal{Q}$.*

Clearly, what we have constructed in this subsection is a minimal dilation of the contractive semi-group $\{p(t)\}_{t\in[0,\infty[}$.

21.2.4 Gaussian path spaces II

In this subsection we describe the main example of Markov path spaces – Gaussian path spaces. They are used to describe free quantum field theories. They are obtained by second quantizing the Markov path system constructed in the previous subsection.

Let $\mathcal{X}$ be a real Hilbert space and $\epsilon > 0$ a self-adjoint operator on $\mathcal{X}$. Let C, $\mathcal{Q}$, $\{j_t\}_{t\in\mathbb{R}}$, $\{u_t\}_{t\in\mathbb{R}}$, r be constructed as in the previous subsection. Let us consider the Gaussian $\mathbf{L^2}$ space with covariance C. According to the notation introduced in Subsect. 5.4.2, it will be denoted

$$\mathbf{L^2}\big(L^2(\mathbb{R},\mathcal{X}), \mathrm{e}^{\phi\cdot C^{-1}\phi}\mathrm{d}\phi\big), \tag{21.9}$$

where we use ϕ as the generic variable in $L^2(\mathbb{R},\mathcal{X})$.

As we discussed in Chap. 5, there are many ways to realize this Gaussian $\mathbf{L}^2$ space as a space $L^2(Q,\mu)$, where (Q,μ) is a probability space. (Note that the notation Q for such a measure space is traditional in a part of the literature, hence the name "Q-space representation".) A class of possible choices, which is in

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fact our favorite, is $Q := B^{\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X})$, where $B \geq 0$ is any self-adjoint operator on $L^2(\mathbb{R}, \mathcal{X})$ such that $B^{-\frac{1}{2}}CB^{-\frac{1}{2}}$ is trace-class. Thus the Gaussian $\mathbf{L^2}$ space (21.9) becomes the concrete space $L^2(Q, \mathrm{d}\mu)$, where μ is a Borel probability measure on Q such that

$$\int_Q \mathrm{e}^{\mathrm{i}\phi(f)}\mathrm{d}\mu(\phi) = \mathrm{e}^{-\frac{1}{2}f\cdot Cf}, \quad f \in B^{-\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X}). \tag{21.10}$$

Following Remark 5.66, we now use ϕ as the generic name for an element of $Q = B^{\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X})$. $\phi(f)$ denotes the pairing of $\phi \in B^{\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X})$ with $f \in B^{-\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X})$.

By Prop. 5.77, we can extend the definition of

$$Q \ni \phi \mapsto \phi(f) \tag{21.11}$$

to $f \in C^{-\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X})$. The function in (21.11) in general needs not to be continuous; however it still belongs to $L^p(Q, \mu)$ for all $1 \leq p < \infty$.

Definition 21.22 *Since the maps j_s defined in (21.7) are isometric, we can define for $s \in \mathbb{R}$, $g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}$, the functions*

$$\phi_s(g) := \phi(\delta_s \otimes g) \in \bigcap_{1\leq p<\infty} L^p(Q, \mu),$$

which are called the sharp-time fields*.*

We can now define the associated path space. We lift r and $\{u_t\}_{t\in\mathbb{R}}$ to $L^2(Q, \mu)$ by setting first

$$R\mathrm{e}^{\mathrm{i}\phi(f)} := \mathrm{e}^{\mathrm{i}\phi(rf)}, \quad U_t\mathrm{e}^{\mathrm{i}\phi(f)} = \mathrm{e}^{\mathrm{i}\phi(u_{-t}f)}, \ f \in B^{-\frac{1}{2}}L^2(\mathbb{R}, \mathcal{X}), \tag{21.12}$$

extending then R and U_t to $L^2(Q, \mu)$ by linearity and density. In particular we have

$$R\phi_s(g) = \phi_{-s}(g), \quad U_t\phi_s(g) = \phi_{s-t}(g), \quad g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}. \tag{21.13}$$

Proposition 21.23 *Let $\mathfrak{S}$ be the completion of the Borel σ-algebra on Q, $\mathfrak{S}_0$ be the σ-algebra generated by the functions $\mathrm{e}^{\mathrm{i}\phi_0(g)}$ for $g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}$. Let R, U_t be defined in (21.12). Then $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ is a Markov path space.*

Definition 21.24 $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ *described in Prop. 21.23 will be called the* Gaussian path space with covariance C*.*

We will later need the following lemma, which follows directly from the results on complex-wave representation in Subsect. 9.2.1.

Lemma 21.25 *Let $\mathcal{Z}$ be a Hilbert space, $\Gamma_{\mathrm{s}}(\mathcal{Z})$ the associated bosonic Fock space and b a self-adjoint operator on $\mathcal{Z}$. Then*

$$\left(\mathrm{e}^{\mathrm{i}\phi(g_1)}\Omega|\mathrm{e}^{-t\mathrm{d}\Gamma(b)}\mathrm{e}^{\mathrm{i}\phi(g_2)}\Omega\right) = \mathrm{e}^{-\frac{1}{2}\|g_1\|^2}\mathrm{e}^{-\frac{1}{2}\|g_2\|^2}\mathrm{e}^{-(g_1|\mathrm{e}^{-tb}g_2)},$$

whenever the r.h.s. is finite.

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Proof of Prop. 21.23. Using that r and u_t preserve C, formula (21.10) and the density of exponentials in $L^2(Q,\mu)$ (see Subsect. 5.2.5) we see that R is unitary on $L^2(Q,\mu)$ and that $t \mapsto U_t$ is a strongly continuous unitary group on $L^2(Q,\mu)$. R and U_t are clearly $*$-automorphisms. By Prop. 5.33, $t \mapsto U_t$ is strongly continuous on $L^\infty(Q,\mu)$ for the σ-weak topology.

From (21.13) we see that the closed vector subspace generated by $\mathrm{e}^{\mathrm{i}\phi_0(g)}$ for $g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}$ is invariant under R, which implies that $RE_0 = E_0$. The fact that $RU_t = U(-t)R$ is obvious.

We now check the Markov property. We unitarily identify $\mathbf{L}^2\big(L^2(\mathbb{R},\mathcal{X}), \mathrm{e}^{\phi\cdot C^{-1}\phi}\mathrm{d}\phi\big)$ with $\Gamma_{\mathrm{s}}(\mathbb{C}\mathcal{Q})$, as in Thm. 9.22. If $I \subset \mathbb{R}$ is a closed interval, then under this identification E_I becomes $\Gamma(e_I)$, where e_I is defined in Lemma 21.15. So the Markov property follows from the pre-Markov property proved in Prop. 21.20 (7).

It remains to check condition (21.5). We note that it is equivalent to the property that the algebra generated by $\big\{U_t f \ : \ f \in L^\infty(Q,\mathfrak{S}_0,\mu),\ t \in \mathbb{R}\big\}$ is dense in $L^2(Q,\mathfrak{S},\mu)$. It is easy to see that finite linear combinations of $\delta_{t_i} \otimes g_i$ for $t_i \in \mathbb{R}$, $g_i \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}$, are dense in $\mathcal{Q}$. It follows that if $f \in \mathcal{Q}$, the function $\mathrm{e}^{\mathrm{i}\phi(f)}$ can be approximated in L^2 by products of $\mathrm{e}^{\mathrm{i}\phi_{t_i}(g_i)}$. Since linear combinations of exponentials are dense in $L^2(Q,\mu)$, we obtain (21.5). □

Theorem 21.26 *There exists a unique unitary map*

$$T_{\mathrm{eucl}} : \mathcal{H} \to \Gamma_{\mathrm{s}}\big(\mathbb{C}(2\epsilon)^{\frac{1}{2}}\mathcal{X}\big)$$

such that

$$\begin{aligned} T_{\mathrm{eucl}} 1 &= \Omega, \\ T_{\mathrm{eucl}} \mathrm{e}^{\mathrm{i}\phi_0(g)} &= \mathrm{e}^{\mathrm{i}(a^*(g)+a(g))} T_{\mathrm{eucl}}, \quad g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}. \end{aligned} \tag{21.14}$$

We have

$$T_{\mathrm{eucl}} \mathrm{e}^{-tH} = \mathrm{e}^{-t\mathrm{d}\Gamma(\epsilon)} T_{\mathrm{eucl}}, \quad t \geq 0.$$

Proof Linear combinations of time-zero exponentials $\mathrm{e}^{\mathrm{i}\phi_0(g)}$, for $g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}$, are dense in $L^2(Q,\mathfrak{S}_0,\mu)$, and

$$\int_Q \mathrm{e}^{\mathrm{i}\phi_0(g)} \mathrm{d}\mu = \mathrm{e}^{-\frac{1}{2}(\delta_0\otimes g|C\delta_0\otimes g)} = \mathrm{e}^{-\frac{1}{2}(g|g)},$$

by Lemma 21.15. Therefore, there exists a unique unitary map $\tilde{T}_{\mathrm{eucl}} : L^2(Q,\mathfrak{S}_0,\mu) \to \mathbf{L}^2\big((2\epsilon)^{\frac{1}{2}}\mathcal{X}, \mathrm{e}^{-x\epsilon x}\mathrm{d}x\big)$ such that

$$\begin{aligned} \tilde{T}_{\mathrm{eucl}} 1 &= 1, \\ \tilde{T}_{\mathrm{eucl}} \mathrm{e}^{\mathrm{i}\phi_0(g)} &= \mathrm{e}^{\mathrm{i}\phi(g)} \tilde{T}_{\mathrm{eucl}}, \quad g \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}. \end{aligned}$$

Composing $\tilde{T}_{\mathrm{eucl}}$ with the map $(T^{\mathrm{rw}})^{-1}$ constructed in Thm. 9.22, we obtain the unitary map T_{eucl} with the first two properties of (21.14). To prove the third one, it suffices by density to check that, for $g_1, g_2 \in (2\epsilon)^{\frac{1}{2}}\mathcal{X}$, one has

$$\int_Q \mathrm{e}^{-\mathrm{i}\phi_0(g_1)}\mathrm{e}^{\mathrm{i}\phi_t(g_2)}\mathrm{d}\mu = \big(\mathrm{e}^{\mathrm{i}(a^*(g_1)+a(g_1))}\Omega|\mathrm{e}^{-t\mathrm{d}\Gamma(\epsilon)}\mathrm{e}^{\mathrm{i}(a^*(g_2)+a(g_2))}\Omega\big). \tag{21.15}$$

The l.h.s. of (21.15) equals

$$\begin{aligned}&\exp\Big(-\frac{1}{2}\big(\delta_t\otimes g_2-\delta_0\otimes g_1|\delta_t\otimes g_2-\delta_0\otimes g_1\big)_{\mathcal{Q}}\Big)\\ &\quad=\exp\Big(-\big(g_2|(4\epsilon)^{-1}g_2\big)_{\mathcal{X}}-\big(g_1|(4\epsilon)^{-1}g_1\big)_{\mathcal{X}}+\big(g_1|(2\epsilon)^{-1}\mathrm{e}^{-t\epsilon}g_2\big)_{\mathcal{X}}\Big),\end{aligned}$$

by Lemma 21.15. Applying Lemma 21.25 to the Hilbert space $\mathbb{C}(2\epsilon)^{\frac{1}{2}}\mathcal{X}$, we see that this equals the r.h.s. of (21.15). □

21.2.5 From a positivity preserving semi-group to a Markov path space

Let (X,ν) be a measure space and $P(t)=\mathrm{e}^{-tH}$ be positivity improving contractive semi-group on $L^2(X,\nu)$. We assume that $0=\inf\operatorname{spec}H$ and $\inf\operatorname{spec}H$ is an eigenvalue. Recall that by the Perron–Frobenius theorem (Thm. 5.25) H has a unique positive ground state. It will be denoted by Ω.

In this subsection we present a construction converse to that of Subsect. 21.2.2.

Theorem 21.27 (1) *There exist*

(i) *a Markov path space* $(Q,\mathfrak{S},\mathfrak{S}_0,U_t,R,\mu)$,

(ii) *a unitary map* $T: L^2(X,\nu)\to L^2(Q,\mathfrak{S}_0,\mu)$ *such that*

$$\begin{aligned}&T\Omega=1,\\ &TL^\infty(X,\nu)T^{-1}=L^\infty(Q,\mathfrak{S}_0,\mu).\end{aligned}$$

(2) *Denoting* TAT^{-1} *by* $\tilde{A}$ *for* $A\in L^\infty(X,\nu)$, *one has*

$$\int_Q \prod_{i=1}^n U_{t_i}(\tilde{A}_i)\mathrm{d}\mu = \big(\Omega|A_1\mathrm{e}^{-(t_2-t_1)H}A_2\cdots\mathrm{e}^{-(t_n-t_{n-1})H}A_n\Omega\big),$$

for $A_i\in L^\infty(X,\nu)$, $i=1,\dots,n$, $t_1\leq\cdots\leq t_n$.

