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ON KILLERS OF CABLE KNOT GROUPS

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Abstract

A killer of a group G is an element that normally generates G. We show that the group of a cable knot contains infinitely many killers such that no two lie in the same automorphic orbit.

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1. Introduction

Let G be an arbitrary group and $S \subseteq G$. We define the normal closure $\langle\langle S \rangle\rangle_G$ of S as the smallest normal subgroup of G containing S or, equivalently,

$$\langle\langle S \rangle\rangle_G = \Big\{\prod_{i=1}^{k} u_i s_i^{\varepsilon_i} u_i^{-1} \mid u_i \in G, \varepsilon_i = \pm 1, s_i \in S, k \in \mathbb{N}\Big\}.$$

Following [5], we call an element $g \in G$ a *killer* if $\langle\langle g \rangle\rangle_G = G$. We say that two killers $g_1, g_2 \in G$ are *equivalent* if there exists an automorphism $\phi : G \to G$ such that $\phi(g_1) = g_2$.

Let $\mathfrak{k}$ be a knot in S^3 and $V(\mathfrak{k})$ a regular neighbourhood of $\mathfrak{k}$. Denote by

$$X(\mathfrak{k}) = S^3 - \mathrm{Int}(V(\mathfrak{k}))$$

the knot manifold of $\mathfrak{k}$ and by $G(\mathfrak{k}) = \pi_1(X(\mathfrak{k}))$ its group. A *meridian* of $\mathfrak{k}$ is an element of $G(\mathfrak{k})$ which can be represented by a simple closed curve on $\partial V(\mathfrak{k})$ that is contractible in $V(\mathfrak{k})$ but not contractible in $\partial V(\mathfrak{k})$. Thus, a meridian is well defined up to conjugacy and inversion.

From a Wirtinger presentation of $G(\mathfrak{k})$, we see that the meridian is a killer. In [6, Theorem 3.11], the author exhibits a knot for which there exists a killer that is not equivalent to the meridian. Silver *et al.* [4, Corollary 1.3] showed that if $\mathfrak{k}$ is a

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hyperbolic 2-bridge knot or a torus knot or a hyperbolic knot with unknotting number one, then its group contains infinitely many pairwise inequivalent killers.

In [4, Conjecture 3.3], it is conjectured that the group of any nontrivial knot has infinitely many inequivalent killers. See also [1, Question 7.1.8]. In this paper we show the following result.

Theorem 1.1. *Let $\mathfrak{k}$ be a cable knot about a nontrivial knot $\mathfrak{k}_1$. Then its group contains infinitely many pairwise inequivalent killers.*

Moreover, we show that having infinitely many inequivalent killers is preserved under connected sums. As a corollary, we show that the group of any nontrivial knot whose exterior is a graph manifold contains infinitely many inequivalent killers.

2. Proof of Theorem 1

Let m, n be coprime integers with $n \geq 2$. The *cable space* $CS(m, n)$ is defined as follows: let $D^2 = \{z \in \mathbb{C} \mid \|z\| \leq 1\}$ and let $\rho : D^2 \to D^2$ be a rotation through an angle of $2\pi(m/n)$ about the origin. Choose a disk $\delta \subset \text{Int}(D^2)$ such that $\rho^i(\delta) \cap \rho^j(\delta) = \emptyset$ for $1 \leq i \neq j \leq n$ and denote by D^2_n the space

$$D^2 - \text{Int}\Big(\bigcup_{i=1}^{n} \rho^i(\delta)\Big).$$

Then ρ induces a homeomorphism $\rho_0 := \rho|_{D^2_n} : D^2_n \to D^2_n$. We define $CS(m, n)$ as the mapping torus of ρ_0, that is,

$$CS(m, n) := D^2_n \times I/(z, 0) \sim (\rho_0(z), 1).$$

Note that $CS(m, n)$ has the structure of a Seifert fibred space. Each fibre is the image of $\{\rho^i(z) \mid 1 \leq i \leq n\} \times I$ under the quotient map, where $z \in D^2_n$. There is exactly one exceptional fibre, namely the image C_0 of the arc $0 \times I$.

