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Littlewood–Paley Characterizations of Second-Order Sobolev Spaces via Averages on Balls

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Abstract. In this paper, the authors characterize second-order Sobolev spaces $W^{2,p}(\mathbb{R}^n)$, with $p \in [2,\infty)$ and $n \in \mathbb{N}$ or $p \in (1,2)$ and $n \in \{1,2,3\}$, via the Lusin area function and the Littlewood–Paley g_λ^*-function in terms of ball means.

1 Introduction

Due to the lack of the differential structure in metric measure spaces, how to introduce suitable derivatives on metric measure spaces is a challenging topic that has attracted a lot of attention in the past two decades (see, for example, [1, 3, 7–9]). Recall that Hajłasz [7] introduced the notion of what are now called Hajłasz gradients, which have proved a suitable substitute, in some aspects, for the usual derivatives used to develop Sobolev spaces of order 1 on metric measure spaces. Later on, Shanmugalingam [9] developed another type of first order Sobolev spaces on metric measure spaces by means of the notion of upper and weak upper gradients, which have the advantage of locality, comparing with Hajłasz gradients.

From then on, a theory of first order Sobolev spaces on metric measure spaces has been thoroughly investigated and achieved great progress (see, for example, the recent monograph [8]).

Recently, Alabern et al. [1] obtained a new characterization of second-order Sobolev spaces $W^{2,p}(\mathbb{R}^n)$ with $p \in (1,\infty)$, which provides a possible way to introduce second-order Sobolev spaces on metric measure spaces. More precisely, for any $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, let $B(x,t)$ denote the ball with center at $x \in \mathbb{R}^n$ and radius $t \in (0,\infty)$, and let $⨍_B g(y)\,dy$ denote the *integral average* of $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ on a ball $B := B(x,t) \subset \mathbb{R}^n$, namely,

$$⨍_{B(x,t)} g(y)\,dy := \frac{1}{|B(x,t)|}\int_{B(x,t)} g(y)\,dy =: B_t g(x). \tag{1.1}$$

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Alabern et al. in [1, Theorem 2] proved the following conclusion.

Theorem A *Let $p \in (1,\infty)$. Then the following are equivalent:*

(i) $f \in W^{2,p}(\mathbb{R}^n)$.

(ii) *$f \in L^p(\mathbb{R}^n)$ and there exists a function $g \in L^p(\mathbb{R}^n)$ such that $S_2(f,g) \in L^p(\mathbb{R}^n)$, where the square function $S_2(f,g)$ is defined by*

$$S_2(f,g)(x) := \left\{ \int_0^\infty \left| \fint_{B(x,t)} \frac{f(y)-f(x)-B_t g(x)|y-x|^2}{t^2}\,dy \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^n.$$

If $f \in W^{2,p}(\mathbb{R}^n)$, then one can take $g = \Delta f/(2n)$, and if (ii) *holds true, then necessarily $g = \Delta f/(2n)$ almost everywhere. In any of the above cases, $\|S_2(f,g)\|_{L^p(\mathbb{R}^n)}$ is equivalent to $\|\Delta f\|_{L^p(\mathbb{R}^n)}$.*

Recall that $\Delta := \sum_{i=1}^n (\frac{\partial}{\partial x_i})^2$, and, for $p \in (1,\infty)$, the *second-order Sobolev space* $W^{2,p}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^p(\mathbb{R}^n)$ such that their weak derivatives $\partial^\alpha f$ with order $|\alpha| \le 2$ belong to $L^p(\mathbb{R}^n)$. Here and hereafter, for $\alpha := (\alpha_1,\dots,\alpha_n) \in (\{0,1,\dots\})^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha := (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$. For any $f \in W^{2,p}(\mathbb{R}^n)$, let

$$\|f\|_{W^{2,p}(\mathbb{R}^n)} := \sum_{|\alpha|\le 2} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