Lemma 21.28 $\mathrm{e}^{-tH}L^\infty(X,\nu)\Omega\subset L^\infty(X,\nu)\Omega,\quad t\geq 0.$

Proof Set $\nu_\Omega=\Omega^2\nu$ and consider the unitary map

$$\begin{aligned}T_\Omega: L^2(X,\nu)&\to L^2(X,\nu_\Omega)\\ f&\mapsto\Omega^{-1}f.\end{aligned}$$

Setting $H_\Omega:=T_\Omega HT_\Omega^{-1}$, we see that e^{-tH_Ω} is positivity preserving, with 1 as the unique strictly positive ground state. Therefore, H_Ω is doubly Markovian. Therefore, by Prop. 5.24, it is a contraction on $L^\infty(X,\nu_\Omega)=L^\infty(X,\nu)$. Now

$$\begin{aligned}&\mathrm{e}^{-tH}L^\infty(X,\nu)\Omega=T_\Omega^{-1}\mathrm{e}^{-tH_\Omega}L^\infty(X,\nu_\Omega)1\\ \subset\ &T_\Omega^{-1}L^\infty(X,\nu_\Omega)1=L^\infty(X,\nu)\Omega.\end{aligned}$$

□

Proof of Thm. 21.27. By the Gelfand–Naimark theorem (Sakai (1971), Thm. 1.2.1), $L^\infty(X,\nu)$ is isomorphic as a C^*-algebra to $C(Q_0)$, where Q_0 is a compact Hausdorff space. In the sequel we will denote by the same letter A an element of $L^\infty(X,\nu)$ and its image in $C(Q_0)$.

Since $L^\infty(X,\nu)$ is a W^*-algebra, we know that Q_0 is a *Stonean space*, i.e. the closure of any open set in Q_0 is open (see Sakai (1971), Prop. 1.3.2). Let Ξ be the set of characteristic functions on Q_0. By Sakai (1971), Prop. 1.3.1, the $*$-algebra generated by Ξ is dense in $C(Q_0)$.

Let $Q := Q_0^{\mathbb{R}}$ be equipped with the product topology, which is also compact by Tychonov's theorem. Note that each $q \in Q$ is a function $\mathbb{R} \ni t \mapsto q_t \in Q_0$. By the Stone–Weierstrass theorem, the $*$-algebra generated by functions f of the form $f(q) = A(q_t)$ for some $t \in \mathbb{R}$ and $A \in C(Q_0)$ is dense in $C(Q)$. By the argument above, the $*$-algebra $\mathcal{L}(Q)$ generated by the functions f of the form $f(q) = A(q_t)$ for some $t \in \mathbb{R}$ and $A \in \Xi$ is also dense in $C(Q)$.

Now let $f \in \mathcal{L}(Q)$. Clearly, f can always be written as

$$f(q) = \sum_{j=1}^{p} a_j \prod_{i=1}^{n} A_{i,j}(q_{t_i}),\ A_{i,j} \in \Xi,\ a_j \in \mathbb{C},$$

for $t_1 \leq \cdots \leq t_n$. Splitting further characteristic functions $A_{i,j}$, we can uniquely rewrite f as

$$f(q) = \sum_{j=1}^{q} b_j \prod_{i=1}^{n} B_{i,j}\left(q_{t_i}\right),\ B_{i,j} \in \Xi,\ b_j \in \mathbb{C}, \tag{21.16}$$

where $B_{i,j}B_{i,k} = 0$ for $j \neq k$. It follows that

$$\rho(f) := \sum_{j=1}^{q} b_j \big(\Omega | B_{1,j} \mathrm{e}^{-(t_2-t_1)H} B_{2,j} \cdots \mathrm{e}^{-(t_n-t_{n-1})H} B_{n,j}\Omega\big), \tag{21.17}$$

defines a linear form on $\mathcal{L}(Q)$ with $\rho(1) = 1$. Now let $F \in \mathcal{L}(Q)$ with $F \geq 0$. Clearly, f can be uniquely written as in (21.16) with $b_j \geq 0$, $B_{i,j} \geq 0$. Since e^{-tH} is positivity preserving and $\Omega \geq 0$, we see that $\rho(f) \geq 0$ and ρ is a positive, hence bounded linear form on $\mathcal{L}(Q)$. We denote by $\mathfrak{S}$ the Baire σ-algebra on Q. Extending ρ to $C(Q)$ by density and using the Riesz–Markov theorem, we obtain a Baire probability measure μ such that

$$\rho(f) = \int_Q f \mathrm{d}\mu,\ f \in \mathcal{L}(Q).$$

We now set

$$rq_s := q_{-s}, \quad u_t q_s := q_{s-t}, \quad t \in \mathbb{R},$$

and

$$Rf(q) := f(rq), \quad \big(U_t f\big)(q) := f(u_{-t}q), \quad t \in \mathbb{R}.$$

Clearly, U_t and R satisfy conditions (3) and (4) of Def. 21.7. Let $\mathfrak{S}_0$ be the sub-σ-algebra of $\mathfrak{S}$ generated by the functions $q \mapsto A(q_0)$, $A \in C(Q_0) = L^\infty(X,\nu)$. Note that $\mathfrak{S} := \bigvee_{t\in\mathbb{R}} U_t\mathfrak{S}_0$. We can rewrite (21.17) as

$$\int_Q \prod_{i=1}^n A_i(q(t_i))\mathrm{d}\mu(q) = \big(\Omega|A_1 \mathrm{e}^{-(t_2-t_1)H} A_2 \cdots \mathrm{e}^{-(t_n-t_{n-1})H} A_n\Omega\big), \qquad (21.18)$$

for $A_i \in L^\infty(X,\nu)$, $t_1 \le \cdots \le t_n$.

It remains to prove that $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ is a Markov path space. Property (1) of Def. 21.9 is obvious. To prove property (2) of Def. 21.9, i.e. that $E_{[0,+\infty[}E_{]-\infty,0]} = E_{\{0\}}$, it suffices by linearity and density to show that for

$$f(q) = \prod_{i=1}^n A_i(q_{t_i}),\ A_i \in L^\infty(Q,\mathfrak{S}_0),\ t_1 \le \cdots \le t_n \le 0 \qquad (21.19)$$

$E_{[0,+\infty[}f$ is $\mathfrak{S}_0$-measurable. Recall that $E_{[0,+\infty[}f = g$ iff g is $\mathfrak{S}_{[0,+\infty[}$-measurable and

$$\int_Q \overline{f} h \mathrm{d}\mu = \int_Q \overline{g} h \mathrm{d}\mu, \qquad (21.20)$$

for all $\mathfrak{S}_{[0,+\infty[}$-measurable functions h. Again by linearity and density, it suffices to check (21.20) for $h(q) = \prod_{i=1}^p B_i(q_{s_i})$, $B_i \in L^\infty(Q,\mathfrak{S}_0)$ and $0 \le s_1 \le \cdots \le s_p$.

For f as in (21.19), we have, using (21.18),

$$\begin{aligned}
&\int_Q \overline{f} h \mathrm{d}\mu \\
&= \int_Q \prod_{i=1}^n \overline{A_i(q_{t_i})} \prod_{i=1}^p B_i(q_{s_i})\mathrm{d}\mu \\
&= \big(\Omega|\overline{A_1}\mathrm{e}^{(t_1-t_2)H} \cdots \mathrm{e}^{(t_{n-1}-t_n)H}\overline{A_n}\mathrm{e}^{(t_n-s_1)H}B_1\mathrm{e}^{(s_1-s_2)H}\cdots \mathrm{e}^{(s_{p-1}-s_p)H}B_p\Omega\big) \\
&= \big(\mathrm{e}^{t_nH}A_n\mathrm{e}^{(t_{n-1}-t_n)H}\cdots \mathrm{e}^{(t_1-t_2)H}A_1\Omega|\mathrm{e}^{-s_1H}B_1\mathrm{e}^{(s_1-s_2)H}\cdots \mathrm{e}^{(s_{p-1}-s_p)H}B_p\Omega\big).
\end{aligned}$$

By Lemma 21.28, there exists $C \in L^\infty(X,\nu)$ such that

$$\mathrm{e}^{t_nH}A_n\mathrm{e}^{(t_{n-1}-t_n)H}A_{n-1}\cdots \mathrm{e}^{(t_1-t_2)H}A_1\Omega = C\Omega,$$

and hence

$$\begin{aligned}
&\int_Q \overline{f} h \mathrm{d}\mu \\
&\quad = \big(\Omega|\overline{C}\mathrm{e}^{-s_1H}B_1\mathrm{e}^{(s_1-s_2)H}B_2\cdots \mathrm{e}^{(s_{p-1}-s_p)H}B_p\Omega\big) \\
&\quad = \int_Q \overline{C(q_0)} \prod_{i=1}^p B_i(q_{s_i})\mathrm{d}\mu.
\end{aligned}$$

Therefore, by (21.20) we have $(E_{[0,\infty[}f)(q) = C(q_0)$, which proves that $E_{[0,+\infty[}f$ is $\mathfrak{S}_0$-measurable and completes the proof of the Markov property.

To complete the proof of the theorem it remains to construct the unitary operator T. We first note that, since Ω is a.e. positive, $L^\infty(X,\nu)\Omega$ is dense in

$L^2(X,\nu)$. Using (21.18), it follows that the map

$$T : L^2(X,\nu) \to L^2(Q,\mathfrak{S}_0,\mu)$$
$$(TA\Omega)(q) := A(q_0), \quad A \in L^\infty(X,\nu), \quad q \in Q,$$

extends to a unitary operator. □

21.3 Perturbations of Markov path spaces

We fix a Markov path space $(Q,\mathfrak{S},\mathfrak{S}_0,U_t,R,\mu)$. Recall that this leads to a construction of a physical space $\mathcal{H}$ equipped with a Hamiltonian H. We will show how to perturb this Hamiltonian using the framework of Markov path spaces. Perturbations that can be treated by Euclidean methods are those corresponding to operators of multiplication by real $\mathfrak{S}_0$-measurable functions, i.e. by functions of time-zero fields. Sometimes the perturbation itself does not even make sense as an operator, although a perturbed Hamiltonian can be defined. These singular cases can be handled using the so-called *Feynman–Kac–Nelson kernels.*

21.3.1 Feynman–Kac–Nelson kernels

Definition 21.29 *Let $\delta \in [0,+\infty]$. A* local Feynman–Kac–Nelson (FKN) kernel *is a family $\{F_{[a,b]}\}_{0\le b-a<\delta}$ of $\mathfrak{S}$-measurable functions on Q such that*

(1) *$F_{[a,b]} > 0$, $F_{[a,b]} \in L^1(Q,\mathfrak{S}_{[a,b]})$,*
(2) *for $a \in \mathbb{R}$, the map $[a,a+\delta[\ni b \mapsto F_{[a,b]} \in L^1(Q)$ is continuous,*
(3) *$F_{[a,b]}F_{[b,c]} = F_{[a,c]}$, for $a \le b \le c$, $c-a<\delta$,*
(4) *$U_s\left(F_{[a,b]}\right) = F_{[a+s,b+s]}$, for $s \in \mathbb{R}$,*
(5) *$R\left(F_{[a,b]}\right) = F_{[-b,-a]}$.*

*If $\delta=\infty$ in the above definition, we will drop the word "local" and use the name "*FKN kernel*".*

Remark 21.30 *Let us mention a certain notational problem. Let F be a measurable function on Q. $U_t(F)$ denotes the image of F under the action of U_t. It is also a function on Q.*

The symbols F, resp. $U_t(F)$ are often understood as multiplication operators. Using this meaning, we have the identity

$$U_t(F) = U_t F U_t^*,$$

where now U_t on the r.h.s. is understood as a unitary operator on $L^2(Q,\mu)$. Clearly, if we use the latter interpretation of the FKN kernel, (4) of Def. 21.29 can be rewritten as $U_s F_{[a,b]} U_{-s}^* = F_{[a+s,b+s]}$.

