Canad. Math. Bull. Vol. **25** (1), 1982

# IDEMPOTENT IDEALS AND NOETHERIAN POLYNOMIAL RINGS

BY
CHARLES LANSKI

ABSTRACT. If $R$ is a commutative Noetherian ring and $I$ is a nonzero ideal of $R$, it is known that $R+I[x]$ is a Noetherian ring exactly when $I$ is idempotent, and so, when $R$ is a domain, $I=R$ and $R$ has identity. In this paper, the noncommutative analogues of these results, and the corresponding ones for power series rings, are proved. In the general case, the ideal $I$ must satisfy the idempotent condition that $TI=T$ for each right ideal $T$ of $R$ contained in $I$. It is also shown that when every ideal of $R$ satisfies this condition, and when $R$ satisfies the descending chain condition on right annihilators, $R$ must be a finite direct sum of simple rings with identity.

In this paper we find a necessary and sufficient condition on an ideal $I$ of a ring $R$ so that the subring $R+I[x]$ of $R[x]$ is a right Noetherian ring. One formulation of this condition is that $I=Re$ for $e$ an idempotent of $R$ satisfying $eR(1-e)=0$, where $R(1-e)=\{r\in R \mid re=0\}$. Our first theorem gives an equivalent condition which shows that $I$ must have strong idempotent properties. Results of Gilmer ([1] and [2]) for commutative rings are consequences of those presented here for arbitrary Noetherian rings.

If $\mathrm{Lat}_A(M)$ denotes the lattice of submodules of the right $A$ module $M$, then for $S=R+I[x]$, it is clear that $\mathrm{Lat}_S(S/xI[x])\cong\mathrm{Lat}_R(R)$. Therefore, $R$ must be a right Noetherian ring when $S$ is. With this same notation, we proceed to our first theorem which is quite easy.

THEOREM 1. *If $R$ is a right Noetherian ring, $I$ is a nonzero ideal of $R$, and $S=R+I[x]$, then the following are equivalent:*

(i) *$S$ is a right Noetherian ring;*

(ii) *$TI=T$ for each right ideal $T$ of $R$ contained in $I$; and*

(iii) *$I[x]$ is a right Noetherian $S$ module.*

**Proof.** To see that (i) implies (ii), set $J=xT[x]$ for $T\subset I$, $T$ a nonzero right ideal of $R$. Clearly $J$ and $JI$ are right ideals of $S$, so $J/JI$ is a Noetherian $S$ module. Since $\mathrm{Lat}_S(J/JI)\cong\mathrm{Lat}_R(\oplus_{i=1}^{\infty} T/TI)$, it follows that $TI=T$. Assuming (ii), a standard Hilbert Basis Theorem argument shows that (iii) holds. That is,

Received by the editors February 22, 1980
AMS (1979) Subject Classification Nos: Primary: 16A33
Secondary: 16A34, 16A12, 16A48,

if $M$ is an $S$ submodule of $I[x]$, let $M_j$ be the set of leading coefficients of polynomials in $M$ of degree $j$, together with zero. Then $M_j$ is a right ideal of $R$, $M_j \subset I$ for $j \geq 1$, and by (ii) $M_j \subset M_{j+1}$ for $j \geq 1$. If $M_k$ is maximal in $\{M_j \mid j \geq 1\}$, then the finite set of polynomials in $M$ whose leading coefficients generate $M_0 + M_1 + \cdots + M_k$ is an $S$ generating set for $M$. Finally, since $\mathrm{Lat}_S(S/I[x]) \cong \mathrm{Lat}_R(R/I)$ and $R$ is a right Noetherian ring, $S/I[x]$ is a Noetherian $S$ module. Thus (i) is an immediate consequence of (iii).

We note that Theorem 1, as well as all of our subsequent results for polynomial rings, hold also for the power series rings $R + I[[x]]$. The proofs in the power series case are trivial modifications of those presented. When $R$ is a commutative ring we obtain results of Gilmer ([2] and [1]) as a corollary of Theorem 1.

Corollary. *If $R$ is a commutative Noetherian ring, $I$ is a nonzero ideal of $R$, and $S = R + I[x]$, then the following are equivalent:*

(i) *$S$ is a Noetherian ring;*

(ii) *$I^2 = I$; and*

(iii) *$I = eR$ for $e^2 = e \in R$.*

*In particular, if $R[x]$ is a Noetherian ring, then $R$ has an identity.*

Before exploring further when an ideal $I$ of $R$ satisfies condition (ii) of Theorem 1, we make a generalization to right ideals and present some examples.

Definition. A nonzero right ideal $I$ of a ring $R$ satisfies ($SI$) if for each right ideal $T$ of $R$ contained in $I$, $TI = T$.

Example 1. A nonnilpotent minimal right ideal in any ring satisfies ($SI$).