Notice that $S_2(f,g)$ as above can be reformulated as follows:

$$\mathcal{G}(f,g)(x) := \left\{ \int_0^\infty \left| \frac{B_t f(x) - f(x)}{t^2} - B_t g(x) \right|^2 \frac{dt}{t} \right\}^{\frac{1}{2}}, \quad x \in \mathbb{R}^n, \tag{1.2}$$

with a different g (multiplied a positive constant), which can be seen as the *Littlewood–Paley g-function* of $\frac{B_t f - f}{t^2} - B_t g$. Therefore, it is a natural question to ask whether or not the corresponding Lusin area function and the corresponding Littlewood–Paley g_λ^*-function can characterize $W^{2,p}(\mathbb{R}^n)$. Here, the Lusin area function and the Littlewood–Paley g_λ^*-function are defined, respectively, by setting, for any f, $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{S}(f,g)(x) := \left\{ \int_0^\infty \int_{B(x,t)} \left| \frac{B_t f(y) - f(y)}{t^2} - B_t g(y) \right|^2 dy\, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}} \tag{1.3}$$

and

$$\mathcal{G}_\lambda^*(f,g)(x) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{B_t f(y) - f(y)}{t^2} - B_t g(y) \right|^2 \left(\frac{t}{t+|x-y|} \right)^{\lambda n} dy\, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}}, \tag{1.4}$$

where $\lambda \in (1,\infty)$. For $\lambda \in (1,\infty)$ and $f \in W^{2,p}(\mathbb{R}^n)$, we simply write

$$\mathcal{G}(f) := \mathcal{G}\Big(f, \frac{\Delta f}{2n+4}\Big), \quad \mathcal{S}(f) := \mathcal{S}\Big(f, \frac{\Delta f}{2n+4}\Big), \quad \text{and} \quad \mathcal{G}_\lambda^*(f) := \mathcal{G}_\lambda^*\Big(f, \frac{\Delta f}{2n+4}\Big).$$

The main purpose of this article is to answer the above question, and we have the following conclusions.

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Theorem 1.1 *Let $n \in \mathbb{N} := \{1, 2, \ldots\}$, $p \in (1, \infty)$, and $f \in L^p(\mathbb{R}^n)$. Then the following statements are equivalent:*

(i) $f \in W^{2,p}(\mathbb{R}^n)$;
(ii) *there exists $g \in L^p(\mathbb{R}^n)$ such that $\mathcal{S}(f, g) \in L^p(\mathbb{R}^n)$;*
(iii) *there exists $g \in L^p(\mathbb{R}^n)$ such that $\mathcal{G}_\lambda^*(f, g) \in L^p(\mathbb{R}^n)$, provided that $p \in [2, \infty)$, $n \in \mathbb{N}$, and $\lambda \in (1, \infty)$ or $p \in (1, 2)$, $n \in \{1, 2, 3\}$, and $\lambda \in (2/p, \infty)$.*

Moreover, if $f \in W^{2,p}(\mathbb{R}^n)$, then g in (ii) *and* (iii) *can be taken as $g = \Delta f/(2n + 4)$; while if either of* (ii) *and* (iii) *holds true, then $g = \Delta f/(2n + 4)$ almost everywhere. In any case, $\|\mathcal{S}(f, g)\|_{L^p(\mathbb{R}^n)}$ and $\|\mathcal{G}_\lambda^*(f, g)\|_{L^p(\mathbb{R}^n)}$ are equivalent to $\|\Delta f\|_{L^p(\mathbb{R}^n)}$, respectively.*

The proof of Theorem 1.1 is presented in Section 2. Observe that in Theorem 1.1, when $p \in (1, 2)$, we require $n \in \{1, 2, 3\}$. It is still unclear whether one can remove this restriction; see Remark 2.7 below for more details.

