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Lightness of Induced Maps and Homeomorphisms

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Abstract. An example is given of a map f defined between arcwise connected continua such that $C(f)$ is light and 2^f is not light, giving a negative answer to a question of Charatonik and Charatonik. Furthermore, given a positive integer n, we study when the lightness of the induced map 2^f or $C_n(f)$ implies that f is a homeomorphism. Finally, we show a result in relation with the lightness of $C(C(f))$.

1 Introduction

Let $f\colon X \to Y$ be a map between continua. J. J. Charatonik and W. J. Charatonik [2] studied the relations between the following three statements:

(i) f is light;
(ii) $C(f)$ is light;
(iii) 2^f is light.

They proved that (iii) implies (ii) and (ii) implies (i) and showed examples where the other implications do not hold. Also, they asked the following question.

Question 1.1 ([2, 5.1]) *Let $f\colon X \to Y$ be a map between arcwise connected continua. Are lightness of the induced maps $C(f)$ and 2^f equivalent conditions?*

In the Section 3, we give a map $f\colon X \to Y$ such that X is an arcwise connected continuum, $C(f)$ is light, but 2^f is not light, giving a negative answer to Question 1.1.

We study the lightness of the induced map $C_n(f)$ for any $n \in \mathbb{N}$, and the interrelation with the lightness of 2^f and f. We show that if $C_n(f)$ is a surjective and light map for some $n \geq 2$, then f is a homeomorphism.

Finally, in the Section 4, we show that if f is a confluent map such that $C(C(f))$ is light, then f is a homeomorphism.

2 Definitions

If (X, d) is a metric space, then given $A \subset X$ the closure of A is denoted by $Cl_X(A)$. The cardinality of A is denoted by $|A|$. The symbol $\mathbb{N}$ denotes the set of positive integer. A *continuum* is a nonempty, compact, connected and metric space. A *map* is assumed to be a continuous function. The symbol $A \subsetneq B$ means that $A \subset B$ and $A \neq B$. Given a continuum X, we consider the following hyperspaces of X:

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(i) $2^X = \{A \subset X : A \text{ is closed and nonempty}\}$;
(ii) $C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$, $n \in \mathbb{N}$.

Here, 2^X is topologized with the Vietoris topology [6, p. 3], which is generated by the collection of sets $\langle U_1, U_2, \dots, U_l \rangle$, where $U_1, U_2, \dots, U_l$ are open sets in X and

$$\langle U_1, U_2, \dots, U_l \rangle = \{A \in 2^X : A \subset \bigcup_{i=1}^{l} U_i \text{ and } A \cap U_i \neq \varnothing \text{ for each } i\}.$$

The set $C_n(X)$ is a subspace of 2^X. The reader may see [6, 7] for general information about hyperspaces.

Let $f : X \to Y$ be a map between continua. Then the function $2^f : 2^X \to 2^Y$ given by $2^f(A) = f(A)$ for each $A \in 2^X$, is called the *induced map between* 2^X *and* 2^Y. The function $2^f|_{C_n(X)}$ is denoted by $C_n(f)$ and it is called the *induced map between the hyperspaces* $C_n(X)$ *and* $C_n(Y)$. In [6, p. 106], it was shown that 2^f is a map. Since $2^f(C_n(X)) \subset C_n(Y)$, $C_n(f)$ is a map between $C_n(X)$ and $C_n(Y)$, for each $n \in \mathbb{N}$.

Definition 2.1 Let $f : X \to Y$ be a map between continua. Then f is said to be

(i) *light* if $f^{-1}(f(x))$ is totally disconnected for each $x \in X$;
(ii) *monotone* if the inverse image of any point in Y is connected;
(iii) *confluent* if for each subcontinuum Q of Y, each component of $f^{-1}(Q)$ is mapped onto Q by f;
(iv) *weakly confluent* if for each subcontinuum Q of Y, there exists a component P of $f^{-1}(Q)$ such that $f(P) = Q$.

Notice that by definition every monotone map is confluent, every confluent map is weakly confluent, and every weakly confluent map is surjective. Moreover, it is easy to prove that f is a weakly confluent map if and only if $C_n(f)$ is surjective, for any $n \in \mathbb{N}$ [3, Proposition 1].

3 Lightness of the Induced Map $C_n(f)$

The next proposition is a generalization of [11, (1.212.3) p. 158].

Proposition 3.1 *Let $f : X \to Y$ be a map between continua and let $n \in \mathbb{N}$. Then $C_n(f)$ is light if and only if for each two points A and B of $C_n(X)$ such that $A \subsetneq B$ and each component of B intersects A, we have that $f(A) \subsetneq f(B)$.*

Proof Suppose first that there are two points A and B in $C_n(X)$ such that $A \subsetneq B$, each component of B intersects A, and $f(A) = f(B)$. Hence, there is an order arc α from A to B in $C_n(X)$ [7, Theorem 1.8.20]. Clearly, $C_n(f)(\alpha) = \{f(A)\}$. Therefore, $C_n(f)$ is not light.