Example 2. Any right ideal in the socle of a semi-prime ring satisfies ($SI$).

Example 3. Any right ideal in a simple ring with 1 satisfies ($SI$).

Example 4. If $M_n(R)$ is the ring of $n \times n$ matrices over $R$ with 1, and $\{e_{ij}\}$ are the usual matrix units, then $e_{ii}M_n(R)$ satisfies ($SI$), but $ce_{ii}M_n(R)$ will not if $c$ is a regular nonunit in the center of $R$.

Example 5. For $e$ a nonzero idempotent in $R$, $eR$ is always idempotent, but may not satisfy ($SI$). For example, if $A$ is a ring with 1 containing an ideal $J$ with $J^2 \neq J$, then set

$$R = \begin{pmatrix} A & J \\ A & A \end{pmatrix} \quad \text{and} \quad I = e_{11}R.$$

Then

$$\begin{pmatrix} J & J \\ 0 & 0 \end{pmatrix} I = \begin{pmatrix} J & J^2 \\ 0 & 0 \end{pmatrix}.$$

https://doi.org/10.4153/CMB-1982-006-5 Published online by Cambridge University Press

EXAMPLE 6. If $I=I^2$ is a right ideal of $R$ so that $RI$ has a right identity $e$, then $I$ satisfies $(SI)$. For if $T\subset I$ is a right ideal of $R$, then $T=Te\subset TRI\subset TI\subset T$.

To characterize those (right) ideals of $R$ satisfying $(SI)$, we need to assume that $R$ is a semi-prime ring and satisfies the descending chain condition (DCC) on right annihilators. Recall that for a nonempty subset $Y\subset R$, the right annihilator of $Y$ in $R$, denoted $r_R(Y)$, is $\{r\in R \mid yr=0 \text{ for all } y\in Y\}$. The left annihilator of $Y$ is defined similarly. When $R$ is a semi-prime ring, the DCC on right annihilators will hold when $R$ is either a left or a right Noetherian ring, or more generally, when $R$ is either a left or a right Goldie ring.

Our next theorem gives the characterization we seek. Note the similarity to Example 6.

THEOREM 2. *Let $R$ be a semi-prime ring with DCC on right annihilators. If $I$ is a nonzero right ideal of $R$ satisfying $(SI)$, then $RI$ is a direct summand of $R$, as an ideal, and $RI$ has an identity. When $R$ is a prime ring, $RI=R$ and $R$ has* 1.

**Proof.** The assumption on right annihilators guarantees the existence of a finite subset $X=\{x_1,\ldots,x_n\}\subset I-\{0\}$ with the properties that $r_R(X)$ is minimal among the right annihilators of finite subsets of $I-\{0\}$, and each proper nonempty subset of $X$ has strictly larger right annihilator than does $X$. Thus, for any $y\in I-\{0\}$, $r(X\cup\{y\})=r(X)$, so that $Ir(X)=0$. The $(SI)$ condition applied to $x_1R\subset I$ gives $x_1R=x_1RI$, and it follows that $R=RI+r_R(\{x_1\})$. If $\{x_1\}\neq X$, then $x_jr_R(\{x_1\})\neq 0$ for some $x_j\in X-\{x_1\}$, by the minimality of $X$, and as above $x_jr_R(\{x_1\})=x_jr_R(\{x_1\})I$. Therefore, $r_R(\{x_1\})=r_R(\{x_1\})I+r_R(\{x_1,x_j\})$, and so $R=RI+r_R(\{x_1,x_j\})$. Continuing in this manner, one obtains $R=RI+r_R(X)$. Since $R$ is a semiprime ring and $r_R(X)\subset r_R(I)\subset r_R(RI)$, it is easy to see that $R=RI\oplus r_R(I)$.

Clearly, $I=I^2\subset RI$, our hypotheses on $R$ are inherited by the summand $RI$, and $I$ satisfies $(SI)$ considered as a right ideal of $RI$. Furthermore, $(RI)I=RI^2=RI$. Therefore, to show that $RI$ has an identity we may as well show that $R$ has 1 when $RI=R$. Consider the Cartesian product $W=R\times Z$ of $R$ with the ring of integers. Make $W$ into a ring with 1 by using componentwise addition and the usual "cross multiplication". Let $J$ be the left annihilator of $(R,0)$ in $W$, and identify $R$ with its image in $R^*=W/J$. It is straightforward to check that $R^*$ is a semi-prime ring with DCC on right annihilators, and that $I$ satisfies $(SI)$ as a right ideal in $R^*$. Thus, $R^*I=RI=R$ is a summand of $R^*$ by the first part of the proof. This is impossible unless $R^*=R$, which is to say, $R$ has 1. The conclusion for prime rings is immediate, completing the proof of the theorem.