Remark 21.31 *The simplest example of a FKN kernel is given by*

$$F_{[a,b]} := \mathrm{e}^{-\int_a^b U_s(V)\mathrm{d}s}, \tag{21.21}$$

where $V \in L^\infty(Q, \mathfrak{S}_0)$. At least formally, all FKN kernels are of this form. In fact, by (3), the operators of multiplication by $F_{[s,t]}$ form a two parameter semi-group. Their generators $V(t)$ are also operators of multiplication by $\mathfrak{S}_{\{t\}}$-measurable functions, commute with one another, and satisfy $U_s\big(V(t)\big) = V(t+s)$ by (4). Setting $V = V(0)$ we see that $F_{[a,b]}$ is formally given by (21.21).

Properties of FKN kernels obtained from formula (21.21) are described in the following lemma.

Lemma 21.32 *Let $1 \le p < \infty$ and $V \in L^p(Q)$. Then the following hold:*

(1) $\int_a^b U_s(V)\mathrm{d}s \in L^p(Q)$.
(2) $\|\mathrm{e}^{-\int_a^b U_s(V)\mathrm{d}s}\|_p \le \|\mathrm{e}^{-(b-a)V}\|_p = \|\mathrm{e}^{-p(b-a)V}\|_1^{1/p}$.
(3) *Let $V \in L^p(Q)$ for some $p > 1$, and $\mathrm{e}^{-\delta V} \in L^1(Q)$ for some $\delta > 0$. Set $F_{[a,b]} = e^{-\int_a^b U_s(V)\mathrm{d}s}$. Then $\{F_{[a,b]}\}_{0 \le b-a<\delta}$ is a local FKN kernel.*
(4) *Let $V \in L^1(Q)$ and $V \ge 0$. Then $\{F_{[a,b]}\}_{0 \le b-a<\infty}$ is a FKN kernel.*

Proof (1) follows from the strong continuity of U_t on $L^p(Q)$.

To prove (2), we apply Jensen's inequality,

$$\mathrm{e}^{-\int_a^b U_s(V)\mathrm{d}s} \le \frac{1}{b-a}\int_a^b \mathrm{e}^{-(b-a)U_s(V)}\mathrm{d}s,$$

and obtain

$$\|\mathrm{e}^{-\int_a^b U_s(V)\mathrm{d}s}\|_p \le \frac{1}{b-a}\int_a^b \|\mathrm{e}^{-(b-a)U_s(V)}\|_p\mathrm{d}s = \|\mathrm{e}^{-(b-a)V}\|_p,$$

since $\mathrm{e}^{-(b-a)U_s(V)} = U_s(\mathrm{e}^{-(b-a)V})$ and U_s is measure preserving.

To prove (3), we will use Subsect. 5.1.9. Write

$$F_{[a,b+\epsilon]} - F_{[a,b]} = (F_{[b,b+\epsilon]} - 1)F_{[a,b]}.$$

Since $F_{[a,b]} \in L^{\delta/(b-a)}(Q)$, it suffices by Hölder's inequality to prove that $F_{[b,b+\epsilon]} \to 1$ in $L^q(Q)$ for $q = \delta/(\delta - b + a)$. Since U_b is isometric on L^q, we may assume that $b = 0$. Clearly, $F_{[0,\epsilon]} \to 1$ a.e., when $\epsilon \to 0$. Hence, $F_{[0,\epsilon]} \to 1$ in measure. Using (2), we see that, for all $p' > 1$, $\|F_{[0,\epsilon]}\|_{p'} \le C$ uniformly for $0 \le \epsilon \le \delta/p'$. Hence, $\{F_{[0,\epsilon]} \ : \ 0 \le \epsilon \le \delta/p'\}$ is an equi-integrable family. By the Lebesgue–Vitali theorem (Thm. 5.32), $F_{[0,\epsilon]} \to 1$ in $L^q(Q)$.

Finally, statement (4) is immediate, since $F_{[a,b]} \le 1$ for all $a \le b$. □

21.3.2 Feynman–Kac–Nelson formula

We now describe the construction of a perturbed Hamiltonian associated with a FKN kernel.

We recall that local Hermitian semi-groups were defined in Subsect. 2.3.6.

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Proposition 21.33 *Let $\{F_{[a,b]}\}_{0\leq b-a\leq\delta}$, $\delta>0$, be a FKN kernel. For $0\leq t<\delta/2$, set*

$$\begin{aligned}\mathcal{D}_t &:= E_0\mathrm{Span}\left(\bigcup_{0\leq s<\delta/2-t}F_{[0,s]}L^\infty(Q,\mathfrak{S}_{[0,+\infty[})\right),\\ P_F(t) &:= E_0F_{[0,t]}U_t\big|_{\mathcal{D}_t}.\end{aligned}$$

Then $\{P_F(t),\mathcal{D}_t\}_{t\in[0,\delta/2]}$ is a local Hermitian semi-group.

Proof We check the conditions of Def. 2.67. Since $F_{[0,s]}$ belongs only to $L^1(Q)$ it is not obvious that $\mathcal{D}_t\subset L^2(Q,\mathfrak{S}_0)=\mathcal{H}$. To prove that this is the case, we write, for $f=E_0F_{[0,s]}g\in\mathcal{D}_t$, $0\leq s<\delta/2-t$,

$$\begin{aligned}\|f\|^2 &= (F_{[0,s]}g|E_0F_{[0,s]}g)=(F_{[0,s]}g|RE_0F_{[0,s]}g)\\ &=(F_{[-s,0]}Rg|E_0F_{[0,s]}g)=(F_{[-s,0]}Rg|E_{]-\infty,0]}E_{[0,+\infty[}F_{[0,s]}g)\\ &=(E_{]-\infty,0]}F_{[-s,0]}Rg|E_{]-\infty,0]}E_{[0,+\infty[}F_{[0,s]}g)=(F_{[-s,0]}Rg|F_{[0,s]}g)\\ &=(Rg|F_{[-s,0]}F_{[0,s]}g)=(Rg|F_{[-s,s]}g)\leq\|F_{[-s,s]}\|_1\|g\|_\infty^2.\end{aligned}\tag{21.22}$$

Since $0\leq s\leq\delta/2$, $F_{[-s,s]}\in L^1(Q)$ and the r.h.s. is finite. Since $L^\infty(Q,\mathfrak{S}_0)\subset\mathcal{D}_t$, $\mathcal{D}_t$ is dense in $\mathcal{H}$. We now claim that $P_F(s)\ \mathcal{D}_t\subset\mathcal{D}_{t-s}$ for $0\leq s\leq t\leq\delta/2$. In fact, if $f=E_0F_{[0,s_1]}g\in\mathcal{D}_t$, for $0\leq s_1\leq\delta/2-t$, we have

$$\begin{aligned}P_F(s)f &= E_0F_{[0,s]}U_sE_0F_{[0,s_1]}g \;=\; E_0F_{[0,s]}E_{\{s\}}F_{[s,s+s_1]}U_sg\\ &=E_0F_{[0,s]}E_{]-\infty,s]}E_{[s,+\infty[}F_{[s,s+s_1]}U_sg \;=\; E_0F_{[0,s]}E_{]-\infty,s]}F_{[s,s+s_1]}U_sg\\ &=E_0E_{]-\infty,s]}F_{[0,s]}F_{[s,s+s_1]}U_sg \;=\; E_0E_{]-\infty,s]}F_{[0,s+s_1]}U_sg\\ &=E_0F_{[0,s+s_1]}U_sg\in\mathcal{D}_{t-s},\end{aligned}\tag{21.23}$$

where we have used the properties of $F_{[a,b]}$ and the Markov property. The identity (21.23) also proves that if $f=E_0F_{[0,s_1]}g\in\mathcal{D}_{t+s}$ for $0\leq s_1\leq\delta/2-(t+s)$, then $P_F(t)P_F(s)f=P_F(t+s)f$.

Let us prove the weak continuity of $P_F(t)$. For $f=E_0F_{[0,s_1]}g\in\mathcal{D}_s$ and $0\leq s_1\leq\delta/2-s$ as above, we have

$$\big(f|P_F(t)f\big)_{\mathcal{H}}=\big(F_{[0,s_1]}g|E_0F_{[0,s_1+t]}U_tg\big)=\big(Rg|F_{[-s_1,s_1+t]}U_tg\big),$$

by the same arguments as in (21.22) and (21.23). Hence,

$$\begin{aligned}\big(f|P_F(t+\epsilon)f\big)-\big(f|P_F(t)f\big) &= \Big(Rg|\big(F_{[-s_1,s_1+t+\epsilon]}-F_{[-s_1,s_1+t]}\big)U_{t+\epsilon}g\Big)\\ &\quad+\Big(Rg|F_{[-s_1,s_1+t]}\big(U_{t+\epsilon}g-U_tg\big)\Big).\end{aligned}$$

The first term tends to 0 when $\epsilon\to0$ by Def. 21.29. The second term tends to 0 when $\epsilon\to0$ by the σ-weak continuity of $t\mapsto U_t$ on $L^\infty(Q)$. □

In the next two propositions we give examples of FKN kernels obtained from a real $\mathfrak{S}_0$-measurable function V as in Lemma 21.32. Note that the Hermitian

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operators $P_F(t)$ are now denoted by $P_V(t)$ and have slightly bigger domains. This choice will be convenient in the next subsection.

The case of positive perturbations is easier:

Proposition 21.34 *Let V be a real $\mathfrak{S}_0$-measurable function such that $V \in L^1(Q)$ and $V(x) \geq 0$ a.e. Then*

$$P_V(t) = E_0 \mathrm{e}^{-\int_0^t U_s(V)\mathrm{d}s} U_t, \quad t \geq 0,$$

is a strongly continuous semi-group of bounded self-adjoint operators on $L^2(Q, \mathfrak{S}_0, \mu)$.

In the case of arbitrary perturbations we need to use the notion of a Hermitian semi-group.

Proposition 21.35 *Let V be a real $\mathfrak{S}_0$-measurable function such that $V \in L^{p_0}(Q)$ for some $p_0 > 2$, and $\mathrm{e}^{-\delta V} \in L^1(Q)$ for some $\delta > 0$. Set $p(t)^{-1} := 1/2 - t/\delta$ for $0 \leq t \leq \delta/2$, and*

$$P_V(t) = E_0 \mathrm{e}^{-\int_0^t U_s(V)\mathrm{d}s} U_t \big|_{L^{p(t)}(Q,\mathfrak{S}_0,\mu)}.$$

Then $\left\{P_V(t), L^{p(t)}(Q, \mathfrak{S}_0, \mu)\right\}_{t\in[0,\delta/2]}$ is a local Hermitian semi-group.

Proof It follows from Lemma 21.32 that $\int_0^t U_s(V)\mathrm{d}s$ is well defined in $L^{p_0}(Q)$, and that $\mathrm{e}^{-\int_0^t U_s(V)\mathrm{d}s} \in L^p(Q)$ for $0 \leq t \leq \delta/p$. By Hölder's inequality, $P_V(t)$ maps $L^{p(t)}(Q)$ into $L^2(Q)$, so $P_V(t)$ is well defined on $L^{p(t)}(Q, \mathfrak{S}_0)$. The fact that $P_V(t)$ maps $L^{p(t+s)}$ into $L^{p(s)}$ follows also from Hölder's inequality. The proofs of the semi-group and weak continuity properties are completely analogous to those of Prop. 21.33. □

Remark 21.36 *Let us write the physical Hilbert space as $\mathcal{H} = L^2(X, \nu)$. We treat paths (elements of Q) as functions $\mathbb{R} \ni t \mapsto q_t \in X$. The expectation E_t is written as*

$$E_t G(x) =: \int G(q)\mathrm{d}\mu_{t,x}(q), \quad G \in L^1(Q, \mathrm{d}\mu), \quad x \in X.$$

Let V be a real function on X. Under some conditions on V (see for example Thms. 21.37, 21.38) one can show that $P_V(t) = E_0 \mathrm{e}^{\int_0^t U_s(V)\mathrm{d}s} U_t = \mathrm{e}^{-t(H+V)}$ for $t \geq 0$. This can be formally rewritten as

$$\mathrm{e}^{-t(H_0+V(x))} f(x) = \int \exp\Big(-\int_0^t V(q_s)\mathrm{d}s\Big) f(q_t)\mathrm{d}\mu_{0,x}(q). \tag{21.24}$$

Recall that in Thm. 21.4 we described the Feynman–Kac formula for the integral kernel of $\mathrm{e}^{-t(-\frac{1}{2}\Delta+V(x))}$. The generalization (21.24) of the Feynman–Kac formula to quantum field theory was first given by Nelson. Therefore, in this context, (21.24) is usually called the Feynman–Kac–Nelson formula.