Now we assume that $C_n(f)$ is not light. Thus, there is a nondegenerate subcontinuum $\mathcal{A}$ of $C_n(X)$ such that $C_n(f)(\mathcal{A}) = \{D\}$ for some $D \in C_n(Y)$. By [7, Lemma 6.1.1], $\bigcup \mathcal{A} \in C_n(X)$. Since $\mathcal{A}$ is nondegenerate, there is $A \in \mathcal{A}$ such that $A \neq \bigcup \mathcal{A}$. Moreover, each component of $\cup \mathcal{A}$ intersects A, by [5, Lemma 3.1]. Therefore, $A \subsetneq \bigcup \mathcal{A}$ and $f(A) = f(\bigcup \mathcal{A}) = D$. ∎

We use the following simple two facts.

Fact 3.2 If $f\colon X \to Y$ is a light map and A is a proper subcontinuum of X, then $f|_A$ is also light.

Fact 3.3 Let $f\colon X \to Y$ be a map between continua such that there exists a point $y \in Y$ where $f^{-1}(y)$ is not connected. If P and Q are two different components of $f^{-1}(y)$, then there is an open subset W of Y such that $y \in W$ and, P and Q belong to different components of $f^{-1}(W)$.

Remark 3.4 Notice that since $C_n(f) = 2^f|_{C_n(X)}$, we have that if 2^f is light, then $C_n(f)$ is light, by Fact 3.2. Let $m < n$. It is not difficult to prove that $C_m(f) = C_n(f)|_{C_m(X)}$. Thus, if $C_n(f)$ is light, then $C_m(f)$ is light.

The next theorem shows a necessary and sufficient condition for the lightness of $C_n(f)$ for any $n \geq 2$.

Theorem 3.5 *Let $f\colon X \to Y$ be a map between continua and let $n \geq 2$. The following are equivalent conditions:*

(i) *For every two nondegenerate and disjoint subcontinua P and Q of X, we have that $f(P) \setminus f(Q) \neq \varnothing$ and $f(Q) \setminus f(P) \neq \varnothing$;*
(ii) *$C_n(f)$ is light.*

Proof Suppose that f satisfies (i). We show that $C_n(f)$ is light. Let A and B be points of $C_n(X)$ such that $A \subsetneq B$ and each component of B intersects A. Let $A_1, A_2, \dots, A_m$ be disjoint subcontinua of X such that $A = A_1 \cup A_2 \cup \dots \cup A_m$ for some $m \leq n$. Since $A \subsetneq B$, without loss of generality, we may suppose that $A_1 \subsetneq B_1$ for some component B_1 of B. We prove that $f(B_1) \setminus f(A) \neq \varnothing$.

First, we show that $f(B_1) \setminus f(A_1) \neq \varnothing$. Since $A_1 \subsetneq B_1$, there is a nondegenerate subcontinuum L_1 in $B_1 \setminus A_1$ by [9, Corollary 5.5]. Clearly, $L_1 \cap A_1 = \varnothing$. Since f satisfies (i), $f(L_1) \setminus f(A_1) \neq \varnothing$. Therefore, $f(B_1) \setminus f(A_1) \neq \varnothing$.

Now we suppose that $f(B_1) \setminus f(A_1 \cup A_2 \cup \dots \cup A_k) \neq \varnothing$, for some

$$k \in \{1, 2, \dots, m-1\}.$$

We prove that $f(B_1) \setminus f(A_1 \cup A_2 \cup \dots \cup A_k \cup A_{k+1}) \neq \varnothing$.

We show first that

$$B_1 \setminus (f^{-1}(f(A_1 \cup A_2 \cup \dots \cup A_k)) \cup A_{k+1}) \neq \varnothing. \tag{3.1}$$

Suppose that $B_1 \subset f^{-1}(f(A_1 \cup A_2 \cup \dots \cup A_k)) \cup A_{k+1}$. Since $A_1 \subsetneq B_1$, B_1 is a continuum, and A_i is a component of A for each $i \in \{1, 2, \dots, k+1\}$, we have that $B_1 \setminus (A_1 \cup A_2 \cup \dots \cup A_{k+1}) \neq \varnothing$. Hence, by [9, Corollary 5.5], there exists a nondegenerate subcontinuum K of $B_1 \setminus (A_1 \cup A_2 \cup \dots \cup A_{k+1})$. Since $B_1 \subset f^{-1}(f(A_1 \cup A_2 \cup \dots \cup A_k)) \cup A_{k+1}$,

$$K \subset f^{-1}(f(A_1 \cup A_2 \cup \dots \cup A_k)) \setminus (A_1 \cup A_2 \cup \dots \cup A_k). \tag{3.2}$$

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Thus, it is easy to show that

$$(3.3)\qquad K = (f^{-1}(f(A_1)) \cap K) \cup (f^{-1}(f(A_2)) \cap K) \cup \cdots \cup (f^{-1}(f(A_k)) \cap K).$$

Claim 3.6 *If $i \in \{1, 2, \ldots, k\}$, then $f^{-1}(f(A_i)) \cap K$ is closed and totally disconnected.*

Let $j \in \{1, 2, \ldots, k\}$. Clearly, $f^{-1}(f(A_j)) \cap K$ is closed. Suppose that $f^{-1}(f(A_j)) \cap K$ has a nondegenerate component R. Notice that $R \cap A_j = \varnothing$, by (3.2). But, R and A_j contradict condition (i). Therefore, $f^{-1}(f(A_i)) \cap K$ is totally disconnected and Claim 3.6 is proved.