Two consequences of Theorem 2 are worth stating as corollaries.

COROLLARY 1. *If $R$ is a semi-prime ring with DCC on right annihilators, then a nonzero right ideal $I$ of $R$ satisfies $(SI)$ if and only if $RI$ has an identity.*

https://doi.org/10.4153/CMB-1982-006-5 Published online by Cambridge University Press

**Proof.** One direction is just Theorem 2. Suppose that $e$ is the identity of $RI$. Then $RI(1-e)=0$ and the semi-primeness of $R$ forces $I(1-e)=0$. Thus $e$ is a right identity for $I$, and the argument in Example 6 shows that $I$ satisfies $(SI)$.

COROLLARY 2. *Let $R$ be a prime right Noetherian ring and $I$ a nonzero ideal of $R$. If $R+I[x]$ is a right Noetherian ring, then $I=R$.*

Using Theorem 1 and Theorem 2, results corresponding to those for commutative rings given in the Corollary to Theorem 1 could be stated for semi-prime rings. We prefer to wait for these until after our next theorem in order to state the general noncommutative results. Before Theorem 3, we give two more examples.

EXAMPLE 7. If $R$ is a right Noetherian ring with 1, then unlike the commutative case, $I=I^2$ for $I$ an ideal of $R$ is not sufficient to force $I$ to satisfy $(SI)$. Let $I$ be a simple right Noetherian domain without 1 which is an algebra over the finite field $F$ [3]. Then $R=I+F=I^*$, as constructed in Theorem 2, is a right Noetherian domain with 1, $I=I^2$, but $I$ cannot satisfy $(SI)$ by Theorem 2.

The next example shows that the conclusion of Theorem 2 requires some condition on $R$ in addition to semi-primeness.

EXAMPLE 8. Let $W$ be the ring of countable by countable matrices having only finitely many nonzero entries over a field $F$, and set $R=W+F$, the algebra generated by $W$ and the scalar matrices. $R$ is a primitive ring, $W$ satisfies $(SI)$ since it is the socle of $R$, but $W=RW$ is not a summand of $R$.

Our next theorem is a one-sided version of Theorem 2 for general right Noetherian rings.

THEOREM 3. *Let $R$ be a right Noetherian ring. A right ideal $I$ of $R$ satisfies $(SI)$ if and only if $I^2=I$ and $RI=Re$ for $e^2=e\in R$ with $eR(1-e)=0$. In this case, $RI$ also satisfies $(SI)$.*

**Proof.** If $RI=Re$, then $e$ is a right identity for $RI$, so $I$ and $RI$ satisfy $(SI)$ by Example 6. Now assume that $I$ satisfies $(SI)$ and let $N$ be the nilpotent radical of $R$. Since $N$ is nilpotent and $I^2=I$, $I\not\subset N$. For any right ideal of $R$ satisfying $N\subset T\subset I+N$, the modular law gives $T=T\cap(I+N)=(T\cap I)+N$. Since the $(SI)$ condition shows that $(I\cap T)I=I\cap T$, it follows that in $R/N$, $(T+N)(I+N)=T+N$, so $(I+N)/N$ satisfies $(SI)$ in $R/N$. Applying Theorem 2 yields a central idempotent $\bar{e}$ of $R/N$ which generates $(RI+N)/N$. The nilpotence of $N$ allows us to lift $\bar{e}$ to an idempotent $e\in RI$. Therefore, $R=Re\oplus R(1-e)$ as left $R$ moldules, $Re\subset RI$, and $eR\cap R(1-e)=eR(1-e)\subset N$, because $e+N$ is central in $R/N$. Of course, $I+N\subset eR+N$, so $I(1-e)\subset N$, and $RI(1-e)\subset N$. If $I(1-e)\neq 0$, choose $k$ minimal so that $I(1-e)(RI(1-e))^k=0$. Set $A=I(1-e)(RI(1-e))^{k-1}$, or $A=I(1-e)$ if $k=1$, and let $B$ be the right ideal of $R$ generated by $A$. Note that $BI(1-e)\subset AI(1-e)+ARI(1-e)$, that

$ARI(1-e)=0$, and $AI(1-e)\subset ARI(1-e)=0$ since $I=I^2\subset RI$. Hence, $BI(1-e)=0$. The $(SI)$ condition gives $B=BI$, so $B(1-e)=0$, and $A\subset B$ forces $A=A(1-e)=0$, contradicting the minimality of $k$. Therefore, $I(1-e)=0$, from which $RI\subset Re$, and so, $RI=Re$ follow. Finally, since $RI$ is an ideal of $R$ and contains $e$, $eR(1-e)\subset Re\cap R(1-e)=0$, completing the proof of the theorem.

Using Theorem 3 and Theorem 1, we can now prove the noncommutative version of the Corollary to Theorem 1.