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21.3.3 Perturbed Hamiltonians

We recall that H is the positive self-adjoint operator generating the group $\{P(t)\}_{t\in[0,\infty[}$ constructed in Thm. 21.11.

Let V be a real $\mathfrak{S}_0$-measurable function. The self-adjoint operator of multiplication by V on $L^2(Q,\mathfrak{S}_0,\mu)$ is also denoted by V. Under the hypotheses of Prop. 21.34 we can define a unique positive self-adjoint operator H_V such that $P_V(t)=\mathrm{e}^{-tH_V}$. Similarly, using Thm. 2.69, under the hypotheses of Prop. 21.35, we can define a unique self-adjoint operator H_V such that $P_V(t)\subset \mathrm{e}^{-tH_V}$. We now give without proof some results about the Hamiltonian H_V.

Theorem 21.37 (Positive perturbations) *Assume the hypotheses of Prop. 21.34. Then:*

(1) *H_V is bounded below.*
(2) *If $V\in L^p(Q)$ for $p>1$, then H_V is a restriction of the form sum $H+V$.*
(3) *If $V\in L^p(Q)$ for $p\geq 2$, then H_V is the closure of $H+V$.*

Theorem 21.38 (Arbitrary perturbations) *Assume the hypotheses of Prop. 21.35. Then:*

(1) *H_V is the closure of $H+V$.*
(2) *Assume that e^{-tH} is hyper-contractive on $L^2(Q,\mathfrak{S}_0)$ and let $T>0$ be such that e^{-TH} maps $L^2(Q)$ into $L^r(Q)$, $r>2$. Then if $\mathrm{e}^{-\delta V}\in L^1(Q)$ for $\delta=r'T$, $1/r+1/r'=1/2$, H_V is bounded below.*

Remark 21.39 *The main examples of models with local interaction that can be treated by the methods of this chapter are the (space-cutoff) $P(\varphi)_2$ and $(\mathrm{e}^{\alpha\varphi})_2$ models (both at 0 and at positive temperature). The $P(\varphi)_2$ model was the first model with a local interaction to be rigorously constructed. It will be further studied in Chap. 22.*

The $(\mathrm{e}^{\alpha\varphi})_2$ model is also called the Høgh-Krohn model. *Although not physical, it has the pedagogical advantage that the interaction term $\int g(x)\mathrm{e}^{\alpha\varphi(x)}\mathrm{d}x$ is positive, even after Wick ordering. It provides an example of where Feynman–Kac–Nelson kernels can be used even if the formal interaction does not exist. In fact, one can show that the formal interaction*

$$\int g(x):\mathrm{e}^{\alpha\varphi(x)}:\mathrm{d}x$$

for g a positive compactly supported function can be given a rigorous meaning iff $|\alpha|<\sqrt{2\pi}$, although the FKN kernels

$$\int_a^b\int g(x):\mathrm{e}^{\alpha\varphi(t,x)}:\mathrm{d}x\mathrm{d}t$$

are well defined iff $|\alpha|<\sqrt{4\pi}$; see Simon (1974).

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Another example where one has to use FKN kernels is the $P(\varphi)(0)$ model, obtained by replacing the space-cutoff $g(x)$ with the delta function δ_0; see Klein–Landau (1975).

21.4 Euclidean approach at positive temperatures

There exists a version of the Euclidean approach for bosonic fields at positive temperatures. The "Euclidean time", which at the zero temperature took values in $\mathbb{R}$, now belongs to the circle of length β. The number β has the meaning of inverse temperature. Given a β-Markov path space, we construct a von Neumann algebra equipped with a W^*-dynamics and a KMS state.

21.4.1 β-Markov path spaces

Definition 21.40 *The* circle of length β*, that is,* $\mathbb{R}/\beta\mathbb{Z}$*, is denoted by* S_β*, and is sometimes identified with* $]-\beta/2, \beta/2]$*.* t *will still denote the generic variable in* S_β*.*

Definition 21.41 *Let* $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ *be a generalized path space as in Def. 21.7. The path space is called*

(1) β-periodic *if* $U_\beta = \mathbb{1}$*, so that* $S_\beta \ni t \mapsto U_t$ *is a strongly continuous unitary group.*
(2) β-Markov *if in addition it satisfies*
   (i) *the* β-reflection property $RE_{\{0,\beta/2\}} = E_{\{0,\beta/2\}}$,
   (ii) *the* β-Markov property $E_{[0,\beta/2]}E_{[-\beta/2,0]} = E_{\{0,\beta/2\}}$.

It is easy to show that in a β-Markov path space we have

$$E_{\{0,\beta/2\}} = E_{[0,\beta/2]}RE_{[0,\beta/2]}. \tag{21.25}$$

21.4.2 Reconstruction theorem

We assume that we are given a β-Markov path space $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$. As in Subsect. 21.2.1, we now proceed to the construction of the corresponding physical objects.

Definition 21.42 *The* physical Hilbert space *is*

$$\mathcal{H} := E_{\{0,\beta/2\}}L^2(Q, \mathfrak{S}, \mu) = L^2(Q, \mathfrak{S}_{\{0,\beta/2\}}, \mu),$$

and the vector $1 \in \mathcal{H}$ *will be denoted by* Ω*. The Abelian von Neumann algebra* $L^\infty(Q, \mathfrak{S}_{\{0\}}, \mu)$ *acting on* $\mathcal{H}$ *will be denoted by* $\mathfrak{A}$*.*

The construction of the generator of the dynamics on $\mathcal{H}$ is now more delicate than in Thm. 21.11, because U_t does not preserve $L^2(Q, \mathfrak{S}_{[0,\beta/2]}, \mu)$. In fact, U_t sends $L^2(Q, \mathfrak{S}_{[0,\beta/2]}, \mu)$ into $L^2(Q, \mathfrak{S}_{[t,t+\beta/2]}, \mu)$. In the construction the crucial role is played by the concept of a *local Hermitian semi-group* introduced in Def. 2.67.

Theorem 21.43 *Set, for* $0 \leq t < \beta/2$,

$$\begin{aligned}\mathcal{M}_t &:= L^2(Q, \mathfrak{S}_{[0,\beta/2-t]}, \mu),\\ \mathcal{D}_t &:= E_{\{0,\beta/2\}}\mathcal{M}_t \subset \mathcal{H}.\end{aligned}$$

Then, for any $0 \leq s \leq t \leq \beta/2$, *there exists a unique* $P(s) : \mathcal{D}_t \to \mathcal{D}_{t-s}$ *such that*

$$P(s)E_{\{0,\beta/2\}}f = E_{\{0,\beta/2\}}U_s f, \quad f \in L^2(Q, \mu),$$

and $\big\{\mathcal{D}_t, P(t)\big\}_{t\in[0,\beta/2]}$ *is a local Hermitian semi-group on* $\mathcal{H}$ *preserving* Ω.

Proof If $s \leq t$, one has $\mathcal{M}_t \subset \mathcal{M}_s$, hence $\mathcal{D}_t \subset \mathcal{D}_s$. From the definition of $\mathfrak{S}_I$ as $\bigvee_{t\in I} U_t\mathfrak{S}_0$ and from the strong continuity of U_t on $L^2(Q, \mathfrak{S}, \mu)$, we see that $\underset{0<t<\beta/2}{\cup} \mathcal{M}_t$ is dense in $L^2(Q, \mathfrak{S}_{[0,\beta/2]}, \mu)$, which implies the density of $\underset{0<t<\beta/2}{\cup} \mathcal{D}_t$ in $\mathcal{H}$.

We now have to check that, for $0 \leq t < \beta/2$, $P(t)$ is well defined as a linear operator on $\mathcal{D}_t$.

Let us fix $0 < r \leq s < t < \beta/2$ with $r + s \leq t$. Let $f \in \mathcal{M}_t$. We have

$$\begin{aligned}&\|E_{\{0,\beta/2\}}U_s f\|^2 = (U_s f|RU_s f)\\ &= (U_{s-r}f|U_{-r}RU_s f) = (U_{s-r}f|RU_{s+r}f)\\ &= (U_{s-r}f|E_{\{0,\beta/2\}}U_{s+r}f) \leq \|E_{\{0,\beta/2\}}U_{s-r}f\|\|f\|.\end{aligned} \tag{21.26}$$

In the first line we use (21.25) and the fact that $U_s f$ is $\mathfrak{S}_{[0,\beta/2]}$-measurable. In the second line we use the unitarity of U_{-r} and $U_{-r}R = RU_r$. In the third line we apply (21.25) again, the Cauchy–Schwarz inequality and the fact that $E_{\{0,\beta/2\}}$ and U_{s+r} are contractions.

Taking $r = s$, we obtain that $\|E_{\{0,\beta/2\}}U_s f\| \leq \|E_{\{0,\beta/2\}}f\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}}$ for $2s \leq t$. If $\frac{n+1}{n}s \leq t$, for $n \in \mathbb{N}$, taking $r = s/n$ and applying recursively (21.26), we obtain

$$\begin{aligned}\|E_{\{0,\beta/2\}}U_s f\| &\leq \|E_{\{0,\beta/2\}}U_{s-r}f\|^{\frac{1}{2}}\|f\|^{\frac{1}{2}}\\ &\leq \|E_{\{0,\beta/2\}}U_{s-pr}f\|^{2^{-p}}\|f\|^{(2^{-1}+\cdots+2^{-p})}\\ &\leq \|E_{\{0,\beta/2\}}f\|^{2^{-n}}\|f\|^{(1-2^{-n})}.\end{aligned}$$

This shows that $E_{\{0,\beta/2\}}f = 0$ implies $E_{\{0,\beta/2\}}U_s f = 0$ for all $0 \leq s < t$. By the strong continuity of U_s, this extends to $s = t$. Thus we have proved

$$E_{\{0,\beta/2\}}f = 0,\ f \in \mathcal{M}_t \ \Rightarrow\ E_{\{0,\beta/2\}}U_t f = 0, \tag{21.27}$$

which means that $P(t)$ is well defined.

The semi-group property of $P(t)$ and the fact that $P(s)\mathcal{D}_t \subset \mathcal{D}_{t-s}$ are immediate. To prove that $P(t)$ is Hermitian, we write, for $f, g \in \mathcal{M}_t$,

$$\begin{aligned}&\big(E_{\{0,\beta/2\}}f|P(t)E_{\{0,\beta/2\}}g\big) = (f|RU_t g) = (U_t f|Rg)\\ &= \big(E_{\{0,\beta/2\}}U_t f|E_{\{0,\beta/2\}}g\big) = \big(P(t)E_{\{0,\beta/2\}}f|E_{\{0,\beta/2\}}g\big).\end{aligned}$$

Finally, the weak continuity of $P(t)$ follows from the strong continuity of U_t. $\square$

Definition 21.44 *The unique self-adjoint operator* L *on* $\mathcal{H}$ *such that* $\mathrm{e}^{-tL}\big|_{\mathcal{D}_t} = P(t)$ *is called the* Liouvillean.

Clearly, $L\Omega = 0$.

Definition 21.45 *We denote by* $\mathfrak{F} \subset B(\mathcal{H})$ *the von Neumann algebra defined by*

$$\mathfrak{F} := \left\{\mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathfrak{A}, \quad t \in \mathbb{R}\right\}''. \tag{21.28}$$

Let

$$R_{\beta/4} := U_{\beta/2} R = U_{\beta/4} R U_{-\beta/4}$$

be the reflection around $s = \beta/4$. Clearly,

$$R_{\beta/4} E_{\{0,\beta/2\}} = E_{\{0,\beta/2\}} R_{\beta/4}. \tag{21.29}$$

Definition 21.46 *By (21.29),*

$$J E_{\{0,\beta/2\}} f := E_{\{0,\beta/2\}} R_{\beta/4} \overline{f}, \quad f \in L^2(Q, \mathfrak{S}, \mu), \tag{21.30}$$

defines an anti-unitary operator J *on* $\mathcal{H}$*. We also introduce a state and a* W^**-dynamics on* $\mathfrak{F}$*:*

$$\omega(A) := (\Omega|A\Omega), \quad \tau_t(A) = \mathrm{e}^{\mathrm{i}tL} A \mathrm{e}^{-\mathrm{i}tL}, \quad A \in \mathfrak{F}.$$

The next theorem will be proven in the following two subsections.