Notice that $f^{-1}(f(A_i)) \cap K$ is 0-dimensional for each $i \in \{1, 2, \ldots, k\}$, by [10, Theorem 4.7]. Since K is a finite union of 0-dimensional and closed sets, K is 0-dimensional by [10, Theorem 5.2] (see (3.3)). But this contradicts the fact that K is a nondegenerate continuum. Therefore, we have that (3.1) is true.

Let L_k be a nondegenerate subcontinuum of $B_1 \setminus (f^{-1}(f(A_1 \cup A_2 \cup \cdots \cup A_k)) \cup A_{k+1})$ [9, Corollary 5.5]. Clearly, $f(L_k) \cap f(A_1 \cup A_2 \cup \cdots \cup A_k) = \varnothing$. Thus, L_k and A_{k+1} are nondegenerate subcontinua of X such that $L_k \cap A_{k+1} = \varnothing$. Hence, $f(L_k) \setminus f(A_{k+1}) \neq \varnothing$ by condition (i).Since $f(L_k) \subset f(B_1)$, $f(B_1) \setminus f(A_1 \cup A_2 \cup \cdots \cup A_{k+1}) \neq \varnothing$. Thus, $f(B_1) \setminus f(A) \neq \varnothing$. Since $f(B_1) \subset f(B)$, $f(B) \setminus f(A) \neq \varnothing$ and $f(A) \subsetneq f(B)$. Therefore, $C_n(f)$ is a light map, by Proposition 3.1.

Conversely, we suppose that condition (i) does not hold. Let A and B be nondegenerate subcontinua of X, such that $A \cap B = \varnothing$ and $f(A) \subset f(B)$. Let $a \in A$ and $n \geq 2$. We define P and Q in $C_n(X)$ by $P = \{a\} \cup B$ and $Q = A \cup B$. Clearly, $P \subsetneq Q$, each component of Q intersects P and $f(P) = f(Q)$. Therefore, $C_n(f)$ is not light by Proposition 3.1. ■

The next corollary follows from Theorem 3.5.

Corollary 3.7 *Let $f\colon X \to Y$ be a map between continua and let n and m be positive integers greater than 1. Then $C_n(f)$ is light if and only if $C_m(f)$ is light.*

By Theorem 3.5 and [2, Corollary 5.5], we have the following proposition.

Proposition 3.8 *Let $f\colon X \to Y$ be a map between continua. Consider the following conditions:*

(i) *2^f is light;*
(ii) *$C_n(f)$ is light for every $n \geq 2$;*
(iii) *$C(f)$ is light;*
(iv) *f is light.*

Then (i) *implies* (ii), (ii) *implies* (iii), *and* (iii) *implies* (iv).

The reader can find examples where the other implications are not true in [2]. Regarding the implications: (ii) implies (i) and (iii) implies (ii) the reader needs to use Theorem 3.5 to find the appropriate examples.

Theorem 3.9 *Let $f\colon X \to Y$ be a weakly confluent map between continua and let $n \geq 2$. If $C_n(f)$ is light, then f is a homeomorphism.*

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Proof We prove that f is monotone. Suppose that there are two points p_1 and p_2 of X such that $f(p_1) = f(p_2)$. Since $C_n(f)$ is light, f is light by Proposition 3.8. Thus, $\{p_1\}$ and $\{p_2\}$ are components of $f^{-1}(f(p_1))$.

By Fact 3.3, there is an open subset W of Y such that $f(p_1) \in W$ and both p_1 and p_2 belong to different components of $f^{-1}(W)$. Let P_1 and P_2 be nondegenerate subcontinua of $f^{-1}(W)$ such that $p_1 \in P_1$ and $p_2 \in P_2$ [9, Corollary 5.5]. Since f is light, $f(P_1)$ and $f(P_2)$ are nondegenerate subcontinua of W.

Let $K = f(P_1) \cup f(P_2)$. Since $f(p_1) \in f(P_1) \cap f(P_2)$, K is a subcontinuum of W. Since f is weakly confluent, there exists a component Q of $f^{-1}(K)$ such that $f(Q) = K$. Notice that since p_1 and p_2 belong to different components of $f^{-1}(W)$, we have that either $Q \cap P_1 = \varnothing$ or $Q \cap P_2 = \varnothing$. Clearly, $f(P_1) \subset f(Q)$ and $f(P_2) \subset f(Q)$. But this contradicts the fact that $C_n(f)$ is light by Theorem 3.5. Therefore, f is monotone. It is not difficult to show that a monotone and light map between continua is a homeomorphism and the proof is complete. ■

In [2, Example 4.5], a map f between continua is given such that $C(f)$ is light, surjective and f is not monotone.

