RINGS CHARACTERIZED BY CYCLIC MODULES

by DINH VAN HUYNH and PHAN DAN

(Received 22 October, 1987; revised 10 June, 1988)

1. Introduction. A ring R is called right PCI if every proper cyclic right R-module is injective, i.e. if C is a cyclic right R-module then $C_R \cong R_R$ or C_R is injective. By [**2**] and [**3**], if R is a non-artinian right PCI ring then R is a right hereditary right noetherian simple domain. Such a domain is called a right PCI domain. The existence of right PCI domains is guaranteed by an example given in [**2**]. As generalizations of right PCI rings, several classes of rings have been introduced and investigated, for example right CDPI rings, right CPOI rings (see [**8**], [**6**]). In Section 2 we define right PCS, right CPOS and right CPS rings and study the relationship between all these rings.

In Section 3 we consider rings each of whose cyclic right modules is a direct sum of an injective module and a semisimple module (briefly, CIS rings). By [**4**, Theorem 2.6], a ring R is CIS if and only if R is a generalized uniserial ring with $J^2 = 0$, where J is the Jacobson radical of R. We shall prove that a ring R is CIS if and only if every right R-module is a direct sum of a projective module and a semisimple module.

2. Definitions and results. Throughout, rings mean associative rings with identity and all modules are unitary. For a module M over a ring R, we write M_R to indicate that M is a right R-module, the socle of M is denoted by $\mathrm{Soc}(M)$. If $M = \mathrm{Soc}(M)$, M is called semisimple. Following Smith [**8**], a ring R is called right CPOI (resp. right CDPI) if every cyclic right R-module is projective or injective (resp. a direct sum of a projective module and an injective module). Clearly there are implications:

$$\text{right PCI} \quad \Rightarrow \quad \text{right CPOI} \quad \Rightarrow \quad \text{right CDPI}.$$

Now, a ring R is called right CPS if every cyclic right R-module is a direct sum of a projective module and a semisimple module. Further, a ring R is called right PCS if every proper cyclic right R-module is semisimple. Finally, a ring R is called right CPOS if every cyclic right R-module is projective or semisimple. From the definitions we easily see the implications:

$$\text{right PCS} \quad \Rightarrow \quad \text{right CPOS} \quad \Rightarrow \quad \text{right CPS}.$$

THEOREM 1. *Let R be a ring with Jacobson radical J.*

(I) *The following statements hold*:

(a) *R is a right CPOI ring if and only if R is right CPOS with $J = 0$;*

(b) *R is right CPOS with $J \neq 0$ if and only if $R_R = U \oplus S$, where U_R is uniform having* (*composition*) *length* 2 *and S_R is semisimple*;

(c) *R is right PCI if and only if R is right PCS with $J = 0$;*

(d) *R is right PCS with $J \neq 0$ if and only if J is the maximal and minimal right ideal of R.*

(II) *If R is right CPS then the following conditions are equivalent*:
(i) *R is right CDPI*;
(ii) *J_R is projective*;
(iii) *R is right non-singular.*

For the proof of Theorem 1 we need the following lemma.

LEMMA 2. *Let R be a right CPS ring with Jacobson radical J. Then*:
(a) *R_R is a direct sum of noetherian uniform modules*;
(b) *J_R is semisimple.*

Proof. By [**1**, Theorem 3.1], every right CPS ring is right noetherian. Hence R has the following decomposition

$$R_R = e_1R \oplus \ldots \oplus e_nR,$$

where $\{e_i\}_{i=1}^n$ is a set of orthogonal idempotents of R and each e_iR is indecomposable. We shall show that each e_iR is uniform.