Theorem 21.47 ω *is a faithful state, it satisfies the* β*-KMS condition for the dynamics* τ*,* J *is the modular conjugation corresponding to* ω *and* L *is the standard Liouvillean for the dynamics* τ*.*

21.4.3 Proof of the KMS condition

In this subsection we prove the part of Thm. 21.47 saying that ω is β-KMS. We first need to introduce additional notation. For $n \in \mathbb{N}$, we set

$$J_\beta(n) := \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n \ : \ t_j \geq 0, \ \sum_{j=1}^n t_j \leq \beta/2\right\},$$

$$I_\beta(n) := \left\{(z_1, \ldots, z_n) \in \mathbb{C}^n \ : \ \mathrm{Re}\, z_j > 0, \ \sum_{j=1}^n \mathrm{Re}\, z_j < \beta/2\right\}.$$

Note that $J_\beta(n) \subset I_\beta(n)^{\mathrm{cl}}$. We denote by $\mathrm{Hol}_\beta(n)$ the space of functions (with values in $\mathcal{H}$ or in $\mathbb{C}$, depending on the context) which are holomorphic in $I_\beta(n)$ and continuous in $I_\beta(n)^{\mathrm{cl}}$.

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Proposition 21.48 (1) *For* $(t_1,\dots,t_n)\in J_\beta(n)$, $A_1,\dots,A_n\in\mathfrak{A}$, *the vector*

$$A_n\prod_{j=n-1}^{1}(\mathrm{e}^{-t_jL}A_j)\Omega \tag{21.31}$$

belongs to $\mathrm{Dom}\,\mathrm{e}^{-t_nL}$.

(2) *The linear span of such vectors is dense in* $\mathcal{H}$.

(3) *Let* $(t_1,\dots,t_n)\in J_\beta(n)$, $(s_1,\dots,s_m)\in J_\beta(m)$ *and* $A_1,\dots,A_n,B_1,\dots,B_m\in\mathfrak{A}$. *Set* $t_0:=\beta/2-(t_n+\cdots+t_1)$, $s_0:=\beta/2-(s_n+\cdots+s_1)$. *Then one has*

$$\begin{aligned}&\Big(\prod_{j=n}^{1}(\mathrm{e}^{-t_jL}A_j)\Omega|\prod_{i=m}^{1}(\mathrm{e}^{-s_iL}B_i)\Omega\Big)\\&=\Big(\prod_{i=0}^{m-1}(\mathrm{e}^{-s_iL}B_{i+1}^*)\Omega|\prod_{j=0}^{n-1}(\mathrm{e}^{-t_jL}A_{j+1}^*)\Omega\Big).\end{aligned}\tag{21.32}$$

Proof For $A\in\mathfrak{A}$, we set $A(t)=U_t(A)$. First let us show that

$$\prod_{j=n}^{1}(\mathrm{e}^{-t_jL}A_j)\Omega=E_{\{0,\beta/2\}}\prod_{j=n}^{1}A_j(t_j+\cdots+t_n).\tag{21.33}$$

We use induction. (21.33) is clear for $n=1$. Assume that it is true for $n-1$, that is,

$$\prod_{j=n-1}^{1}(\mathrm{e}^{-t_jL}A_j)\Omega=E_{\{0,\beta/2\}}\prod_{j=n-1}^{1}A_j(t_j+\cdots+t_{n-1}).$$

Then

$$\begin{aligned}\mathrm{e}^{-t_nL}A_n\prod_{j=n-1}^{1}(\mathrm{e}^{-t_jL}A_j)\Omega&=E_{\{0,\beta/2\}}U_{t_n}A_n\prod_{j=n-1}^{1}A_j(t_j+\cdots+t_{n-1})\\&=E_{\{0,\beta/2\}}\prod_{j=n}^{1}A_j(t_j+\cdots+t_n),\end{aligned}$$

which proves (21.33) for n.

Since $A_n\prod_{j=n-1}^{1}A_j(t_j+\cdots+t_{n-1})$ belongs to $\mathcal{M}_{t_n}$, this proves that (21.31) belongs to $\mathrm{Dom}\,\mathrm{e}^{-t_nL}$. Hence, (1) is true.

The linear span of vectors as on the r.h.s. of (21.33) is dense in $L^2(Q,\mathfrak{S}_{[0,\beta/2]},\mu)$, which proves (2).

We have

$$\begin{aligned}&\Big(\prod_{j=n}^{1}(\mathrm{e}^{-t_jL}A_j)\Omega|\prod_{j=m}^{1}(\mathrm{e}^{-s_iL}B_i)\Omega\Big)&&(21.34)\\&=\Big(\prod_{j=n}^{1}A_j(\tau_j)|R\prod_{i=m}^{1}B_i(\sigma_i)\Big)_{L^2(Q)}\\&=\Big(\prod_{j=n}^{1}A_j(\tau_j)|\prod_{i=m}^{1}B_i(-\sigma_i)\Big)_{L^2(Q)},&&(21.35)\end{aligned}$$

where

$$\tau_j=\sum_{k=j}^{n}t_k,\quad 1\le j\le n,\qquad \sigma_i=\sum_{k=i}^{m}t_k,\quad 1\le i\le m.\tag{21.36}$$

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Since $U_\beta = \mathbb{1}$, we have

$$(f|g) = (U_{-\beta/2}f|U_{\beta/2}g)_{L^2(Q)}, \quad f, g \in L^2(Q).$$

Hence, (21.35) equals

$$\begin{aligned}
&\Big(\prod_{j=n}^{1} A_j(-\beta/2+\tau_j)|\prod_{j=m}^{1} B_i(\beta/2-\sigma_i)\Big)_{L^2(Q)} \\
&= \Big(\prod_{i=1}^{m} \overline{B_i(\beta/2-\sigma_i)}|\prod_{j=1}^{n} \overline{A_j(-\beta/2+\tau_j)}\Big)_{L^2(Q)} \\
&= \Big(\prod_{i=0}^{m-1}(\mathrm{e}^{-s_i L}B_{i+1}^*)\Omega|\prod_{j=0}^{n-1}(\mathrm{e}^{-t_j L}A_{j+1}^*\Omega\Big).
\end{aligned}$$

This proves (3). □

Proposition 21.49 *For $(z_1, \dots, z_n) \in I_\beta(n)^{\mathrm{cl}}$ and $A_1, \dots, A_n \in \mathfrak{A}$, the vector*

$$A_n \prod_{n-1}^{1}(\mathrm{e}^{-z_j L}A_j)\Omega \tag{21.37}$$

belongs to $\mathrm{Dom}\,\mathrm{e}^{-z_n L}$. *Furthermore, the function*

$$I_\beta(n)^{\mathrm{cl}} \ni (z_1, \dots, z_n) \mapsto \prod_{j=n}^{1}(\mathrm{e}^{z_j L}A_j)\Omega$$

belongs to $\mathrm{Hol}_\beta(n)$ *and is bounded by* $\prod_{j=1}^{n}\|A_j\|$.

Proof We prove the result by induction in n.

By Prop. 21.48 (1), $A_1\Omega \in \mathrm{Dom}\,\mathrm{e}^{-\beta L/2}$. Therefore, Prop. 2.63 implies that the map $I_\beta(1) \ni z_1 \mapsto \mathrm{e}^{-z_1 L}A_1\Omega$ belongs to $\mathrm{Hol}_\beta(1)$. Moreover, for $z_1 \in I(1)$, we have

$$\|\mathrm{e}^{\mathrm{i}z_1 L}A_1\Omega\| = \|\mathrm{e}^{-(\mathrm{Re}\,z_1)L}A_1\Omega\| = \|A_1^*\Omega\| \le \|A_1\|,$$

again using Prop. 2.63. This proves the result for $n = 1$.

Assume that the result holds for $n-1$. For $(z_1, \dots, z_{n-1}) \in I_\beta(n-1)$, set

$$\begin{aligned}
g(z_1, \dots, z_{n-1}) &= A_n \prod_{j=n-1}^{1}(\mathrm{e}^{z_j L}A_j)\Omega, \\
h(z_1, \dots, z_{n-1}) &= \prod_{j=1}^{n-1}(A_j^*\mathrm{e}^{z_j L})A_n^*\Omega.
\end{aligned}$$

By the induction assumption, g, h belong to $\mathrm{Hol}_\beta(n-1)$ and are bounded by $\prod_{j=1}^{n-1}\|A_j\|$. Moreover, using (3) of Prop. 21.48 with $m = n$, $B_j = A_j$ and

$$s_n = \beta/2 - \sum_{i=1}^{n-1} s_i, \quad t_n = \beta/2 - \sum_{j=1}^{n-1} t_j,$$

we obtain that

$$g(t_1, \dots, t_{n-1}) \in \mathrm{Dom}\,\mathrm{e}^{-t_n L}, \qquad g(s_1, \dots, s_{n-1}) \in \mathrm{Dom}\,\mathrm{e}^{-s_n L},$$

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and

$$\begin{aligned}&\big(\mathrm{e}^{-t_nL}g(t_1,\dots,t_{n-1})|\mathrm{e}^{-s_nL}g(s_1,\dots,s_{n-1})\big)\\&\quad=\big(h(s_1,\dots,s_{n-1})|h(t_1,\dots,t_{n-1})\big),\end{aligned}\tag{21.38}$$

for $(s_1,\dots,s_n)$, $(t_1,\dots,t_{n-1})\in J_\beta(n-1)$. Denote by $\mathcal{H}_f$, resp. $\mathcal{H}_h$ the closed subspaces of $\mathcal{H}$ generated by the vectors $\mathrm{e}^{-t_nL}g(t_1,\cdots,t_{n-1})$, resp. $h(t_1,\cdots,t_{n-1})$, for $(t_1,\dots,t_{n-1})\in J_\beta(n-1)$.

Note that $h(z_1,\dots,z_{n-1})$ belongs to $\mathcal{H}_h$ for $(z_1,\dots,z_{n-1})\in I_\beta(n-1)^{\mathrm{cl}}$. In fact, let $\Psi\perp\mathcal{H}_h$. Then

$$\big(\Psi|h(z_1,\dots,z_{n-1})\big)=0,\quad z_1,\dots,z_{n-1}\in J_\beta(n-1).\tag{21.39}$$

Hence, by analyticity and continuity, (21.39) is true for $z_1,\dots,z_{n-1}\in I_\beta(n-1)^{\mathrm{cl}}$.

From (21.38) we see that there exists a unique anti-unitary map $T:\mathcal{H}_f\to\mathcal{H}_h$ such that

$$T\mathrm{e}^{-t_nL}g(t_1,\dots t_{n-1})=h(t_1,\dots,t_{n-1}).$$

It follows that

$$f(z_1,\dots,z_{n-1}):=T^{-1}h(\overline{z_1},\dots,\overline{z_{n-1}})$$

belongs to $\mathrm{Hol}_\beta(n-1)$. Note that, by the definition of T, for $t_1,\dots,t_{n-1}\in J_\beta(n-1)$ one has

$$f(t_1,\dots,t_{n-1})=\mathrm{e}^{-t_nL}g(t_1,\dots,t_{n-1}).\tag{21.40}$$

We claim that, for $z_1,\dots,z_{n-1}\in I_\beta(n-1)$,

$$g(z_1,\dots,z_{n-1})\in\mathrm{Dom}\,\mathrm{e}^{-(\beta/2-\sum_{j=1}^{n-1}z_j)L}\tag{21.41}$$

and

$$f(z_1,\dots,z_{n-1})=\mathrm{e}^{-(\beta/2-\sum_{j=1}^{n-1}z_j)L}g(z_1,\cdots,z_{n-1}).\tag{21.42}$$

In fact, the scalar products of the above two functions with a fixed vector $\Psi\in\mathrm{Dom}\,\mathrm{e}^{-\beta L/2}$ belong to $\mathrm{Hol}_\beta(n-1)$ and coincide on $J_\beta(n-1)$ by (21.40). By analytic continuation it follows that $g(z_1,\dots,z_{n-1})$ belongs to $\mathrm{Dom}\,\mathrm{e}^{-(\beta/2-\sum_{j=1}^{n-1}z_j)L}$, and that (21.41) and (21.42) are true.