Corollary 3.10 *Let $f: X \to Y$ be a weakly confluent map between continua. If 2^f is light, then f is a homeomorphism.*

Proof Since 2^f is light, $C_n(f)$ is light for each $n \geq 2$ by Proposition 3.8. Now the corollary follows from Theorem 3.9. ■

By Theorem 3.5 and [2, Corollary 5.7], we have the following result.

Theorem 3.11 *Let X be an arcwise connected continuum, let $f: X \to Y$ be a map between continua and let $n \geq 2$. Then $C_n(f)$ is light if and only if $C(f)$ is light.*

Theorems 3.9 and 3.11 imply the following corollary.

Corollary 3.12 *Let X be an arcwise connected continuum and let $f: X \to Y$ be a weakly confluent map. If $C(f)$ is light, then f is a homeomorphism.*

A *dendroid* is an arcwise connected and hereditarily unicoherent continuum. A point p in a dendroid X is called a *ramification* point, if $X \setminus \{p\}$ has three or more components. A *dendrite* is a locally connected dendroid. For general information about dendroids or dendrites, the reader may see [7, 9].

Proposition 3.13 *Let Y be a dendrite with only a finite number of ramification points, and let $f: [0,1] \to Y$ be a surjective map. If $C(f)$ is light, then f is a homeomorphism.*

Proof We prove that f is monotone. Suppose that there are two different points a and b in $[0,1]$ such that $f(a) = f(b)$. Since f is light (see Proposition 3.8), $\{a\}$ and $\{b\}$ are components of $f^{-1}(f(a))$. Suppose that $a < b$. Observe that $f([a,b])$ is a nondegenerate subdendrite of Y [9, Corollary 10.6].

Let $c \in [a,b]$ such that $f(c)$ is an end point and different from $f(a) = f(b)$. Since Y has only a finite number of ramification points, there exists a point $y_0 \in Y$ such that the arc from y_0 to $f(c)$, denoted by β, is a free arc in Y. Notice that $\beta \subset f([a,c]) \cap f([c,b])$. Let $t_0 = \max\{f^{-1}(y_0) \cap [a,c]\}$. It is easy to show that

$f([t_0, c]) = \beta$. Hence, $[c, b] \subsetneq [t_0, b]$ and $f([c, b]) = f([x_0, b])$. Thus, $C(f)$ is not light by Proposition 3.1. Hence, f is monotone and a light map. Therefore, f is a homeomorphism. ■

Theorem 3.14 *Let $f: X \to Y$ be a map, where X is an arcwise connected continuum and Y is a dendroid with only a finite number of ramification points. If $C(f)$ is light, then f is a homeomorphism.*

Proof We prove that f is monotone. Suppose there exist two points a and b in X such that $f(a) = f(b)$ and a and b belong to different components of $f^{-1}(f(a))$. Let α be an arc in X, where a and b are the end points of α. Since $C(f)$ is light, $C(f)|_{C(\alpha)}$ is light by Fact 3.2. It is easy to show that $C(f)|_{C(\alpha)} = C(f|_\alpha)$. Hence, $f|_\alpha$ is a homeomorphism by Proposition 3.13. This contradicts the fact that $f(a) = f(b)$. Thus, f is monotone. Since $C(f)$ is light, f is light by Proposition 3.8. Therefore, f is a homeomorphism. ■

Corollary 3.15 *Let $f: X \to Y$ be a map, where X is an arcwise connected continuum and Y is a dendroid with only a finite number of ramification points. Then the following are equivalent:*

(i) *2^f is light;*
(ii) *$C_n(f)$ is light, for some $n \geq 2$;*
(iii) *$C(f)$ is light;*
(iv) *f is a homeomorphism.*

Proof (i) implies (ii) and (ii) implies (iii) follows from Proposition 3.8. If $C(f)$ is light, then f is a homeomorphism by Theorem 3.14. Finally, it is known that since f is a homeomorphism, 2^f is a homeomorphism. Therefore, 2^f is light and our proof is complete. ■

The next example shows that the condition Y has only a finite number of ramification points may not be removed.

Example 3.16 There is a map $f: [0, 1] \to X$, where X is a dendrite such that 2^f is light and f is not a homeomorphism.

Let $X_1 = \{(x, 0) : -1 \leq x \leq 1\}$. Define $f_1: [0, 1] \to X_1$ such that:

- $f_1(0) = f_1(1) = (-1, 0)$ and $f_1(\frac{1}{2}) = (1, 0)$;
- $f_1|_{[0,\frac{1}{2}]}$ and $f_1|_{[\frac{1}{2},1]}$ are homeomorphisms.

Now let $X_2 = X_1 \cup J_{11}$, where $J_{11} = \{(0, y) : -\frac{1}{2} \leq y \leq \frac{1}{2}\}$, *i.e.,* J_{11} is an arc whose midpoint divides X_1 into two equal parts, and the size of J_{11} is half of the size of X_1. Notice that X_2 has four maximal free arcs. We divide $[0, 1]$ into 8 equal parts, *i.e.,* $\{[\frac{i}{8}, \frac{i+1}{8}] : i = 0, 1, \dots, 7\}$, and define $f_2: [0, 1] \to X_2$ an a surjective map, such that:

- $f_2(0) = f_2(1) = (-1, 0)$ and $f_2(\frac{1}{2}) = (1, 0)$.
- $f_2|_{[\frac{i}{8},\frac{i+1}{8}]}$ is a homeomorphism from $[\frac{i}{8}, \frac{i+1}{8}]$ onto a maximal free arc of X_2, for each $i = 0, 1, \dots, 7$. We do this counter clockwise.