In order to compute the fundamental group A of $CS(m, n)$, denote the free generators of $\pi_1(D^2_n)$ corresponding to the boundary paths of the removed disks $\rho_0(\delta), \ldots, \rho_0^n(\delta)$ by $x_1, \ldots, x_n$, respectively. From the definition of $CS(m, n)$, we see that we can write A as the semi-direct product $F(x_1, \ldots, x_n) \rtimes \mathbb{Z}$, where the action of $\mathbb{Z} = \langle t \rangle$ on $\pi_1(D^2_n) = F(x_1, \ldots, x_n)$ is given by

$$t x_i t^{-1} = x_{\sigma(i)} \quad \text{for } 1 \leq i \leq n.$$

The element t is represented by the exceptional fibre of $CS(m, n)$ and the permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is given by $i \mapsto i + m \bmod n$. Thus,

$$A = \langle x_1, \ldots, x_n, t \mid t x_i t^{-1} = x_{\sigma(i)} \text{ for } 1 \leq i \leq n\rangle.$$

We finally remark that any element $a \in A$ is uniquely written as $w \cdot t^z$ for some $w \in F(x_1, \ldots, x_n)$ and $z \in \mathbb{Z}$.

We next define cable knots. Let V_0 be the solid torus $D^2 \times I/(z, 0) \sim (\rho(z), 1)$ and, for some $z_0 \in \text{Int}(D^2) - 0$, let $\mathfrak{k}_0$ be the image of $\{\rho^i(z_0) \mid 1 \leq i \leq n\} \times I$ under the quotient

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map. Note that $\mathfrak{k}_0$ is a simple closed curve contained in the interior of V_0. Let $\mathfrak{k}_1$ be a nontrivial knot in S^3 and $V(\mathfrak{k}_1)$ a regular neighbourhood of $\mathfrak{k}_1$ in S^3. Further, let $h : V_0 \to V(\mathfrak{k}_1)$ be a homeomorphism which maps the meridian $\partial D^2 \times 1$ of V_0 to a meridian of $\mathfrak{k}_1$. The knot $\mathfrak{k} := h(\mathfrak{k}_0)$ is called an (m, n)-*cable knot* about $\mathfrak{k}_1$.

Thus, the knot manifold $X(\mathfrak{k})$ of an (m, n)-cable knot $\mathfrak{k}$ decomposes as

$$X(\mathfrak{k}) = CS(m, n) \cup X(\mathfrak{k}_1)$$

with $\partial X(\mathfrak{k}_1) = CS(m, n) \cap X(\mathfrak{k}_1)$ an incompressible torus in $X(\mathfrak{k})$. It follows from the theorem of Seifert and van Kampen that

$$G(\mathfrak{k}) = A *_C B,$$

where $B = G(\mathfrak{k}_1)$ and $C = \pi_1(\partial X(\mathfrak{k}_1))$. Denote by m_1 the meridian of $\mathfrak{k}_1$ and note that in A we have $m_1 = x_1 \cdot \ldots \cdot x_n$. In turn, the meridian $m \in G(\mathfrak{k})$ of $\mathfrak{k}$ is written as $m = x_1 \in A$.

The proof of Theorem 1.1 is divided into two steps. In Lemma 2.1, we exhibit elements that normally generate the group of the cable knot and, next, in Lemma 2.2, we prove that these killers are indeed inequivalent.

Choose $s \in \{1, \ldots, n-1\}$ such that $\sigma^s(1) = 2$. Since $\sigma^s(i) = i + sm \bmod n$, it follows that $\sigma^s = (1\ \ 2\ \ 3\ \ \ldots\ \ n-1\ \ n)$.

Lemma 2.1. *Let $\mathfrak{k}$ be an (m, n)-cable knot about a nontrivial knot $\mathfrak{k}_1$. Then, for each $l \geq 1$, the element*

$$g_l := x_1^l x_2^{-(l-1)} = x_1^l \cdot (t^s x_1 t^{-s})^{-(l-1)}$$

normally generates the group of $\mathfrak{k}$.

Proof. The first step of the proof is to show that the group of the companion knot is contained in $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$.

Claim 1. The meridian $m_1 = x_1 \cdot \ldots \cdot x_n$ of $\mathfrak{k}_1$ belongs to $\langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$. Consequently, $B = \langle\langle m_1 \rangle\rangle_B = \langle\langle m_1 \rangle\rangle_{G(\mathfrak{k})} \cap B \subseteq \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$.