By Prop. 2.63, we obtain that the function

$$\Big\{0\le z_n\le\beta/2-\sum_{j=1}^{n-1}\mathrm{Im}\,z_j\Big\}\ni z_n\mapsto\mathrm{e}^{z_nL}g(z_1,\dots,z_{n-1})$$

is continuous and analytic in the interior of its domain. For $\mathrm{Re}\,z_n=0$, we have

$$\|\mathrm{e}^{z_nL}g(z_1,\dots,z_{n-1})\|=\|g(z_1,\dots,z_{n-1})\|\le\prod_{j=1}^{n}\|A_j\|.$$

For $\operatorname{Re} z_n = \beta/2 - \sum_{j=1}^{n-1} \operatorname{Im} z_j$,

$$\begin{aligned}\|e^{z_n L} g(z_1, \dots, z_{n-1})\| &= \|f(z_1, \dots, z_{n-1})\| \\ &= \|h(\overline{z}_1, \dots, \overline{z}_{n-1})\| \le \prod_{j=1}^{n} \|A_j\|.\end{aligned}$$

Therefore, by Prop. 2.63, for $0 \le z_n \le \beta/2 - \sum_{j=1}^{n-1} \operatorname{Im} z_j$,

$$\|e^{z_n L} g(z_1, \dots, z_{n-1})\| \le \prod_{j=1}^{n} \|A_j\|,$$

which ends the proof of the induction step. □

Proof of Thm. 21.47, Part 1. By Prop. 21.49, we can analytically continue (21.42) to obtain

$$\begin{aligned}&\Big(\prod_{j=n}^{1} (e^{-it_j L} A_j)\Omega | \prod_{i=m}^{1} (e^{-is_i L} B_i)\Omega\Big) \\ &\quad = \Big(e^{(-\beta/2 + is_m + \cdots + is_1)L} \prod_{i=1}^{m} (B_i^* e^{is_i L})\Omega | e^{(-\beta/2 + it_n + \cdots + it_1)L} \prod_{j=1}^{n} (A_j^* e^{it_j L})\Omega\Big).\end{aligned}$$

Changing variables, this can be rewritten as

$$\begin{aligned}(A\Omega|B\Omega) &= \big(e^{-\beta L/2} B^*\Omega | e^{-\beta L/2} A^*\Omega\big), \\ A &:= \prod_{j=n}^{1} \tau_{\tau_j}(A_j), \\ B &:= \prod_{i=m}^{1} \tau_{\sigma_i}(B_i).\end{aligned}$$

This identity implies that the (τ, β)-KMS condition (6.7) is satisfied in the $*$-algebra $\mathfrak{F}_0$ generated by $\{\tau_t(A) \ : \ A \in \mathfrak{A}, \ t \in \mathbb{R}\}$. But $\mathfrak{F}_0$ is weakly dense in $\mathfrak{F}$. By Prop. 6.64, the (τ, β)-KMS condition is satisfied for all $A, B \in \mathfrak{F}$. □

21.4.4 Identification of the modular conjugation

To complete the proof of Thm. 21.47, we need the following lemma:

Lemma 21.50 (1) *$JA\Omega = e^{-\beta L/2} A^*\Omega$, for all $A \in \mathfrak{F}$;*
(2) *$Je^{itL} = e^{itL} J$, for all $t \in \mathbb{R}$;*
(3) *$J\mathfrak{F}J \subset \mathfrak{F}'$.*

Proof Let $(t_1, \dots, t_n) \in J_\beta(n)$ and $A_1, \dots, A_n \in \mathfrak{A}$. Then

$$\begin{aligned}J \prod_{j=n}^{1} (e^{-t_j L} A_j)\Omega &= E_{\{0,\beta/2\}} R_{\beta/4} \prod_{j=n}^{1} \overline{A_j(\tau_j)} \\ &= E_{\{0,\beta/2\}} \prod_{j=n}^{1} \overline{A_j(\beta/2 - \tau_j)} \\ &= e^{-(\beta/2 - \sum_{j=1}^{n} t_j)L} \prod_{j=1}^{n-1} (A_j^* e^{-t_j L}) A_n^* \Omega,\end{aligned} \tag{21.43}$$

where τ_j are defined in (21.36).

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By Prop. 21.49, we can apply the analytic continuation to the above identity and obtain

$$J \prod_{j=n}^{1} (\mathrm{e}^{\mathrm{i}t_j L} A_j)\Omega = \mathrm{e}^{-\beta L/2} \mathrm{e}^{\mathrm{i}\sum_{j=1}^{n} t_j L} \prod_{j=1}^{n-1} (A_j^* \mathrm{e}^{-\mathrm{i}t_j L}) A_n^* \Omega.$$

Changing variables, this can be rewritten as

$$J A\Omega = \mathrm{e}^{-\beta L/2} A^* \Omega, \tag{21.44}$$

for

$$A = \prod_{j=n}^{1} \tau_{t_j}(A_j),$$

which proves (1) on $\mathfrak{F}_0$.

Now let $A \in \mathfrak{F}$. Since $\mathfrak{F}$ is the strong closure of $\mathfrak{F}_0$, by the Kaplansky density theorem there exists a sequence $A_n \in \mathfrak{F}_0$ such that $A_n \to A$, $A_n^* \to A^*$ strongly. Applying (21.44) to A_n, we obtain that $A_n\Omega \to A\Omega$ and $\mathrm{e}^{-\beta L/2} A_n \Omega \to J A^* \Omega$. Since $\mathrm{e}^{-\beta L/2}$ is closed, this implies that

$$\mathrm{e}^{-\beta L/2} A\Omega = J A^* \Omega, \;\; A \in \mathfrak{F}. \tag{21.45}$$

This proves (1) on $\mathfrak{F}$.

Let us now prove (2). Let $\Psi = E_{\{0,\beta/2\}} F$ for $F \in L^2(Q, \mathfrak{S}_{[\epsilon,\beta/2-\epsilon]}, \mu)$ and $\epsilon > 0$. For $0 \leq s < \epsilon$, we have

$$J\mathrm{e}^{-sL}\Psi = E_{\{0,\beta/2\}} R_{\beta/4} \overline{U_s F} = E_{\{0,\beta/2\}} U_{-s} R_{\beta/4} \overline{F}.$$

Since $U_{-s} R_{\beta/4} \overline{F} \in L^2(Q, \mathfrak{S}_{[\epsilon-s,\beta/2-\epsilon+s]})$, it follows that $J\mathrm{e}^{-sL}\Psi \in \operatorname{Dom} \mathrm{e}^{-sL}$ and

$$\mathrm{e}^{-sL} J \mathrm{e}^{-sL} \Psi = E_{\{0,\beta/2\}} R_{\beta/4} \overline{F} = J\Psi,$$

or equivalently

$$J\mathrm{e}^{-sL}\Psi = \mathrm{e}^{sL} J \Psi. \tag{21.46}$$

We note that $\Psi, J\Psi \in \bigcap_{|s|<\epsilon} \operatorname{Dom} \mathrm{e}^{sL}$, hence they are analytic vectors for L. Therefore, we can analytically continue (21.46), using that J is anti-linear, to obtain

$$J\mathrm{e}^{\mathrm{i}tL}\Psi = \mathrm{e}^{\mathrm{i}tL} J \Psi.$$

Since the set of such vectors Ψ is dense in $\mathcal{H}$, this proves (2).

Let us now prove (3). Since $\mathfrak{F}$ is the strong closure of $\mathfrak{F}_0$, it suffices to show that, for $A, B \in \mathfrak{F}_0$, one has

$$[JAJ, B] = 0. \tag{21.47}$$

To prove (21.47), it suffices, using (2), to prove that

$$[JAJ, \mathrm{e}^{\mathrm{i}tL} B \mathrm{e}^{-\mathrm{i}tL}] = 0, \quad t \in \mathbb{R}, \quad A, B \in \mathfrak{A}. \tag{21.48}$$

Let us now prove (21.48).

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First note that, for any $A_0, B_{\beta/2} \in L^\infty(Q)$, we have

$$A_0 U_{-s} B_{\beta/2} U_s = U_{-s} B_{\beta/2} U_s A_0. \tag{21.49}$$

Assume now that $A_0 \in L^\infty(Q, \mathfrak{S}_{\{0\}}, \mu)$ and $B_{\beta/2} \in L^\infty(Q, \mathfrak{S}_{\{\beta/2\}}, \mu)$. Let $\Psi = E_{\{0,\beta/2\}} f$ for $f \in L^2(Q, \mathfrak{S}_{[\epsilon,\beta/2]}, \mu)$, $0 < \epsilon < \beta/2$. Since $B_{\beta/2} U_s f$ and $B_{\beta/2} U_s A_0 f$ belong to $L^2(Q, \mathfrak{S}_{[s,\beta/2]}, \mu)$, we see that

$$\begin{aligned} E_{\{0,\beta/2\}} B_{\beta/2} U_s f &= B_{\beta/2} \mathrm{e}^{-sL} \Psi, \\ E_{\{0,\beta/2\}} B_{\beta/2} U_s A_0 f &= B_{\beta/2} \mathrm{e}^{-sL} A_0 \Psi \end{aligned}$$

belong to $\mathrm{Dom}\, \mathrm{e}^{sL}$ and (21.49) can be rewritten as

$$A_0 \mathrm{e}^{sL} B_{\beta/2} \mathrm{e}^{-sL} \Psi = \mathrm{e}^{sL} B_{\beta/2} \mathrm{e}^{-sL} A_0 \Psi. \tag{21.50}$$

Hence, to prove (21.48) it suffices to show that

$$s \mapsto \mathrm{e}^{sL} B_{\beta/2} \mathrm{e}^{-sL} A_0 \Psi$$

can be holomorphically extended to $\{0 \le \mathrm{Re}\, z \le \epsilon\}$, and that its analytic extension to $s = -\mathrm{i}t$ equals $\mathrm{e}^{-\mathrm{i}tL} B_{\beta/2} \mathrm{e}^{\mathrm{i}tL} A_0 \Psi$.

Let us take a vector Ψ of the form

$$\Psi = \prod_{j=n}^{1} \mathrm{e}^{-t_j L} A_j \Omega$$

for $t_j \ge 0$, $t_1 + \cdots + t_n \le \epsilon$ and $A_j \in \mathfrak{A}$. Recall from Prop. 21.48 that the linear span of such vectors is dense in $\mathcal{H}$.

Let $B_0 \in \mathfrak{A}$ such that $B_{\beta/2} = J B_0 J$. By (2), we have

$$\mathrm{e}^{sL} B_{\beta/2} \mathrm{e}^{-sL} A_0 = J \mathrm{e}^{-sL} B_0 J \mathrm{e}^{-sL} A_0.$$

Hence,

$$\begin{aligned} \mathrm{e}^{sL} B_{\beta/2} \mathrm{e}^{-sL} A_0 \Psi &= J \mathrm{e}^{-sL} B_0 J \mathrm{e}^{-sL} A_0 \Psi \\ &= J \mathrm{e}^{-sL} B_0 J \mathrm{e}^{-sL} A_0 \prod_{j=n}^{1} \mathrm{e}^{-t_j L} A_j \Omega \\ &= J \mathrm{e}^{-sL} B_0 \mathrm{e}^{sL - (\beta/2 - \sum_{j=1}^{n} t_j) L} \prod_{j=1}^{n} (A_j^* \mathrm{e}^{-t_j L}) A_0^* \Omega, \end{aligned}$$

using (21.43). By Prop. 21.49, this can be analytically continued to $s = \mathrm{i}t$ to give

$$\begin{aligned} & J \mathrm{e}^{\mathrm{i}tL} B_0 \mathrm{e}^{-\mathrm{i}tL - (\beta/2 - \sum_{j=1}^{n} t_j) L} \prod_{j=1}^{n} (A_j^* \mathrm{e}^{-t_j L}) A_0^* \Omega \\ &= J \mathrm{e}^{\mathrm{i}tL} B_0 \mathrm{e}^{\mathrm{i}tL} J A_0 \prod_{j=n}^{1} \mathrm{e}^{-t_j L} A_j \Omega \\ &= \mathrm{e}^{\mathrm{i}tL} J B_0 J \mathrm{e}^{-\mathrm{i}tL} A_0 \Psi \; = \; \mathrm{e}^{\mathrm{i}tL} B_{\beta/2} \mathrm{e}^{-\mathrm{i}tL} A_0 \Psi, \end{aligned}$$

once again using (21.43). This completes the proof of (3). □

Proof of Thm. 21.47, Part 2. We will use the Tomita–Takesaki theory described in Subsect. 6.4.2. Let us check first that Ω is cyclic and separating for $\mathfrak{F}$. Let

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$\Psi \in \{\mathfrak{F}_0\Omega\}^\perp$. It follows that, for all $t_1, \cdots, t_n \in \mathbb{R}$, $A_i \in \mathfrak{A}$, one has

$$\left(\Psi | \prod_{j=n}^{1} (A_j \mathrm{e}^{\mathrm{i}t_j L})\Omega\right) = 0.$$

By analytic continuation and Prop. 21.49, this implies that for all $(t_1, \ldots, t_n) \in J_\beta(n)$ one has

$$\left(\Psi | \prod_{j=n}^{1} (A_j \mathrm{e}^{-t_j L})\Omega\right) = 0.$$

Since the vectors of the form $\prod_{j=n}^{1} (A_j \mathrm{e}^{-t_j L})\Omega$ span $\mathcal{H}$, this implies that $\Psi = 0$, and hence Ω is cyclic for $\mathfrak{F}$.