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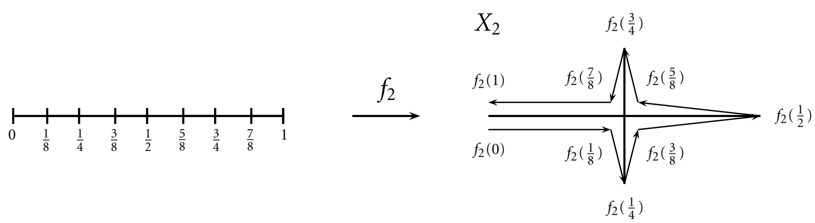

Figure 1

Figure 1 may clarify the definition of f_2.

It is important to note that if $f_2(t)$ is an end point of X_2 different of $(-1,0)$, then $f_2^{-1}(f_2(t)) = \{t\}$.

We do one more step. Let $X_3 = X_2 \cup (J_{21} \cup J_{22} \cup J_{23} \cup J_{24})$, where for each $i \in \{1,2,3,4\}$ J_{2i} is an arc whose midpoint divides each maximal free arc of X_2 into two equal parts, and the size of J_{2i} is half of the size of the arc which it divides. The continuum X_3 has 16 maximal free arcs. We divide $[0,1]$ into 32 equal parts, *i.e.*, $\{[\frac{i}{32}, \frac{i+1}{32}] : i \in \{0,1,\dots,31\}\}$ and define $f_3\colon [0,1] \to X_3$ to be a surjective map such that:

- $f_3(\frac{i}{8}) = f_2(\frac{i}{8})$ for each $i \in \{0,1,\dots,8\}$;
- $f|_{[\frac{i}{32},\frac{i+1}{32}]}$ is a homeomorphism from $[\frac{i}{32}, \frac{i+1}{32}]$ onto a maximal free arc of X_3, for each $i \in \{0,1,\dots,31\}$. We do this counter clockwise (see Figure 2).

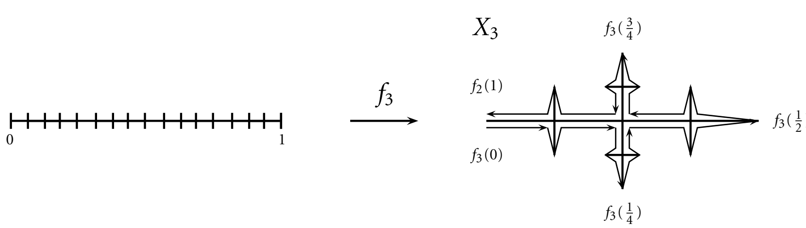

Figure 2

Inductively, suppose we have defined a dendrite X_{n-1} and a surjective map

$$f_{n-1}\colon [0,1] \to X_{n-1},$$

such that X_{n-1} has 4^{n-2} maximal free arcs and $f_{n-1}|_{[\frac{i}{2(4^{n-2})},\frac{i+1}{2(4^{n-2})}]}$ is a homeomorphism from $[\frac{i}{2(4^{n-2})}, \frac{i+1}{2(4^{n-2})}]$ onto a maximal free arc of X_{n-1}, for each $i \in \{0,1,\dots,2(4^{n-2})-1\}$. Let $X_n = X_{n-1} \cup \{J_{n-1i} : 1 \le i \le 4^{n-2}\}$, where J_{n-1i}

is an arc whose midpoint divides each maximal free arc of X_{n-1} into two equal parts, and the size of J_{n-1i} is half the size of the arc which it divides, for each $i \in \{1, 2, \dots, 4^{n-2}\}$. We divide $[0,1]$ into $2(4^{n-1})$ equal parts, *i.e.*, $\{[\frac{i}{2(4^{n-1})}, \frac{i+1}{2(4^{n-1})}] : 0 \le i \le 2(4^{n-1}) - 1\}$ and define $f_n \colon [0,1] \to X_n$ an a surjective map, such that:

- $f_n(\frac{i}{2(4^{n-2})}) = f_{n-1}(\frac{i}{2(4^{n-2})})$, for each $i \in \{0, 1, \dots, 2(4^{n-2})\}$;
- $f_n|_{[\frac{i}{2(4^{n-1})}, \frac{i+1}{2(4^{n-1})}]}$ is a homeomorphism from $[\frac{i}{2(4^{n-1})}, \frac{i+1}{2(4^{n-1})}]$ onto a maximal free arc of X_n, for each $i = 0, 1, \dots, 2(4^{n-1}) - 1$. We do this counter clockwise.

It is important to emphasize that for every interval $[\frac{i}{4^{n-1}}, \frac{i+1}{4^{n-1}}]$, there exists a point $t \in [\frac{i}{4^{n-1}}, \frac{i+1}{4^{n-1}}]$ such that $f_n(t)$ is an end point of X_n for each $i \in \{0, 1, \dots, 4^{n-1} - 1\}$. Thus, $f_n^{-1}(f_n(t)) = \{t\}$. Also, if $f_k(t)$ is an end point of X_k, then $f_m(t) = f_k(t)$ for every $m > k$ and $f_m(t)$ is an end point of X_m.