To show Theorem 1.1, following the proof of [1, Theorem 2], we rewrite $B_t f(y) - f(y) - t^2 B_t g(y)$ with $g = \frac{\Delta f}{2n+4}$ as a convolution operator $K_t * (-\Delta f)$, and prove that the kernel K_t satisfies the vector-valued Hörmander condition. Different from the proof of [1, Theorem 2] for $\mathcal{G}(f, g)$, since, comparing with $\mathcal{G}(f, g)$, there exists an additional integral on y in $\mathcal{S}(f, g)$ and $\mathcal{G}_\lambda^*(f, g)$, to show K_t satisfy the vector-valued Hörmander condition, we strongly depend on the local integrability of the kernel K_t, which leads to the restriction $n \in \{1, 2, 3\}$ when $p \in (1, 2)$. On the other hand, in the proof of [1, Theorem 2], the norm estimate $\|\Delta f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{G}(f)\|_{L^p(\mathbb{R}^n)}$ was obtained via the polarization and a duality argument, which is not feasible for us to obtain the norm estimates that

$$\|\Delta f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)} \quad \text{and} \quad \|\Delta f\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{G}_\lambda^*(f)\|_{L^p(\mathbb{R}^n)}$$

in the case when $p \in [2, \infty)$ and $n \geq 4$. To overcome this difficulty, we make use of the fact $\dot{W}^{2,p}(\mathbb{R}^n) = \dot{F}^2_{p,2}(\mathbb{R}^n)$ and prove that

$$\|f\|_{\dot{F}^2_{p,2}(\mathbb{R}^n)} \lesssim \|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)}$$

by means of the usual Lusin area function characterization of the Triebel–Lizorkin space $\dot{F}^2_{p,2}(\mathbb{R}^n)$ (see, for example, [12]) and the Fefferman–Stein vector-valued inequality from [5].

It will be very interesting to clarify whether (ii) of Theorem A is equivalent to (ii) and (iii) of Theorem 1.1 on metric measure spaces. Also, notice that, very recently, Dai et al. [3] established various pointwise characterizations of $W^{2,p}(\mathbb{R}^n)$ via ball means closer to Hajłasz gradients in spirit. It is also unclear, on metric measure spaces, whether those pointwise characterizations and these Littlewood–Paley characterizations are equivalent.

Throughout the article, for any function ϕ, $t \in (0, \infty)$ and $x \in \mathbb{R}^n$, let $\phi_t(x) := t^{-n}\phi(x/t)$. We use C to denote a *positive constant* that is independent of the main parameters involved but whose value may differ from line to line. If $f \leq Cg$, we then write $f \lesssim g$ and, if $f \lesssim g \lesssim f$, we then write $f \sim g$.

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2 Proof of Theorem 1.1

Before we give the proof of Theorem 1.1, we consider the relationship among the Littlewood–Paley g-function in (1.2), the Lusin area function in (1.3), and the Littlewood–Paley g_λ^*-function in (1.4). To this end, let $F: \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ be a non-negative measurable function and, for $x \in \mathbb{R}^n$, define

$$G(F)(x) := \Big\{ \int_0^\infty |F(x,t)|^2 \frac{dt}{t} \Big\}^{\frac12},$$

$$S(F)(x) := \Big\{ \int_0^\infty \int_{B(x,t)} |F(y,t)|^2 \, dy \, \frac{dt}{t^{n+1}} \Big\}^{\frac12},$$

$$G_\lambda^*(F)(x) := \Big\{ \int_0^\infty \int_{\mathbb{R}^n} |F(y,t)|^2 \Big(\frac{t}{t+|x-y|} \Big)^{\lambda n} dy \, \frac{dt}{t^{n+1}} \Big\}^{\frac12},$$

where $\lambda \in (1,\infty)$. Obviously, taking $f,\ g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and setting

$$F(x,t) := \Big| \frac{B_t f(x) - f(x)}{t^2} - B_t g(x) \Big|, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \tag{2.1}$$

we then see that

$$G(F) = \mathcal{G}(f,g), \quad S(F) = \mathcal{S}(f,g), \quad \text{and} \quad G_\lambda^*(F) = \mathcal{G}_\lambda^*(f,g)$$

for $\lambda \in (1,\infty)$. Concerning the relations among the functions $G(F)$, $S(F)$ and $G_\lambda^*(F)$, we have the following conclusions.