Since $J\Omega = \Omega$, Ω is also cyclic for $J\mathfrak{F}J$. By (3) of Lemma 21.50, this implies that Ω is separating for $\mathfrak{F}$.

By (1) of Lemma 21.50,

$$\mathrm{e}^{-\beta L/2} B\Omega = JB^*\Omega, \quad B \in \mathfrak{F}. \tag{21.51}$$

Therefore, the operator S of the Tomita–Takesaki theory is

$$S = J\mathrm{e}^{-\beta L/2}.$$

By the uniqueness of the polar decomposition of S, this implies that J is the modular conjugation and $\mathrm{e}^{-\beta L/2}$ the modular operator for the state Ω. This completes the proof of the theorem. □

21.4.5 Gaussian β-Markov path spaces I

We would like to describe a β-periodic version of the construction described in Subsect. 21.2.4. Let $\mathcal{X}$ be a real Hilbert space and $\epsilon > 0$ a self-adjoint operator on $\mathcal{X}$. (Again, we assume the reality just for definiteness.) Consider the real Hilbert space

$$L^2(S_\beta, \mathcal{X}) \simeq L^2(S_\beta, \mathbb{R}) \otimes \mathcal{X}$$

and the covariance

$$C = (D_t^2 + \epsilon^2)^{-1}$$

with *β-periodic boundary conditions*. (This means $-D_t^2$ is the Laplacian on the circle S_β.)

Consider also the space $\mathcal{Q} := C^{-\frac{1}{2}} L^2(S_\beta, \mathcal{X})$ and its dual, that is, $\mathcal{Q}^\#$, which can be identified with $C^{\frac{1}{2}} L^2(S_\beta, \mathcal{X})$.

Lemma 21.51 *Let us define for $t \in S_\beta$ the map*

$$j_t : \big(2\epsilon \tanh(\beta\epsilon/2)\big)^{\frac{1}{2}} \mathcal{X} \ni g \mapsto \delta_t \otimes g \in \mathcal{Q}. \tag{21.52}$$

Then

$$(j_{t_1}g_1|j_{t_2}g_2)_{\mathcal{Q}} = \Big(g_1|\frac{\mathrm{e}^{-|t_1-t_2|\epsilon}+\mathrm{e}^{-(\beta-|t_1-t_2|)\epsilon}}{2\epsilon(1\!\!1-\mathrm{e}^{-\beta\epsilon})}g_2\Big)_{\mathcal{X}}.$$

In particular j_t is isometric.

Proof The proof is analogous to the proof of Lemma 21.15. In particular, we use the discrete unitary Fourier transform

$$L^2(S_\beta)\ni f\mapsto (f_n)\in l^2(\mathbb{Z}),\quad f_n=\beta^{-\frac12}\int_{S_\beta}\mathrm{e}^{-\mathrm{i}2\pi nt/\beta}f(t)\mathrm{d}t,$$

and apply

$$\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}2\pi nt/\beta}}{(2\pi n/\beta)^2 1\!\!1+\epsilon^2}=\frac{\mathrm{e}^{-|t|\epsilon}+\mathrm{e}^{-(\beta-|t|)\epsilon}}{2\epsilon(1\!\!1-\mathrm{e}^{-\beta\epsilon})}\tag{21.53}$$

instead of (21.8). □

Definition 21.52 *For $t\in\mathbb{R}$, resp. for $I\subset\mathbb{R}$ we define $\mathcal{Q}_t$, e_t, e^t, resp. $\mathcal{Q}_I$, e_I, e^I, as in Subsect. 21.2.4.*

Definition 21.53 *We define*

$$rf(s):=f(-s),\quad u_tf(s)=f(s-t),\quad f\in\mathcal{Q},\quad s,t\in S_\beta.$$

Proposition 21.54 (1) *Let $t,t_1,t_2\in S_\beta$, $t_1<t_2$ and $f\in\mathcal{Q}^\#$. We have*

$$\begin{aligned}e^tf(s)&=(\mathrm{e}^{\beta\epsilon}-\mathrm{e}^{-\beta\epsilon})^{-1}\big(\mathrm{e}^{-|t-s|\epsilon}(\mathrm{e}^{\beta\epsilon}-1\!\!1)+\mathrm{e}^{|t-s|\epsilon}(1\!\!1-\mathrm{e}^{-\beta\epsilon})\big)f(t),\\ e^{[t_1,t_2]}f(s)&=1\!\!1_{[t_1,t_2]}(s)f(s)+\big(\sinh(\beta+t_1-t_2)\epsilon\big)^{-1}\\ &\quad\times\Bigg(1\!\!1_{]-\beta/2,t_1[}(s)\Big(\sinh\big((s+\beta-t_2)\epsilon\big)f(t_1)-\sinh\big((s-t_1)\epsilon\big)f(t_2)\Big)\\ &\quad+1\!\!1_{]t_2,\beta/2[}(s)\Big(\sinh\big((s-t_2)\epsilon\big)f(t_1)-\sinh\big((s-\beta-t_1)\epsilon\big)f(t_2)\Big)\Bigg).\end{aligned}$$

(2) *$C_\mathrm{c}^\infty\big(]t_1,t_2[,\mathrm{Dom}\,\epsilon\big)$ is dense in $\mathcal{Q}_{]t_1,t_2[}$.*
(3) *$\mathbb{R}\ni t\mapsto u_t$ is an orthogonal β-periodic C_0-group on $\mathcal{Q}$.*
(4) *r is an orthogonal operator satisfying $ru_t=u_{-t}r$ and $r^2=1\!\!1$.*
(5) *$\sum\limits_{t\in S_\beta}u_t\mathcal{Q}_0$ is dense in $\mathcal{Q}$.*
(6) *$re_0=e_0$.*
(7) *$e_{[0,\beta/2]}e_{[-\beta/2,0]}=e_{\{0,\beta/2\}}$.*

21.4.6 Gaussian β-Markov path spaces II

As in Subsect. 21.2.4, we consider the Gaussian $\mathbf{L^2}$ space with covariance C. According to the notation introduced in Subsect. 5.4.2, this will be denoted

$$\mathbf{L^2}\big(L^2(S_\beta,\mathcal{X}),\mathrm{e}^{\phi\cdot C^{-1}\phi}\mathrm{d}\phi\big),\tag{21.54}$$

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where we use ϕ as the generic variable in $L^2(S_\beta, \mathcal{X})$. Let $L^2(Q, \mathrm{d}\mu)$ be a concrete realization of (21.54).

Definition 21.55 *For $s \in S_\beta$ and $g \in \big(2\epsilon \tanh(\beta\epsilon/2)\big)^{\frac{1}{2}}\mathcal{X}$, we define*

$$\phi_s(g) := \phi(\delta_s \otimes g) \in \bigcap_{1 \leq p < \infty} L^p(Q),$$

called the sharp-time fields.

Set

$$R\mathrm{e}^{\mathrm{i}\phi(f)} := \mathrm{e}^{\mathrm{i}\phi(rf)}, \quad U_t\mathrm{e}^{\mathrm{i}\phi(f)} := \mathrm{e}^{\mathrm{i}\phi(u_{-t}f)}, \quad f \in \mathcal{Q}, \quad t \in S_\beta, \tag{21.55}$$

and extend R and U_t to $L^2(Q, \mathrm{d}\mu)$ by linearity and density.

We obtain the following proposition, whose proof is completely analogous to Prop. 21.23.

Proposition 21.56 *Let $\mathfrak{S}$ be the completion of the Borel σ-algebra on Q. Let $\mathfrak{S}_0$ be the σ-algebra generated by the functions $\mathrm{e}^{\mathrm{i}\phi_0(g)}$ for $g \in \big(2\epsilon \tanh(\beta\epsilon/2)\big)^{\frac{1}{2}}\mathcal{X}$. Let R, U_t be defined in (21.55). Then $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ is a β-Markov path space.*

Definition 21.57 $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu)$ *defined above will be called the* Gaussian β-Markov path space with covariance C.

The β-KMS system obtained from the Gaussian path space can be interpreted in terms of Araki–Woods CCR representations. We set

$$\rho = (\mathrm{e}^{\beta\epsilon} - \mathbb{1})^{-1}.$$

Recall that in Subsect. 17.1.5 we defined the (left) Araki–Woods CCR representation, denoted $g \mapsto W_{\rho,\mathrm{l}}(g)$. Recall also that J_s denoted the corresponding modular conjugation on the Araki–Woods W^*-algebra.

Theorem 21.58 *There exists a unique unitary map*

$$T_\mathrm{eucl} : \mathcal{H} \to \Gamma_\mathrm{s}\big((2\epsilon)^{\frac{1}{2}}\mathbb{C}\mathcal{X} \oplus (2\overline{\epsilon})^{\frac{1}{2}}\overline{\mathbb{C}\mathcal{X}}\big)$$

intertwining the CCR representation of the time-zero fields with the Araki–Woods CCR representation at density ρ, that is,

$$\begin{aligned} T_\mathrm{eucl}1 &= \Omega, \\ T_\mathrm{eucl}\mathrm{e}^{\mathrm{i}\phi_0(g)}T_\mathrm{eucl}^{-1} &= W_{\rho,\mathrm{l}}(g) = \mathrm{e}^{\mathrm{i}\phi((\mathbb{1}+2\rho)^{\frac{1}{2}}g \oplus \overline{\rho}^{\frac{1}{2}}\overline{g})}, \quad g \in \big(2\epsilon \tanh(\beta\epsilon/2)\big)^{\frac{1}{2}}\mathcal{X}. \end{aligned}$$

It satisfies

$$T_\mathrm{eucl}L = \mathrm{d}\Gamma(\epsilon \oplus -\overline{\epsilon})T_\mathrm{eucl}, \tag{21.56}$$

$$T_\mathrm{eucl}J = J_\mathrm{s}T_\mathrm{eucl}. \tag{21.57}$$

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Proof The proof is similar to Thm. 21.26. To construct T_{eucl}, it suffices by linearity and density to check that

$$\int_Q \mathrm{e}^{\mathrm{i}\phi_0(g)}\mathrm{d}\mu = \mathrm{e}^{-\frac{1}{2}(\delta_0\otimes g|C\delta_0\otimes g)} = \exp\Big(-\frac{1}{2}\big(g|(2\epsilon)^{-1}(\mathbb{1}-\mathrm{e}^{-\beta\epsilon})^{-1}(\mathbb{1}+\mathrm{e}^{-\beta\epsilon})g\big)_{\mathcal{X}}\Big)$$
$$= \big(\Omega|\mathrm{e}^{\mathrm{i}\phi_{\rho,1}(g)}\Omega\big),$$

which is immediate. To check (21.56) we verify using Lemma 21.25 that

$$\int_Q \mathrm{e}^{-\mathrm{i}\phi_0(g_1)}\mathrm{e}^{\mathrm{i}\phi_t(g_2)}\mathrm{d}\mu = \big(W_{\rho,1}(g_1)\Omega|\mathrm{e}^{-t\mathrm{d}\Gamma(\epsilon\oplus-\overline{\epsilon})}W_{\rho,1}(g_2)\Omega\big), \quad 0\leq t\leq \beta/2.$$

(21.57) can be checked similarly. □

21.5 Perturbations of β-Markov path spaces

Let us fix a β-Markov path space $(Q,\mathfrak{S},\mathfrak{S}_0,U_t,R,\mu)$. In this section we describe a large class of perturbations of the measure μ that still satisfy the axioms of a β-Markov path space. We also describe the corresponding new physical space and Liouvillean.

We will restrict ourselves to perturbations given by a real $\mathfrak{S}_0$-measurable function V such that

$$V,\ \mathrm{e}^{-\beta V}\in L^1(Q). \tag{21.58}$$

As in Sect. 21.3, it is also possible to consider more singular perturbations associated with the equivalent of a Feynman–Kac–Nelson kernel; see Klein–Landau (1981b).

21.5.1 Perturbed path spaces

By Lemma 21.32, we know that the function

$$F := \mathrm{e}^{-\int_{S_\beta} U_s(V)\mathrm{d}s}$$

belongs to $L^1(Q)$.