Let $X = \varprojlim\{X_n, \phi_n\}$, where $\phi_n \colon X_n \to X_{n-1}$ defined by

$$\phi_n(x) = \begin{cases} x & \text{if } x \notin J_i, \\ p_i & \text{if } x \in J_i, \quad \text{where } \{p_i\} = J_{n-1i} \cap X_{n-1}. \end{cases}$$

Remember that $X_n = X_{n-1} \cup \{J_{n-1i} : i \in \{1, 2, \dots, 4^{n-2}\}\}$. Since ϕ_n is monotone, for each $n \in \mathbb{N}$, X is a dendrite [7, Corollaries 2.1.14, 2.1.26]. Let $I = \varprojlim\{[0,1]_n, \varphi_n\}$, where $[0,1]_n = [0,1]$ and $\varphi_n \colon [0,1] \to [0,1]$ is defined such that $\phi_n \circ f_n = f_{n-1} \circ \varphi_n$ for each $n \in \mathbb{N}$. It is possible to check that φ_n is monotone, for each $n \in \mathbb{N}$. Thus, I is homeomorphic to $[0,1]$.

Let $f \colon I \to X$ be defined by $f\big(\{t_n\}_{n=1}^{\infty}\big) = \{f_n(t_n)\}_{n=1}^{\infty}$. Then the map f is a surjective map by [7, Theorem 2.1.48]. Clearly, f is not a homeomorphism.

Claim 3.17 *The set $\{t \in I : f^{-1}(f(t)) = \{t\}\}$ is dense in I.*

Let U be an open subset of I. Then there exists a positive integer n such that $[\frac{i}{4^{n-1}}, \frac{i+1}{4^{n-1}}] \subset U$ for some $i \in \{0, 1, \dots, 4^{n-1} - 1\}$. Hence, there exists a point $t \in [\frac{i}{4^{n-1}}, \frac{i+1}{4^{n-1}}]$ such that $f_k(t) = f_n(t)$ and $f_k(t)$ is an end point of X_k for every $k \ge n$. Furthermore, $f_k^{-1}(f_k(t)) = \{t\}$ for every $k \ge n$. Therefore, $f^{-1}(f(t)) = \{t\}$ and the claim is proved.

Finally, we prove that 2^f is light. Let A and B be points in 2^I such that $A \subsetneq B$ and each component of B intersects A. It is not difficult to show that there is an open subset U of I such that $U \subset B \setminus A$. By Claim 3.17, there is $t \in U$ such that $f^{-1}(f(t)) = \{t\}$. Thus, $f(t) \in f(B) \setminus f(A)$. Therefore, $f(A) \subsetneq f(B)$ and 2^f is a light map, by [2, (3.6) p. 183].

The idea of the next example is similar to [2, Example 5.2]. It gives a map f defined between arcwise connected continua such that $C_n(f)$ is light for every $n \in \mathbb{N}$ and 2^f is not light, giving a negative answer to [2, Questions 5.1 and 5.9].

Example 3.18 Let C be the Cantor set and let $Z = C \times [0,1]$. Let ρ be the Cantor function from C onto $[0,1]$ defined in [4, Figure 3-19, p.131]. We define the relation R on Z by

$$(x_1, y_1)R(x_2, y_2) \text{ if and only if} (x_1, y_1) = (x_2, y_2)$$

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or

$$(y_1 = y_2 = 1 \text{ and } \rho(x_1) = \rho(x_2)).$$

Let $X = Z/R$. Clearly, X is a dendroid (in particular, it is arcwise connected). Similarly, let $Y = Z/R'$, where R' is a relation on Z defined by

$$(x_1, y_1)R'(x_2, y_2) \text{ if and only if} (x_1, y_1) = (x_2, y_2)$$

or

$$(y_1, y_2 \in \{0, 1\} \text{ and } \rho(x_1) = \rho(x_2)).$$

Notice that $R \subset R'$. Let f be the natural map from X onto Y induced by the quotient maps q_R and $q_{R'}$, *i.e.*, $f \circ q_R = q_{R'}$. Let A and B be the closed subsets of X defined by

$$A = (C \times \{0\}) \cup \{(\tfrac{1}{4}, 1)\} \text{ and } B = (C \times \{0\}) \cup q_R(\{(x, 1) : x \in C\}).$$

Clearly, $A \subsetneq B$ and each component of B intersects A. Moreover, $f(A) = f(B)$. Let α be an order arc in 2^X from A to B [7, Theorem 1.8.20]. Clearly, $2^f(\alpha) = \{f(A)\}$. Therefore, 2^f is not light.