COROLLARY. *Let $R$ be a right Noetherian ring and $I$ a nonzero ideal of $R$. Then the following are equivalent:*

(i) *$R+I[x]$ is a right Noetherian ring;*

(ii) *$I$ is a right Noetherian ring with right identity;*

(iii) *$I[x]$ is a right Noetherian ring.*

*In particular, $R[x]$ is a right Noetherian ring if and only if $R$ is a right Noetherian ring with right identity.*

**Proof.** Assuming (i), Theorem 1 implies that $I$ satisfies the $(SI)$ condition, so Theorem 3 gives $I=RI=Re$ for $eR(1-e)=0$. If $T$ is a right ideal of $I$, then $TR=TRe+TR(1-e)=TI\subset T$, since $TR(1-e)\subset ReR(1-e)=0$. Therefore, $T$ is a right ideal of $R$, so $I$ must satisfy the ascending chain condition on right ideals. That (ii) implies (iii) is just a restatement of the Hilbert Basis Theorem. Finally, if (iii) holds, then clearly $I[x]$ is a (right) Noetherian $R+I[x]$ module, so $R+I[x]$ is a right Noetherian ring by Theorem 1.

For a simple example illustrating the last statement of the Corollary, let $R=e_{11}M_n(A)$ for $A$ a right Noetherian ring with 1. Then $R$ is a right Noetherian ring with a left identity and $R[x]$ is not a right Noetherian ring since $R$ has a nonzero left annihilator.

We also observe that the condition $I=I^2$ in Theorem 3 is required. Let $R=Fe+Fx$ for $F=GF(p^k)$ with the relations $e^2=e$ and $ex=xe=x^2=0$. Then $R$ is a right Noetherian ring and $I=R$ satisfies $RI=Re$ with $e^2=e$ and $eR(1-e)=0$, but $R$ does not satisfy $(SI)$. However, the assumption $I=I^2$ is not required when $R$ is a semi-prime ring by Corollary 1 of Theorem 2.

Our last result is another application of Theorem 2.

THEOREM 4. *If $R$ is a ring with DCC on right annihilators, then the following are equivalent:*

(i) *every ideal of $R$ satisfies $(SI)$;*

(ii) *$R$ is a finite direct sum of simple rings with identity;*

(iii) *every right ideal of $R$ satisfies $(SI)$;*

(iv) *every right ideal of $R$ is idempotent.*

*Furthermore, each of (i)–(iv) implies that every left ideal of $R$ is idempotent.*

**Proof.** Observe first that any of (i)–(iv) guarantees that $R$ is a semi-prime ring. Example 3 shows that (ii) implies (iii), and the corresponding statement for left ideals. It is obvious that (iii) implies (iv) and that (iv) implies (i). Therefore, it suffices to show that (i) implies (ii). Theorem 2 shows that each nonzero ideal $I$ of $R$ has an identity and is a summand of $R$. Thus $I$, and any complement, inherit the hypothesis on $R$. Consequently, since $R$ is a semi-prime ring, it follows from the annihilator chain condition that $R$ must satisfy the descending chain condition on ideals. In particular, $R$ contains nonzero minimal ideals, each of which is a simple ring with identity. The annihilator condition now forces $R$ to be a finite direct sum of such minimal ideals, completing the proof of the theorem.

It is easy to find examples of rings having every (right) ideal idempotent but satisfying no chain condition. For example, one can take a ring which is equal to its socle, or any infinite direct sum of simple rings with identity. We end the paper with such an example which has no socle and which is not built up from an infinite direct sum of simple rings.

EXAMPLE 9. Let $D$ be a division ring and $V$ a countable dimensional vector space over $D$ with basis $B=\{v_1, v_2, \ldots\}$. Denote by $R$, the subring of all $D$-linear transformations on $V$ whose matrices in the basis $V$ have the form of a $2^n \times 2^n$ diagonal matrix over $D$, repeated infinitely "down the diagonal". It is easy to check that every (right) ideal of $R$ is generated by a set of central orthogonal idempotents. Therefore, each right ideal of $R$ is idempotent, but $R$ contains no minimal right ideals. For an example without 1, take a direct sum of $R$ with itself an infinite number of times.

## REFERENCES

1. R. W. Gilmer, *If R[x] is Noetherian, R contains an identity*, Amer. Math. Monthly, **74** (1967), p. 700.

2. R. W. Gilmer, *An existence theorem for non-Noetherian rings*, Amer. Math. Monthly, **77** (1970), 621–623.

3. J. C. Robson, '*Simple Noetherian rings need not have unity elements*, Bull. London Math. Soc., **7** (1975), 269–270.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CA 90007

https://doi.org/10.4153/CMB-1982-006-5 Published online by Cambridge University Press