Note that for $0 \leq i \leq n-1$, $t^{is} g_l t^{-is} = x_{i+1}^l x_{i+2}^{-(l-1)}$, where indices are taken mod n. Thus,

$$\begin{aligned}
x_1 \cdot \ldots \cdot x_n &= x_1^{-(l-1)}(x_1^l x_2 \cdot \ldots \cdot x_n x_1^{-(l-1)}) x_1^{l-1} \\
&= x_1^{-(l-1)} \Big(\prod_{i=0}^{n-1} x_{i+1}^l x_{i+2}^{-(l-1)} \Big) x_1^{l-1} \\
&= x_1^{-(l-1)} \Big(\prod_{i=0}^{n-1} t^{is} \cdot g_l \cdot t^{-is} \Big) x_1^{l-1} \\
&= \prod_{i=0}^{n-1} (x_n^{-(l-1)} t^{is} \cdot g_l \cdot t^{-is} x_1^{l-1}) \\
&= \prod_{i=0}^{n-1} ((x_1^{-(l-1)} t^{is}) \cdot g_l \cdot (x_1^{(-l-1)} t^{is})^{-1}),
\end{aligned}$$

which implies that $m_1 \in \langle\langle g_l \rangle\rangle_{G(\mathfrak{k})}$. Thus, Claim 1 is proved.

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From Claim 1, it follows that the peripheral subgroup $C = \pi_1(\partial X(\mathfrak{k}_1))$ of $\mathfrak{k}_1$ is contained in $\langle\langle g_l\rangle\rangle_{G(\mathfrak{k})}$ since $C \subseteq B$ and consequently

$$G(\mathfrak{k})/\langle\langle g_l\rangle\rangle_{G(\mathfrak{k})} = A *_C B/\langle\langle g_l\rangle\rangle_{G(\mathfrak{k})} \cong A/\langle\langle g_l, C\rangle\rangle_A.$$

Thus, we need to show that $A/\langle\langle g_l, C\rangle\rangle_A = 1$. It is easy to see that $A/\langle\langle C\rangle\rangle_A$ is cyclically generated by $\pi(x_1)$, where $\pi : A \to A/\langle\langle C\rangle\rangle_A$ is the canonical projection. The result now follows from the fact that $\pi(g_l) = \pi(x_1^l \cdot t^s x_1^{(l-1)} t^{-s}) = \pi(x_1)$. □

Lemma 2.2. *If $k \neq l$, then g_k is not equivalent to g_l.*

Proof. Assume that $\phi : G(\mathfrak{k}) \to G(\mathfrak{k})$ is an automorphism such that $\phi(g_l) = g_k$ and let $f : X(\mathfrak{k}) \to X(\mathfrak{k})$ be a homotopy equivalence inducing ϕ. From [3, Theorem 14.6], it follows that f can be deformed into $\hat{f} : X(\mathfrak{k}) \to X(\mathfrak{k})$ so that $\hat{f}$ sends $X(\mathfrak{k}_1)$ homeomorphically onto $X(\mathfrak{k}_1)$ and $\hat{f}|_{CS(m,n)} : CS(m,n) \to CS(m,n)$ is a homotopy equivalence. Thus, $\phi(A)$ is conjugated to A, that is, $\phi(A) = gAg^{-1}$ for some $g \in G(\mathfrak{k})$. Since $\phi(g_l) = g_k$, this implies that $g_k \in gAg^{-1}$. As g_k is not conjugated (in A) to an element of C, it follows that $g \in A$ and so $\phi(A) = A$. By [3, Proposition 28.4], we may assume that $\hat{f}|_{CS(m,n)}$ is fibre preserving. Since $CS(m,n)$ has exactly one exceptional fibre, which represents t, we must have $\phi(t) = at^\eta a^{-1}$ for some $a = v \cdot t^{z_1} \in A$ and some $\eta \in \{\pm 1\}$.