Lemma 2.1 *Let $p \in (1,\infty)$, $\lambda \in (1,\infty)$, and $a \in (0,\infty)$. Then there exists a positive constant C such that for all measurable functions F on $\mathbb{R}^n \times (0,\infty)$,*

(i) *for all $x \in \mathbb{R}^n$, $S(F)(x) \le C G_\lambda^*(F)(x)$, where C is independent of F and x;*

(ii) *$\|S_a(F)\|_{L^p(\mathbb{R}^n)} \le C a^{n/\min\{p,2\}} \|S(F)\|_{L^p(\mathbb{R}^n)}$, where C is independent of a and F, and $S_a(F)$ is defined by*

$$S_a(F)(x) := \Big\{ \int_0^\infty \int_{B(x,at)} \big|F(y,t)\big|^2 dy \frac{dt}{t^{n+1}} \Big\}^{\frac12}, \quad a \in (0,\infty), \ \ x \in \mathbb{R}^n;$$

(iii) *for $p \in [2,\infty)$, $\|G_\lambda^*(F)\|_{L^p(\mathbb{R}^n)} \le C \|G(F)\|_{L^p(\mathbb{R}^n)}$, where C is independent of F.*

The proof of Lemma 2.1(i) is obvious, and the remainder can be found in [11, Theorems 4.3 and 4.4, pp. 315-317] and [10, Theorem 2(b), pp. 91-92]; see also [2, Proposition 4].

From Lemma 2.1, we deduce the following conclusion, which might be well known. For the convenience of the reader, we give some details.

Lemma 2.2 *Let $p \in (1,\infty)$ and $\lambda \in (\max\{2/p,1\},\infty)$. Then, for all measurable functions F on $\mathbb{R}^n \times (0,\infty)$, $S(F) \in L^p(\mathbb{R}^n)$ if and only if $G_\lambda^*(F) \in L^p(\mathbb{R}^n)$. Moreover, the $L^p(\mathbb{R}^n)$-norm of $G_\lambda^*(F)$ is equivalent to that of $S(F)$ with the equivalent positive constants independent of F.*

Proof By Lemma 2.1(i), we immediately have $\|S(F)\|_{L^p(\mathbb{R}^n)} \lesssim \|G_\lambda^*(F)\|_{L^p(\mathbb{R}^n)}$. Conversely, for all $\lambda \in (1,\infty)$, we observe that, for all $x \in \mathbb{R}^n$,

$$[G_\lambda^*(F)(x)]^2 \lesssim \sum_{k=0}^{\infty} 2^{-kn\lambda}[S_{2^k}(F)(x)]^2. \tag{2.2}$$

When $p \in (1,2]$, by (2.2) and the triangle inequality, we find that, for all $x \in \mathbb{R}^n$,

$$[G_\lambda^*(F)(x)]^p \lesssim \sum_{k=0}^{\infty} 2^{-\frac{pkn\lambda}{2}}[S_{2^k}(F)(x)]^p,$$

which, together with the Minkowski inequality, Lemma 2.1(ii), and $\lambda > 2/p$, implies that

$$\begin{aligned}\|G_\lambda^*(F)\|_{L^p(\mathbb{R}^n)}^p &\lesssim \sum_{k=0}^{\infty} 2^{-\frac{pkn\lambda}{2}}\|S_{2^k}(F)\|_{L^p(\mathbb{R}^n)}^p \\ &\lesssim \sum_{k=0}^{\infty} 2^{-kn(\frac{p\lambda}{2}-1)}\|S(F)\|_{L^p(\mathbb{R}^n)}^p \lesssim \|S(F)\|_{L^p(\mathbb{R}^n)}^p.\end{aligned}$$