We set $e = e_i$ for a fixed i and consider the right R-module eR. Let M be a non-zero submodule of eR. By assumption there are submodules M_1, M_2 of eR containing M such that

$$eR/M = M_1/M \oplus M_2/M,$$

where M_1/M is projective and M_2/M is semisimple. It follows that eR/M_2 is projective. Hence $eR = M_2 \oplus C$ for some submodule C of eR. Since eR is indecomposable, C has to be zero, i.e. eR/M is semi-simple. Now let H be a submodule of eR with $M \cap H = 0$. Then there exists a submodule B of eR containing M such that

$$eR/M = B/M \oplus (M+H)/M.$$

From this it follows that $eR = B \oplus H$. Hence $H = 0$, showing that eR is uniform, i.e. (a) holds.

For (b), let $J(e_iR)$ be the Jacobson radical of e_iR. Then

$$J = J(e_1R) \oplus \ldots \oplus J(e_nR).$$

By (a), for each non-zero submodule M of e_iR, e_iR/M is semisimple. Therefore M is an intersection of finitely many maximal submodules of e_iR. It follows that $M \supseteq J(e_iR)$. This shows that $J(e_iR)$ is simple if $J(e_iR)$ is non-zero. Thus J_R is semisimple.

Proof of (I) *of Theorem* 1. (a) Let R be a right CPOI ring. By [**6**, Theorem] or by [**8**, Theorem 2.12], $R = A \oplus B$, where A is a semiprime artinian ring and B is a right CPI domain. Then B is right noetherian and for each non-zero right ideal B_1 of B, B/B_1 is B-injective and each of its submodules is injective too. Therefore B/B_1 is injective and semisimple. From this we can easily see that R is right CPOS with $J = 0$.

Conversely, let R be right CPOS with $J = 0$. Then, in particular, R is right CPS. By Lemma 2,

$$R_R = e_1R \oplus \ldots \oplus e_nR, \tag{1}$$

https://doi.org/10.1017/S0017089500007801 Published online by Cambridge University Press

where $\{e_i\}_{i=1}^n$ is a set of orthogonal idempotents of R and each e_iR is uniform. If R is right artinian R is semisimple, in particular R is right CPOI. Suppose now that R is not right artinian. Then there is an e_iR, e_1R say, which is a non-artinian right R-module. Therefore e_1R contains a non-zero submodule M with $M \neq e_1R$. Since e_1R is uniform, for such a submodule M of e_1R, e_1R/M is semisimple by hypothesis. Hence $\mathrm{Soc}(e_1R)=0$. Since

$$R/M \cong e_1R/M \oplus e_2R \oplus \ldots \oplus e_nR$$

with non-projective e_1R/M, R/M has to be semisimple. Hence

$$A := \mathrm{Soc}(R_R) = e_2R \oplus \ldots \oplus e_nR.$$

Put $B = e_1R$; we have $R_R = A \oplus B$ with $\mathrm{Soc}(B_R) = 0$. Clearly $BA = 0$. Furthermore, since $(AB)^2 = A(BA)B = 0$, it follows that $AB \subseteq J = 0$, proving that B is also an ideal of R. This shows that R is a direct sum of a semiprime artinian ring A and a right CPS domain B. Now, for each non-zero right ideal H of B, B/H is a semisimple right B-module. By **[8]**, it follows that B is a right V-ring, i.e. each simple right B-module is injective. Hence B/H is an injective right B-module, proving that B is a right CPI domain. By **[6**, Theorem] or by **[8**, Theorem 2.12], R is right CPOI.

(b) Let R be a right CPOS ring with $J \neq 0$. As above, R_R has a decomposition (1). Since $J \neq 0$, there is an e_iR, e_1R say, such that e_1R is not simple. Let M be a proper non-zero submodule of e_1R. Then as we have seen above, e_1R/M is semisimple and non-projective. Therefore $S = e_2R \oplus \ldots \oplus e_nR$ is also semisimple. We have $J \subseteq e_1R$. Hence by Lemma 2 and by the uniformity of e_1R, J_R is simple. In particular R is right artinian; so $\mathrm{End}_R(e_1R)$ is a local ring; it follows that e_1R has a unique maximal submodule. Hence J is the maximal and minimal submodule of e_1R, or in other words, e_1R has composition length 2. Put $U = e_1R$ and we have the decomposition of R in (b).