Let P and Q be disjoint and nondegenerate subcontinua of X. Notice that $f|_{X \setminus (C \times \{0\})}$ is a bijection. Since P and Q are nondegenerate subcontinua of X, there are points $p \in P$ and $q \in Q$ such that $\{p, q\} \subset X \setminus (C \times \{0\})$. Hence, $f(p) \in f(P) \setminus f(Q)$ and $f(q) \in f(Q) \setminus f(P)$. Thus, $C_n(f)$ is light for $n \geq 2$ by Theorem 3.5. Therefore, $C_n(f)$ is light for every $n \in \mathbb{N}$, by Theorem 3.11.

A map defined between continua $f\colon X \to Y$ is called *of order smaller than or equal to* k, if $|f^{-1}(y)| \leq k$ for every $y \in Y$. The maps of order smaller than or equal to 2 are said to be *simple* [1, p. 84]. Notice that Example 3.18 gives a map between arcwise connected continua f of order smaller than or equal to 3.

The next theorem shows that there is not a simple map f, such that $C_n(f)$ is light for some $n \geq 2$ and 2^f is not light.

Theorem 3.19 *Let $f\colon X \to Y$ be a simple map between continua. Then 2^f is light if and only if $C_n(f)$ is light for some $n \geq 2$.*

Proof If 2^f is light, then $C_n(f)$ is light, by Proposition 3.8. Let $f\colon X \to Y$ be a simple map between continua. We assume that 2^f is not light. Thus, there exist two points A and B in 2^X such that $A \subsetneq B$, each component of B intersects A and $f(A) = f(B)$ [2, Theorem 3.6].

Let $b \in B \setminus A$. Let B_0 be the component of B such that $b \in B_0$. By [9, Corollary 5.5], there is a nondegenerate subcontinuum D of $B_0 \setminus A$. Since f is simple, f is light. Hence, $f(D)$ is a nondegenerate subcontinuum of Y. Notice that $f(D) \subset f(B)$. Thus, $f(D) \subset f(A)$. Let $E = f^{-1}(f(D)) \cap A$. Clearly, $f(E) = f(D)$.

We show that E is connected. Suppose that E is not connected. Then there exist two closed subsets F_1 and F_2 of E such that $E = F_1 \cup F_2$ and $F_1 \cap F_2 = \varnothing$. Observe that F_1 and F_2 are closed subsets of X, and $f(D) = f(F_1) \cup f(F_2)$. Since $f(D)$ is connected, $f(F_1) \cap f(F_2) \neq \varnothing$. Hence, there are $x_1 \in F_1$ and $x_2 \in F_2$ such that $f(x_1) = f(x_2)$. Since f is simple, either $x_1 \in D$ or $x_2 \in D$. But this contradicts the fact that $D \cap A = \varnothing$. Therefore, E is connected.

Finally, D and E are disjoint and nondegenerate subcontinua of X such that $f(D) = f(E)$. Therefore, $C_n(f)$ is not light by Theorem 3.5. ∎

In [2, Example 4.5] a simple map f is given such that $C(f)$ is light and 2^f is not light.

A continuum X is *decomposable* provided that it can be written as the union of two of its proper subcontinua. We said that X is *indecomposable* if it is not decomposable. We said that X is *hereditarily decomposable* (*hereditarily indecomposable*) if each subcontinuum of X is decomposable (indecomposable, respectively).

Proposition 3.20 *Let $f\colon X \to Y$ be a surjective map between continua where Y is indecomposable. If $C(f)$ is light, then X is indecomposable.*

Proof Let $f\colon X \to Y$ be a map between continua where Y is indecomposable. Suppose that X is decomposable. Thus, there are two proper subcontinua A and B of X such that $X = A \cup B$. Clearly, $Y = f(A) \cup f(B)$. Since Y is indecomposable, either $f(A) = Y$ or $f(B) = Y$. Suppose that $f(B) = Y$. Hence, there exists an order arc α in $C(X)$ from B to X [6, Theorem 14.6]. It is easy to see that $C(f)(\alpha) = \{Y\}$. Therefore, $C(f)$ is not light. Similarly, if $f(A) = Y$. ∎

A similar argument shows the following proposition.

Proposition 3.21 *Let $f\colon X \to Y$ be a map between continua where Y is hereditarily indecomposable. If $C(f)$ is light, then X is hereditarily indecomposable.*

In [2, Example 4.5], a map f between indecomposable continua is given such that $C(f)$ is light and f is not monotone.

Question 3.22 *Let f be a map between hereditarily indecomposable continua. If $C(f)$ is light, then does it follow that f is a homeomorphism?*

4 Lightness of the Induced Map $C(C(f))$

The main result in this section is Theorem 4.2, where we show that if f is a confluent map and $C(C(f))$ is light, then f is a homeomorphism. The following result may be found in [5, Lemma 6.1].

Lemma 4.1 *Let $f\colon X \to Y$ be a confluent map. If α is an arc in $C(Y)$ and β is a component of $C(f)^{-1}(\alpha)$, then $C(f)(\beta) = \alpha$.*

Theorem 4.2 *Let $f\colon X \to Y$ be a confluent map between continua. If $C(C(f))$ is light, then f is a homeomorphism.*

Proof Let $f\colon X \to Y$ be a confluent map between continua such that $C(C(f))$ is light.