When $p \in [2,\infty)$, again by (2.2), the Minkowski inequality, Lemma 2.1(ii), and $\lambda > 1$, we have

$$\begin{aligned}\|G_\lambda^*(F)\|_{L^p(\mathbb{R}^n)}^2 &= \left\|[G_\lambda^*(F)]^2\right\|_{L^{p/2}(\mathbb{R}^n)} \lesssim \sum_{k=0}^{\infty} 2^{-kn\lambda}\left\|[S_{2^k}(F)]^2\right\|_{L^{p/2}(\mathbb{R}^n)} \\ &\sim \sum_{k=0}^{\infty} 2^{-kn\lambda}\|S_{2^k}(F)\|_{L^p(\mathbb{R}^n)}^2 \lesssim \sum_{k=0}^{\infty} 2^{-kn(\lambda-1)}\|S(F)\|_{L^p(\mathbb{R}^n)}^2 \\ &\lesssim \|S(F)\|_{L^p(\mathbb{R}^n)}^2.\end{aligned}$$

This finishes the proof of Lemma 2.2. ∎

As an immediate consequence of Lemma 2.2 with F as in (2.1), we obtain the equivalence between (ii) and (iii) of Theorem 1.1.

Lemma 2.3 *Let $p \in (1,\infty)$, $\lambda \in (\max\{2/p,1\},\infty)$, and $f, g \in L^p(\mathbb{R}^n)$. Then $\mathcal{S}(f,g) \in L^p(\mathbb{R}^n)$ if and only if $\mathcal{G}_\lambda^*(f,g) \in L^p(\mathbb{R}^n)$. Moreover, the $L^p(\mathbb{R}^n)$-norm of $\mathcal{S}(f,g)$ is equivalent to that of $\mathcal{G}_\lambda^*(f,g)$, with the equivalent positive constants independent of f and g.*

As another application of Lemma 2.1, we have the following observation.

Lemma 2.4 *Let $p \in [2,\infty)$ and $\lambda \in (1,\infty)$. Then there exist positive constants C_1 and C_2 such that, for all $f \in W^{2,p}(\mathbb{R}^n)$, $\|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)} \le C_1\|\mathcal{G}_\lambda^*(f)\|_{L^p(\mathbb{R}^n)} \le C_2\|\Delta f\|_{L^p(\mathbb{R}^n)}$. In particular, for all $f \in W^{2,2}(\mathbb{R}^n)$, $\|\mathcal{S}(f)\|_{L^2(\mathbb{R}^n)}$ and $\|\mathcal{G}_\lambda^*(f)\|_{L^2(\mathbb{R}^n)}$ are both equivalent to $\|\Delta f\|_{L^2(\mathbb{R}^n)}$, with the equivalent positive constants independent of f.*

Proof From Lemma 2.3, Lemma 2.1(iii), and Theorem A, we deduce that, for all $p \in [2,\infty)$, $\lambda \in (1,\infty)$ and $f \in W^{2,p}(\mathbb{R}^n)$,

$$\|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{G}_\lambda^*(f)\|_{L^p(\mathbb{R}^n)} \lesssim \|\mathcal{G}(f)\|_{L^p(\mathbb{R}^n)} \sim \|\Delta f\|_{L^p(\mathbb{R}^n)}.$$

In particular, when $p = 2$, for all $f \in W^{2,2}(\mathbb{R}^n)$, by the Fubini theorem and [1, (11)], we know that

$$(2.3)\quad \|\mathcal{S}(f)\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n}\int_0^\infty \int_{B(x,t)} \Big| \frac{B_t f(y) - f(y)}{t^2} - \frac{B_t(\Delta f)(y)}{2n+4}\Big|^2 \, dy \, \frac{dt}{t^{n+1}}\, dx$$
$$= \int_{\mathbb{R}^n}\int_0^\infty \int_{B(y,t)} \Big| \frac{B_t f(y) - f(y)}{t^2} - \frac{B_t(\Delta f)(y)}{2n+4}\Big|^2 \, dx \, \frac{dt}{t^{n+1}}\, dy$$
$$= \widetilde{c_n}\|\mathcal{G}(f)\|^2_{L^2(\mathbb{R}^n)} = c\|\Delta f\|^2_{L^2(\mathbb{R}^n)},$$

where $\widetilde{c_n}$ and c are positive constants depending only on n. This finishes the proof of Lemma 2.4. ∎

For the case when $p \in (1,2)$, we have the following conclusion.