Conversely let R be a ring with a decomposition as in (b). Then the Jacobson radical J of R is the minimal submodule of U_R. Let A be a non-zero right ideal of R. It is enough to show that either A_R is a direct summand of R_R or R/A_R is semisimple. Suppose that A_R is not a direct summand of R_R. Then $A \cap J \neq 0$. It follows that $J \subseteq A$; therefore R/A_R is semisimple.

Since every right PCI (resp. right CPS) ring is a right CPOI (resp. right CPOS) ring, one can easily prove (c) and (d) by using (a) and (b), respectively.

Proof of (II) *of Theorem* 1. (i)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(ii). By Lemma 2, J_R is semisimple. Let M be any minimal submodule of J_R. Then for some non-zero $m \in M$, $M = mR$. Hence $M_R \cong R/r(m)$, where $r(m) = \{x \in R : mx = 0\}$. By (iii), $r(m)$ is not essential in R_R. It follows that $R_R = r(m) \oplus H$ for some right ideal H of R which is R-isomorphic to M_R. Hence M_R is projective, proving that J_R is projective.

(ii)$\Rightarrow$(i). Assume (ii). By Lemma 2, $J_R \subseteq \mathrm{Soc}(R_R)$. Then $J^2 = 0$ and $\mathrm{Soc}(R_R) = J_R \oplus H$, where H_R is a direct sum of finitely many minimal right ideals of R (because a right CPS ring is right noetherian having finite uniform dimension). Thus we can easily check that H_R is generated by an idempotent of R. Hence $\mathrm{Soc}(R_R)$ is projective. To end the proof it is enough to show that any non-projective simple right R-module is injective. Let S be such a module and U a right ideal of R with a homomorphism φ of U_R to S_R. We show that φ can be extended to an R-homomorphism of R to S. Without loss of generality, we can assume that U_R is essential in R_R. If $\varphi(U)=0$, the assertion is trivial. Suppose that $\varphi(U)\neq 0$. We have $S\cong U/\ker\varphi$. If $\ker\varphi$ is not essential in U_R then $U_R=\ker\varphi\oplus T$. Hence $T\subseteq \mathrm{Soc}(R_R)$; so T_R is projective, a contradiction to the non-projectivity of S_R ($\cong T_R$). Hence $\ker\varphi$ has to be essential in U_R. It follows that $\ker\varphi$ is an essential right ideal of R. Since R is right CPS, it follows that $R/\ker\varphi$ is semisimple. Hence there is a submodule L of R_R containing $\ker\varphi$ such that

$$R_R/\ker\varphi = U/\ker\varphi \oplus L/\ker\varphi.$$

Therefore $R=L+U$, $\ker\varphi = L\cap U$, $R/L\cong U/\ker\varphi\cong S$. Combining these facts, we get that φ is extended to a homomorphism in $\mathrm{Hom}_R(R_R, S_R)$, proving the injectivity of S_R.

The proof of Theorem 1 is complete.

EXAMPLES. Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, respectively. Then the matrix ring

$$R=\begin{bmatrix}\mathbb{R} & \mathbb{C}\\ 0 & \mathbb{C}\end{bmatrix}$$

has a decomposition $R_R = U\oplus S$, where $U=\begin{bmatrix}R & \mathbb{C}\\ 0 & 0\end{bmatrix}$, $S=\begin{bmatrix}0 & 0\\ 0 & \mathbb{C}\end{bmatrix}$. Clearly U_R is uniform, having composition length 2, and S_R is simple. Then, by Theorem 1 (I), (b), R is a right CPOS ring. Moreover, R has the following properties:

(1) R is not a direct sum of a semiprime artinian ring and a right PCS domain;
(2) R is not a right PCS ring;
(3) R is not a right CPOI ring;
(4) R is a right CPS ring which is not right CDPI;
(5) the ring theoretic direct sum $R\oplus R$ is a right CPS ring which is not right CPOS.