Definition 21.59 *We introduce the perturbed measure*

$$\mathrm{d}\mu_V := \frac{F\mathrm{d}\mu}{\int_Q F\mathrm{d}\mu}.$$

Clearly, μ_V is a probability measure.

Note that we can write $F = F_{[-\beta/2,0]}F_{[0,\beta/2]}$ for

$$F_{[0,\beta/2]} = \mathrm{e}^{-\int_0^{\beta/2}U_s(V)\mathrm{d}s},$$
$$F_{[-\beta/2,0]} = \mathrm{e}^{-\int_{-\beta/2}^{0}U_s(V)\mathrm{d}s} = R(F_{[0,\beta/2]}).$$

$F_{[0,\beta/2]}$, resp. $F_{[-\beta/2,0]}$ is $\mathfrak{S}_{[0,\beta/2]}$-, resp. $\mathfrak{S}_{[-\beta/2,0]}$-measurable.

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Proposition 21.60 *The perturbed path space $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu_V)$ is β-Markov.*

Proof We first check the properties of Def. 21.41. Since $F > 0$ $\mu-$a.e., the sets of measure 0 for μ and μ_V coincide, so $\mathfrak{S}$ is complete for μ_V and $L^\infty(Q,\mu) = L^\infty(Q,\mu_V)$. The function F is clearly R and U_t invariant, hence R and U_t preserve μ_V.

Approximating F by $F_n = F\mathbb{1}_{[0,n]}(F)$ and using that U_t is strongly continuous in measure for μ, we see that U_t is also strongly continuous in measure for μ_V. By Lemma 5.33, this implies that U_t is strongly continuous on $L^2(Q,\mu_V)$.

Property (21.5) of Def. 21.7 is obvious. It remains to check the Markov property. To simplify notation we set $E_0 = E_{\{0,\beta/2\}}$, $E_+ = E_{[0,\beta/2]}$, $E_- = E_{[-\beta/2,0]}$ and decorate with the superscript V the corresponding objects for μ_V. We also set $F_+ = F_{[0,\beta/2]}$, $F_- = F_{[-\beta/2,0]}$, so that $F = F_+F_-$.

Property (6) of Prop. 5.27 can be rewritten as the following operator identity, where we identify a function and the associated multiplication operator:

$$E_I^V = \big(E_I(F)\big)^{-1}E_IF, \quad I \subset S_\beta.$$

Using $R(F) = F$ and $RE_0 = E_0$, we see that $RE_0^V = E_0^V$.

Then using (2) of Prop. 5.27, we obtain that

$$E_\pm^V = \big(E_\pm(F_+F_-)\big)^{-1}E_\pm F_+F_- = \big(E_\pm(F_\mp)\big)^{-1}E_\pm F_\mp = \big(E_0(F_\mp)\big)^{-1}E_\pm F_\mp,$$

where in the last step we used the β-Markov property for μ. This yields

$$\begin{aligned} E_+^V E_-^V &= \big(E_0(F_-)\big)^{-1}E_+F_-\big(E_0(F_+)\big)^{-1}E_-F_+ \\ &= \big(E_0(F_-)\big)^{-1}\big(E_0(F_+)\big)^{-1}E_+F_-E_-F_+ \\ &= \big(E_0(F_-)\big)^{-1}\big(E_0(F_+)\big)^{-1}E_+E_-F_-F_+ \\ &= \big(E_0(F_-)\big)^{-1}\big(E_0(F_+)\big)^{-1}E_+E_-F \\ &= \big(E_0(F_-)\big)^{-1}\big(E_0(F_+)\big)^{-1}E_0F. \end{aligned} \tag{21.59}$$

Next we compute, as an identity between functions,

$$E_0(F_-)E_0(F_+) = E_+(F_-)E_-(F_+) = (E_+E_-)(F_-F_+) = E_0(F). \tag{21.60}$$

Combining (21.59) and (21.60), we obtain that $E_+^V E_-^V = E_0^V$, which implies the Markov property for μ_V. □

21.5.2 Perturbed Liouvilleans

Applying Subsect. 21.4.2, we can associate with the path space $(Q, \mathfrak{S}, \mathfrak{S}_0, U_t, R, \mu_V)$ a perturbed β-KMS system. In particular, the perturbed physical space is

$$\mathcal{H}_V^{\text{int}} = L^2(Q, \mathfrak{S}_{\{0,\beta/2\}}, \mu_V).$$

It is convenient to relate this perturbed KMS system with the free KMS system obtained with the measure μ and living on the free physical space $\mathcal{H}$. We will decorate with the subscript V and the superscript int the objects obtained by Subsect. 21.4.2 for the path space involving the perturbed measure μ_V. The corresponding objects transported to $\mathcal{H}$ will be decorated with just the subscript V.

Let us first unitarily identify $\mathcal{H}$ with $\mathcal{H}_V^{\mathrm{int}}$.

Lemma 21.61 *Let $T_V : \mathcal{H}_V^{\mathrm{int}} \to \mathcal{H}$ be defined by*

$$T_V \Psi := \frac{1}{(\int_Q F \mathrm{d}\mu)^{\frac{1}{2}}} E_{\{0,\beta/2\}}(F_{[0,\beta/2]}\Psi).$$

Then T_V is unitary.

Proof Without loss of generality we can assume that $\int_Q F\mathrm{d}\mu = 1$. Let $\Psi, \Phi \in L^2(Q, \mathfrak{S}_{\{0,\beta/2\}}, \mu_V) = \mathcal{H}_V^{\mathrm{int}}$. Using the reflection property and (21.60), we have

$$\begin{aligned}
(T_V\Phi|T_V\Psi)_{\mathcal{H}} &= \int_Q \overline{E_{\{0,\beta/2\}}(F_{[0,\beta/2]}\Phi)} E_{\{0,\beta/2\}}(F_{[0,\beta/2]}\Psi)\mathrm{d}\mu \\
&= \int_Q F_{[0,\beta/2]}\overline{\Phi}\Psi E_{\{0,\beta/2\}}(F_{[-\beta/2,0]})\mathrm{d}\mu \\
&= \int_Q F_{[0,\beta/2]}\overline{\Phi}\Psi E_{[0,\beta/2]}E_{[-\beta/2,0]}(F_{[-\beta/2,0]})\mathrm{d}\mu \\
&= \int_Q F_{[0,\beta/2]}\overline{\Phi}\Psi F_{[-\beta/2,0]}\mathrm{d}\mu \\
&= \int_Q F_{[0,\beta]}\overline{\Phi}\Psi\mathrm{d}\mu \;=\; (\Phi|\Psi)_{\mathcal{H}_V^{\mathrm{int}}}.
\end{aligned}$$

□

The following result is shown in Klein–Landau (1981b).

Proposition 21.62 *Let V be a real $\mathfrak{S}_0$-measurable function satisfying (21.58). Set*

$$F_{[0,t]} := \mathrm{e}^{-\int_0^t U_s(V)\mathrm{d}s}$$

and, for $0 < t < \beta/2$,

$$\begin{aligned}
\mathcal{M}_t &:= \mathrm{Span}\left(\bigcup_{0\leq s\leq \beta/2-t} F_{[0,s]}L^\infty(Q, \mathfrak{S}_{[0,\beta/2-t]}, \mu)\right), \\
\mathcal{D}_t &:= E_{\{0,\beta/2\}}\mathcal{M}_t.
\end{aligned}$$

Then, for any $0 \leq s \leq t \leq \beta/2$, there exists a unique $P_V(s) : \mathcal{D}_t \to \mathcal{D}_{t-s}$ such that

$$P_V(s)E_{\{0,\beta/2\}}f = E_{\{0,\beta/2\}}F_{[0,t]}U_s f, \quad f \in L^2(Q, \mu).$$

$\{\mathcal{D}_t, P_V(t)\}_{t\in[0,\beta/2]}$ is a local Hermitian semi-group on $\mathcal{H}$.

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Definition 21.63 *The self-adjoint operator associated with* $\{\mathcal{D}_t, P_V(t)\}_{t\in[0,\beta/2]}$ *is denoted by* $\tilde{L}_V$.

The following theorem is an analog of Thms. 21.37, 21.38.

Theorem 21.64 *Assume in addition to (21.58) that either*

$$V \in L^2(Q, \mathrm{d}\mu), \quad V \geq 0, \tag{21.61}$$

or

$$V \in L^{2+\epsilon}(Q, \mathrm{d}\mu). \tag{21.62}$$

Let L be the free Liouvillean constructed in Def. 21.44. Then

$$\tilde{L}_V = (L + V)^{\mathrm{cl}}.$$

We denote by τ_V^t the dynamics on $\mathfrak{F}$ generated by $\mathrm{e}^{\mathrm{i}t\tilde{L}_V}$. We set

$$\Omega_V := \|\mathrm{e}^{-\beta\tilde{L}_V/2}\Omega\|^{-1}\mathrm{e}^{-\beta\tilde{L}_V/2}\Omega$$

and denote by ω_V the state on $\mathcal{F}$ generated by the vector Ω_{V}. Clearly, $(\mathfrak{F}, \tau_V, \omega_V)$ is a β-KMS system. We denote by L_V the associated standard Liouvillean (see Def. 6.55). Note that both $\tilde{L}_V$ and L_V generate the same dynamics on $\mathfrak{F}$, even though they are different operators.

We have the following result:

Theorem 21.65 *Let V be a real $\mathfrak{S}_0$-measurable function satisfying (21.58) and either (21.61) or*

$$V \in L^p(Q, \mu), \quad \mathrm{e}^{-\frac{\beta}{2}V} \in L^q(Q, \mu), \qquad p^{-1} + q^{-1} = \frac{1}{2}, \quad 2 < p,\, q < \infty. \tag{21.63}$$

Then

$$L_V = (\tilde{L}_V - JVJ)^{\mathrm{cl}}.$$

The relationship between the two kinds of perturbed β-KMS system – $(\mathfrak{F}, \tau_V, \omega_V)$, which lives on the free space, and $(\mathfrak{F}_V^{\mathrm{int}}, \tau_V^{\mathrm{int}}, \omega_V^{\mathrm{int}})$, which lives on on the perturbed space – is described in the following theorem:

Theorem 21.66 *Let V be a real $\mathfrak{S}_0$-measurable function satisfying the assumptions of Thm. 21.64. Then*

(1) $\mathfrak{A}_V^{\mathrm{int}} = T_V^{-1}\mathfrak{A}T_V$, $\mathfrak{F}_V^{\mathrm{int}} = T_V^{-1}\mathfrak{F}T_V$;
(2) $T_V\Omega_V^{\mathrm{int}} = \Omega_V$;
(3) $T_V\tau_V^{\mathrm{int},t}(A)T_V^{-1} = \tau_V^t\left(T_V A T_V^{-1}\right)$, $A \in \mathfrak{F}_V^{\mathrm{int}}$, $t \in \mathbb{R}$;
(4) $T_V J_V^{\mathrm{int}} T_V^{-1} = J$.

21.6 Notes

As explained in the introduction, the name "Euclidean approach" comes from the fact that the Minkowski space $\mathbb{R}^{1,d}$ is turned into the Euclidean space $\mathbb{R}^{1+d}$

by the Wick rotation. Hence, the Euclidean approach is usually associated with relativistic quantum field theory. As we saw, it can also be applied in other situations, as in the usual non-relativistic quantum mechanics. In constructive quantum field theory, the use of the Wick rotation was advocated by Symanzik (1965). The construction of interacting Hamiltonians through the corresponding heat semi-group appeared earlier in works of Nelson (1965) and Segal (1970). The monographs by Glimm–Jaffe (1987) and Simon (1974) contain a more detailed treatment of Euclidean methods at zero temperature, essentially in two space-time dimensions. Osterwalder–Schrader (1973, 1975) formulated a set of axioms for a Euclidean quantum theory, parallel to the Wightman axioms on Minkowski space, allowing the reconstruction of a physical theory in a way similar to the one explained here.

The treatment of this chapter follows a series of interesting papers by Klein (1978) and Klein–Landau (1975, 1981b). In particular, the proof of Thm. 21.37 can be found in Klein–Landau (1975), Thm. 3.4, and the proof of Thm. 21.38 in Klein–Landau (1981a), Sect. 2.

Our treatment of path spaces at positive temperature follows Klein–Landau (1981b) and Gérard–Jaekel (2005). In particular, Thms. 21.43 and 21.47 are due to Klein–Landau (1981b). Thms. 21.65 and 21.66 are proven in Gérard–Jaekel (2005).