Lemma 2.5 *Let $p \in (1,2)$ and $n \in \{1,2,3\}$. Then there exists a positive constant C such that, for all $f \in W^{2,p}(\mathbb{R}^n)$, $\|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)} \le C\|\Delta f\|_{L^p(\mathbb{R}^n)}$.*

Proof For all $x \in \mathbb{R}^n \setminus \{0\}$, let

$$I_2(x) := \begin{cases} -\frac{1}{2}|x|, & n = 1,\\ -\frac{1}{2\pi}\log|x|, & n = 2,\\ c_n|x|^{2-n}, & n \ge 3,\end{cases}$$

where c_n is a constant such that, for all $f \in W^{2,p}(\mathbb{R}^n)$, $I_2 * (-\Delta f) = f$. Indeed, $-I_2$ is the standard fundamental solution of the Laplacian; see, for example, [1, p. 603].

Let $\chi(x) := \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x)$ and $\chi_t(x) := t^{-n}\chi(x/t)$ for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$, where $\chi_{B(0,1)}$ denotes the characteristic function of $B(0,1)$. Take $g := -\Delta f$. Then

$$\frac{B_t f - f}{t^2} - \frac{B_t(\Delta f)}{2n+4} = \frac{\chi_t * I_2 * g - I_2 * g}{t^2} + \frac{1}{2n+4}\chi_t * g =: \frac{1}{t^2} K_t * g,$$

where

$$K_t := \chi_t * I_2 - I_2 + \frac{t^2}{2n+4}\chi_t, \quad t \in (0,\infty).$$

Thus, for all $f \in W^{2,p}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{S}(f)(x) = \Big\{ \int_0^\infty \int_{B(x,t)} |K_t * g(y)|^2 \, dy \, \frac{dt}{t^{n+5}}\Big\}^{\frac{1}{2}}$$
$$= \Big\{ \int_0^\infty \int_{B(0,t)} \big|K_t * g(x+y)\big|^2 \, dy \, \frac{dt}{t^{n+5}}\Big\}^{\frac{1}{2}}$$
$$= \Big\{ \int_0^\infty \int_{B(0,t)} \Big|\int_{\mathbb{R}^n} K_t(x+y-z) g(z)\, dz\Big|^2 \, dy \, \frac{dt}{t^{n+5}}\Big\}^{\frac{1}{2}}$$
$$=: \|Tg(x)\|_{L^2(\Sigma)},$$

where

$$\Sigma := \Big(\mathbb{R}^n \times (0,\infty), \frac{\chi_{B(0,t)}(y)}{t^{n+5}}\, dy\, dt\Big)$$

and

$$Tg(x)(y,t) := \int_{\mathbb{R}^n} K_t(x+y-z) g(z)\, dz, \quad x \in \mathbb{R}^n,\ (y,t) \in \mathbb{R}^n \times (0,\infty).$$

By (2.3), we know that for all $f\in W^{2,2}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\big\|Tg(x)\big\|^2_{L^2(\Sigma)}\,dx\sim\|\Delta f\|^2_{L^2(\mathbb{R}^n)}\sim\|g\|^2_{L^2(\mathbb{R}^n)},$$

which further implies that $T: L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n, L^2(\Sigma))$ is a bounded linear operator.

Notice that, to prove Lemma 2.5, it suffices to show that T is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n, L^2(\Sigma))$. To this end, by the vector-valued Calderón–Zygmund theory (see [6, p. 492, Theorem 3.4]), we only need to prove that there exists a positive constant $\widetilde{C}$ such that, for all $z,\ w\in\mathbb{R}^n$,

$$\int_{|x-z|>2|z-w|}\big\|K_t(x+\cdot-z)-K_t(x+\cdot-w)\big\|_{L^2(\Sigma)}\,dx\le\widetilde{C}.$$