The automorphism $\phi|_A : A \to A$ induces an automorphism ϕ_* on the factor group $A/\langle t^n\rangle = \langle x_1, t \mid t^n = 1\rangle = \mathbb{Z} * \mathbb{Z}_n$ such that $\phi_*(t) = at^\eta a^{-1}$. It is a standard fact about automorphisms of free products that we must have $\phi_*(x_1) = at^{e_0} x_1^\varepsilon t^{e_1} a^{-1}$ for $e_0, e_1 \in \mathbb{Z}$ and $\varepsilon \in \{\pm 1\}$. Thus, for some $d \in \mathbb{Z}$,

$$\phi(x_1) = at^{e_0} x_1^\varepsilon t^{e_1} a^{-1} t^{dn} = at^{e_0} \cdot x_1 t^{e_0+e_1+dn} \cdot t^{-e_0} a^{-1}.$$

Since t has nonzero homology in $H_1(X(\mathfrak{k}))$, it follows that $e_0 + e_1 + dn = 0$. Consequently, $\phi(x_1) = b \cdot x_1^\varepsilon \cdot b^{-1}$, where $b = at^{e_0} = v \cdot t^{z_2} \in A$ and $z_2 = z_1 + e_0$.

Hence,

$$\begin{aligned}
\phi(g_l) &= \phi(x_1^l x_2^{-(l-1)}) \\
&= \phi(x_1^l \cdot t^s x_1^{-(l-1)} t^{-s}) \\
&= bx_1^{\varepsilon l} b^{-1} \cdot at^{\eta s} a^{-1} \cdot bx_1^{-\varepsilon(l-1)} b^{-1} \cdot at^{-\eta s} a^{-1} \\
&= vt^{z_2} x_1^{\varepsilon l} t^{-z_2} v^{-1} \cdot vt^{z_1} t^{\eta s} t^{-z_1} v^{-1} \cdot vt^{z_2} x_1^{-\varepsilon(l-1)} t^{-z_2} v^{-1} \cdot vt^{z_1} t^{-\eta s} t^{-z_1} v^{-1} \\
&= vx_i^{\varepsilon l} x_j^{-\varepsilon(l-1)} v^{-1},
\end{aligned}$$

where $i = \sigma^{z_2}(1)$ and $j = \sigma^{z_2+\eta s}(1)$. Note that $i \neq j$ since $\sigma^s(1) = 2$ and $\sigma^{-s}(1) = n$. Hence, $\phi(g_l) = g_k$ implies that

$$v(x_i^{\varepsilon l} \cdot x_j^{-\varepsilon(l-1)}) v^{-1} = x_1^k \cdot x_2^{-(k-1)}$$

in $F(x_1, \ldots, x_n)$. Thus, in the abelianisation of $F(x_1, \ldots, x_n)$,

$$\varepsilon[lx_i + (1-l)x_j] = kx_1 + (1-k)x_2,$$

which implies that $\{i, j\} = \{1, 2\}$. If $(i, j) = (1, 2)$, then $\varepsilon l = k$ and so $k = |k| = |\varepsilon l| = l$. If $(i, j) = (2, 1)$, then $\varepsilon l = k - 1$ and $\varepsilon(1 - l) = k$. Consequently, $\varepsilon = 1$ and $l + k = 1$, which is impossible since $k, l \geq 1$. □

3. Connected sums and killers

In this section we show that having infinitely many inequivalent killers is preserved under connected sums of knots. This fact, Theorem 1.1 and [4, Corollary 1.3] imply that the group of knots whose exterior is a graph manifold have infinitely many inequivalent killers.

Let $\mathfrak{k}$ be a knot and $\mathfrak{k}_1, \dots, \mathfrak{k}_n$ its prime factors, that is, $\mathfrak{k} = \mathfrak{k}_1 \sharp \dots \sharp \mathfrak{k}_n$ and each $\mathfrak{k}_i$ is a nontrivial prime knot. Assume that $x \in G(\mathfrak{k}_i)$ is a killer of $G(\mathfrak{k}_i)$. It is well known that $G(\mathfrak{k}_i) \leq G(\mathfrak{k})$ and $\langle m \rangle \leq G(\mathfrak{k}_i)$ for all i, where m denotes the meridian of $\mathfrak{k}$. From this, we immediately see that $m \in \langle\langle x \rangle\rangle_{G(\mathfrak{k}_i)} \subseteq \langle\langle x \rangle\rangle_{G(\mathfrak{k})}$, which implies that $G(\mathfrak{k}) = \langle\langle m \rangle\rangle_{G(\mathfrak{k})} \subseteq \langle\langle x \rangle\rangle_{G(\mathfrak{k})}$, that is, x is a killer of $G(\mathfrak{k})$.