The above assertions can be proved easily by using Theorem 1.

Denote by $\mathbb{Z}$ the ring of integers. Then $\mathbb{Z}/4\mathbb{Z}$ is a PCS ring which is not PCI.

We do not know an example of a right noetherian right CDPI ring which is not right CPS. We also do not know whether any right CDPI ring is right noetherian.

3. Characterizations of CIS rings. A module M is called uniserial if the set of submodules of M is linearly ordered. A ring R is said to be *right generalized uniserial* if it is right artinian and R_R is a direct sum of uniserial modules. A generalized uniserial ring is a ring which is right and left uniserial.

THEOREM 3. *For a ring R with Jacobson radical J the following conditions are equivalent*:

(a) *every cyclic right R-module is a direct sum of an injective module and a semisimple module*;

https://doi.org/10.1017/S0017089500007801 Published online by Cambridge University Press

(b) *every right R-module is a direct sum of an injective module and a semisimple module*;

(c) *every right R-module is a direct sum of a projective module and a semisimple module*;

(d) *R is a generalized uniserial ring with* $J^2=0$.

Proof. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(d) are proved in [**4**, Theorem 2.6].

(d)$\Rightarrow$(c). Assume (d). Let M be a right R-module. By [**5**, Theorem 25.4.2], $M=\oplus_i M_i$, where each M_i is a finitely generated uniserial module. By [**5**, Theorem 18.23], for each M_i, there exists a primitive idempotent e_i and a submodule F of e_iR such that $M_i \cong e_iR/F$. Since, by [**4**, Theorem 2.6], the length of e_iR is at most 2, M_i is projective or simple. Hence M is a direct sum of a projective module and a semisimple module.

(c)$\Rightarrow$(d). Assume (c). By a well-known theorem of Kaplansky, every projective module is a direct sum of countably generated modules. Hence (c) says that every right R-module is a direct sum of countably generated modules. Hence by [**5**, Theorem 20.23], R is right artinian. Since R is right CPS, by Lemma 2, J_R is semisimple and hence $J^2=0$. Also, by Lemma 2, $R_R=e_1R\oplus\ldots\oplus e_nR$, where $\{e_i\}_{i=1}^n$ is a set of orthogonal idempotents of R and each e_iR is uniform. Suppose that, for a fixed i, the length of e_iR is greater than 1 and let M be the minimal submodule of e_iR. By (c), e_iR/M is semisimple, i.e. M is an intersection of maximal submodules of e_iR. But since R is right artinian, e_iR has a unique maximal submodule. It follows that M is a maximal submodule of e_iR. This shows that the length of e_iR is 2. Further, denote by $E(e_iR)$ the injective hull of e_iR. By (c), $E(e_iR)$ has to be projective because e_iR is uniform and not simple. By [**5**, Theorem 20.15], $E(e_iR)$ is isomorphic to some e_jR with $1\le j\le n$. Hence $E(e_iR)$ has length 2. It follows that $E(e_iR)=e_iR$, i.e. e_iR is injective. Now (d) follows from [**4**, Theorem 2.6].

The proof of Theorem 3 is complete.

We note that the statements of Theorem 3 are left-right symmetric. For short we call a ring in Theorem 3 a CIS ring. One can prove that a ring R with nilpotent Jacobson radical J is generalized uniserial if and only if R/J^2 is a CIS ring.

Since a right PCI domain is not CIS, Theorem 3 is not true if the condition (c) is only required for cyclic right R-modules.

ACKNOWLEDGEMENT. The authors wish to express their gratitude to Patrick F. Smith for his many helpful comments and suggestions. The first author gratefully acknowledges the support of the Alexander von Humboldt-Stiftung for his research.

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INSTITUTE OF MATHEMATICS
P.O. BOX 631 BO HÔ
HANOI, VIETNAM