We shall prove a strong estimate, say, there exists $\delta\in(0,\infty)$ such that

$$\big\|K_t(x+\cdot-z)-K_t(x+\cdot-w)\big\|_{L^2(\Sigma)}\lesssim\frac{|z-w|^\delta}{|x-z|^{n+\delta}}\quad\text{for all}\ \ |x-z|>2|z-w|. \tag{2.4}$$

If we let $\widetilde{x}:=x-z$ and $\widetilde{z}:=w-z$, then (2.4) becomes

$$\big\|K_t(\widetilde{x}+\cdot)-K_t(\widetilde{x}+\cdot-\widetilde{z})\big\|_{L^2(\Sigma)}\lesssim\frac{|\widetilde{z}|^\delta}{|\widetilde{x}|^{n+\delta}}\quad\text{for all}\ \ |\widetilde{x}|>2|\widetilde{z}|. \tag{2.5}$$

To prove (2.5), we divide K_t into two parts, $\frac{t^2}{2n+4}\chi_t$ and $H_t:=I_2*\chi_t-I_2$, and we estimate these two parts separately.

First we estimate $\frac{t^2}{2n+4}\chi_t$. Notice that if $|\widetilde{x}|>4t$, then, for all $y\in B(0,t)$, $|\widetilde{x}+y|>3t$ and $|\widetilde{x}+y-\widetilde{z}|>t$, and hence $\chi_t(\widetilde{x}+y)=\chi_t(\widetilde{x}+y-\widetilde{z})=0$; while when $|\widetilde{x}|\le 4t$, a simple geometrical observation shows that, for all $t\in(0,\infty)$ and $x,\ y\in\mathbb{R}^n$,

$$\Big|\big[B(x,t)\setminus B(y,t)\big]\cup\big[B(y,t)\setminus B(x,t)\big]\Big|\lesssim t^{n-1}|y-x|,$$

which further implies that

$$\begin{aligned}
&\int_{|\widetilde{x}|/4}^\infty\int_{B(0,t)}t^2\big|\chi_t(\widetilde{x}+y)-\chi_t(\widetilde{x}+y-\widetilde{z})\big|^2\,dy\frac{dt}{t^{n+5}}\\
&\quad\lesssim\int_{|\widetilde{x}|/4}^\infty\Big|\big[B(\widetilde{x},t)\setminus B(\widetilde{x}-\widetilde{z},t)\big]\cup\big[B(\widetilde{x}-\widetilde{z},t)\setminus B(\widetilde{x},t)\big]\Big|\frac{dt}{t^{3n+1}}\\
&\quad\lesssim|\widetilde{z}|\int_{|\widetilde{x}|/4}^\infty\frac{dt}{t^{2n+2}}\sim\frac{|\widetilde{z}|}{|\widetilde{x}|^{2n+1}}.
\end{aligned}$$

Combining the above estimates, we find that when $|\widetilde{x}|>2|\widetilde{z}|$,

$$\Big\|\frac{t^2}{2n+4}\chi_t(\widetilde{x}+\cdot)-\frac{t^2}{2n+4}\chi_t(\widetilde{x}+\cdot-\widetilde{z})\Big\|_{L^2(\Sigma)}\lesssim\frac{|\widetilde{z}|^{1/2}}{|\widetilde{x}|^{n+1/2}}, \tag{2.6}$$

which is desired.

Next we estimate H_t. First we observe that when $|\widetilde{x}|>4t$, we have $|\widetilde{x}+y|>3t$ and $|\widetilde{x}+y-\widetilde{z}|>t$ whenever $y\in B(0,t)$. Thus, there exists $\epsilon_y\in(0,\infty)$ such that $0\notin B(\widetilde{x}+y,t+\epsilon_y)$ and $0\notin B(\widetilde{x}+y-\widetilde{z},t+\epsilon_y)$. Since I_2 is harmonic in $\mathbb{R}^n\setminus\{0\}$, by