Let A and B be nondegenerate subcontinua of X such that $A \cap B = \varnothing$. We prove that $f(A) \setminus f(B) \neq \varnothing$. Suppose that $f(A) \subset f(B)$. Without loss of generality, we may suppose that $f(A) \subsetneq f(B)$, because if $f(A) = f(B)$, then there is a nondegenerate continuum $A_0 \subsetneq A$ and, since $C(f)$ is light by Proposition 3.8, we have that $f(A_0) \subsetneq f(A)$ by Proposition 3.1. Hence, $A_0 \cap B = \varnothing$ and $f(A_0) \subsetneq f(B)$. Therefore, we assume that $f(A) \subsetneq f(B)$.

Let D be the component of $f^{-1}(f(B))$ such that $A \subset D$. Since f is confluent, $f(D) = f(B)$. Observe that if $D \cap B \neq \varnothing$, then $B \subsetneq D$. But this contradicts the fact that $C(f)$ is light by Proposition 3.1. Thus, $D \cap B = \varnothing$.

Let γ be an order arc in $C(X)$ from A to D [6, Theorem 14.6]. Since f is light, it is not difficult to show that $C(f)(\gamma)$ is an arc in $C(Y)$ from $f(A)$ to $f(B)$. Let ζ be the component of $C(f)^{-1}(C(f)(\gamma))$ such that $B \in \zeta$. By Lemma 4.1, $C(f)(\zeta) = C(f)(\gamma)$. Since ζ is a component, we have either $\gamma \subsetneq \zeta$ or $\gamma \cap \zeta = \varnothing$.

Notice that $\gamma \subsetneq \zeta$ contradicts the fact that $C(C(f))$ is light by Proposition 3.1. Hence, suppose that $\gamma \cap \zeta = \varnothing$. Notice that γ and ζ are nondegenerate subcontinua of $C(X)$ and $C(f)(\zeta) = C(f)(\gamma)$. Thus, $C_n(C(f))$ is not light by Theorem 3.5. Since $C(X)$ is arcwise connected, $C(C(f))$ is not light by Theorem 3.11.

Thus, $f(A) \setminus f(B) \neq \varnothing$. Similarly, we show that $f(B) \setminus f(A) \neq \varnothing$. Hence, $C_n(f)$ is light for some $n \geq 2$. Therefore, f is a homeomorphism by Theorem 3.9. ■

Corollary 4.3 *Let $f\colon X \to Y$ be a map between continua where Y is hereditarily indecomposable. If $C(C(f))$ is light, then f is a homeomorphism.*

Proof Let Y be a hereditarily indecomposable continuum and let $f\colon X \to Y$ be a map. By [8, (6.11), p 53], f is confluent. Thus, the corollary follows of Theorem 4.2. ■

Let $f(t) = e^{2\pi i t}$ be a map from $[0,1]$ to S^1. It is not difficult to show that $C(f)|_{C([0,1])\setminus\{\{0\},\{1\}\}}$ is an injective map. Thus, if $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $C([0,1])$ such that $\mathcal{A} \subsetneq \mathcal{B}$, then $C(f)(\mathcal{A}) \subsetneq C(f)(\mathcal{B})$. Therefore, $C(C(f))$ is light and, clearly, f is not a homeomorphism.

Question 4.4 *Let f be a weakly confluent map. If $C(C(f))$ is light, then does it follow that f is a homeomorphism?*

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References

[1] K. Borsuk and R. Molski, *On a class of continuous mappings.* Fund. Math. **45**(1958), 84–98.

[2] J. J. Charatonik and W. J. Charatonik, *Lightness of induced mappings.* Tsukuba J. Math. **22**(1998), no. 1, 179–192.

[3] J. J. Charatonik, A. Illanes, and S. Macías, *Induced mappings on the hyperspaces $C_n(X)$ of a continuum X.* Houston J. Math. **28**(2002), no. 4, 781–805.

[4] J. G. Hocking and G. S. Young, *Topology.* Second edition. Dover Publications, New York, 1988.

[5] H. Hosokawa, *Induced mappings on hyperspaces.* Tsukuba J. Math. **21**(1997), no. 1, 239–250.

[6] A. Illanes and S. B. Nadler, Jr., *Hyperspaces. Fundamentals and Recent Advances.* Monographs and Textbooks in Pure and Applied Mathematics 216, Marcel Dekker, New York, 1999.
[7] S. Macías, *Topics on Continua.* Pure and Applied Mathematics Series 275, Chapman & Hall/CRC, Boca Raton, FL, 2005.
[8] T. Maćkowiak, *Continuos mappings on continua.* Dissertationes Math. (Rozprawy Mat.) **158**(1979), 1–95.
[9] S. B. Nadler, Jr., *Continuum Theory, An Introduction.* Monographs and Textbooks in Pure and Applied Mathematics 158, Marcel Dekker, New York, 1992.
[10] ______, *Dimension Theory: An Introduction with Exercises.* Aportaciones Matemáticas. Textos 18. Sociedad Matemática Mexicana. México, 2002.
[11] ______, *Hyperspaces of Sets. A Text with Research Questions.* Aportaciones Matemáticas, Textos 33, Sociedad Matemática Mexicana, 2006.

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