Now suppose that $x, y \in G(\mathfrak{k}_i)$ are killers of $G(\mathfrak{k}_i)$ and that there exists an automorphism ϕ of $G(\mathfrak{k})$ such that $\phi(x) = y$. Then ϕ is induced by a homotopy equivalence $f : X(\mathfrak{k}) \to X(\mathfrak{k})$. From [3, Theorem 14.6], it follows that f can be deformed into $\hat{f} : X(\mathfrak{k}) \to X(\mathfrak{k})$ so that:

(1) $\hat{f}|_V : V \to V$ is a homotopy equivalence, where $V = S^1 \times (n\text{-punctured disk})$ is the peripheral component of the characteristic submanifold of $X(\mathfrak{k})$;
(2) $\hat{f}|_{\overline{X(\mathfrak{k}) - V}} : \overline{X(\mathfrak{k}) - V} \to \overline{X(\mathfrak{k}) - V}$ is a homeomorphism.

Note that $\overline{X(\mathfrak{k}) - V} = X(\mathfrak{k}_1) \dot{\cup} \cdots \dot{\cup} X(\mathfrak{k}_n)$. Since $\hat{f}|_{\overline{X(\mathfrak{k}) - V}}$ is a homeomorphism, it follows that $\hat{f}$ sends $X(\mathfrak{k}_i)$ homeomorphically onto $X(\mathfrak{k}_{\tau(i)})$ for some permutation τ of $\{1, \dots, n\}$. Consequently, there exists $g' \in G(\mathfrak{k})$ such that

$$\phi(G(\mathfrak{k}_i)) = g' G(\mathfrak{k}_{\tau(i)}) g'^{-1}.$$

If $\tau(i) = i$ and $g' \in G(\mathfrak{k}_i)$, then ϕ induces an automorphism $\psi := \phi|_{G(\mathfrak{k}_i)}$ of $G(\mathfrak{k}_i)$ such that $\psi(x) = y$, that is, x and y are equivalent in $G(\mathfrak{k}_i)$. If $\tau(i) \neq i$ or $g' \notin G(\mathfrak{k}_{\tau(i)})$, then it is not hard to see that y is conjugated (in $G(\mathfrak{k}_i)$) to an element of $\langle m \rangle$ since $y = \phi(x) \in G(\mathfrak{k}_i) \cap g' G(\mathfrak{k}_{\tau(i)}) g'^{-1}$. As $\langle\langle m^k \rangle\rangle \neq G(\mathfrak{k})$ for $|k| \geq 2$ and y normally generates $G(\mathfrak{k})$, we conclude that y is conjugated (in $G(\mathfrak{k}_i)$)) to $m^{\pm 1}$. The same argument applied to ϕ^{-1} shows that x is conjugated (in $G(\mathfrak{k}_i)$)) to $m^{\pm 1}$.

Therefore, if the group of one of the prime factors of $\mathfrak{k}$ has infinitely many inequivalent killers, then so does the group of $\mathfrak{k}$. As a corollary of Theorem 1.1 and the remark made above, we obtain the following result.

Corollary 3.1. *If $\mathfrak{k}$ is a knot such that $X(\mathfrak{k})$ is a graph manifold, then $G(\mathfrak{k})$ contains infinitely many pairwise inequivalent killers.*

Proof. From [2], it follows that the only Seifert-fibred manifolds that can be embedded into a knot manifold with incompressible boundary are torus knot complements, composing spaces and cable spaces. Thus, if $X(\mathfrak{k})$ is a graph manifold, then one of the following holds:

(1) $\mathfrak{k}$ is a torus knot;
(2) $\mathfrak{k}$ is a cable knot;
(3) $\mathfrak{k} = \mathfrak{k}_1 \sharp \dots \sharp \mathfrak{k}_n$, where each $\mathfrak{k}_i$ is either a torus knot or a cable knot.

Now the result follows from Theorem 1.1 and [4, Corollary 1.3]. □

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