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the mean property of harmonic functions, we see that

$$I_2 * \chi_t(\widetilde{x}+y) = \fint_{B(\widetilde{x}+y,t)} I_2(h)\,dh = I_2(\widetilde{x}+y),$$
$$I_2 * \chi_t(\widetilde{x}+y-\widetilde{z}) = \fint_{B(\widetilde{x}+y-\widetilde{z},t)} I_2(h)\,dh = I_2(\widetilde{x}+y-\widetilde{z}).$$

These mean that for all $t < |\widetilde{x}|/4$,

$$\int_{B(0,t)} \big| H_t(\widetilde{x}+y) - H_t(\widetilde{x}+y-\widetilde{z})\big|^2\,dy = 0,$$

and hence

$$\int_0^{|\widetilde{x}|/4} \int_{B(0,t)} \big| H_t(\widetilde{x}+y) - H_t(\widetilde{x}+y-\widetilde{z})\big|^2\,dy\,\frac{dt}{t^{n+5}} = 0.$$

On the other hand, when $|\widetilde{x}| \le 4t$, we see that

$$\begin{aligned}
&\int_{|\widetilde{x}|/4}^{\infty} \int_{B(0,t)} |H_t(\widetilde{x}+y) - H_t(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}}\\
&\quad\le \int_{|\widetilde{x}|/4}^{\infty} \int_{B(0,t)} \big|I_2(\widetilde{x}+y) - I_2(\widetilde{x}+y-\widetilde{z})\big|^2\,dy\,\frac{dt}{t^{n+5}}\\
&\qquad + \int_{|\widetilde{x}|/4}^{\infty} \int_{B(0,t)} |I_2 * \chi_t(\widetilde{x}+y) - I_2 * \chi_t(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}} =: \mathrm{J}_1 + \mathrm{J}_2.
\end{aligned}$$

By the mean value theorem, we know that

$$\big|I_2 * \chi_t(\widetilde{x}+y) - I_2 * \chi_t(\widetilde{x}+y-\widetilde{z})\big| \le |\widetilde{z}| \sup_{\theta\in[0,1]} \big|\nabla I_2 * \chi_t(\widetilde{x}+y-\theta\widetilde{z})\big|,$$

which, together with the fact $|\widetilde{z}| < |\widetilde{x}|/2 \le 2t$ whenever $|\widetilde{x}| \le 4t$, further implies that, for all $y \in B(0,t)$ and $\theta \in [0,1]$,

$$\begin{aligned}
|\nabla I_2 * \chi_t(\widetilde{x}+y-\theta\widetilde{z})| &\lesssim \fint_{B(0,t)} \frac{1}{|\widetilde{x}+y-\theta\widetilde{z}-h|^{n-1}}\,dh\\
&\lesssim \frac{1}{t^n}\int_{B(0,8t)} \frac{1}{|h|^{n-1}}\,dh \lesssim \frac{1}{t^{n-1}},
\end{aligned}$$

since $|\nabla I_2(x)| \lesssim |x|^{1-n}$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Therefore,

$$\mathrm{J}_2 \lesssim \int_{|\widetilde{x}|/4}^{\infty} \int_{B(0,t)} \frac{|\widetilde{z}|^2}{t^{2n-2}}\,dy\,\frac{dt}{t^{n+5}} \sim |\widetilde{z}|^2 \int_{|\widetilde{x}|/4} \frac{1}{t^{2n+3}}\,dt \lesssim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n+2}}. \tag{2.7}$$

To estimate J_1, we notice that when $|y| > 2|\widetilde{x}|$, we have $|\widetilde{x}+y-\theta\widetilde{z}| > |\widetilde{x}|/2$ for all $\theta \in [0,1]$. Hence, by the mean value theorem, we have

$$|I_2(\widetilde{x}+y) - I_2(\widetilde{x}+y-\widetilde{z})| \lesssim \frac{|\widetilde{z}|}{|\widetilde{x}|^{n-1}},$$

and hence

$$\begin{aligned}
&\int_{|\widetilde{x}|/4}^{\infty} \int_{2|\widetilde{x}|<|y|<t} |I_2(\widetilde{x}+y) - I_2(\widetilde{x}+y-\widetilde{z})|^2\,dy\,\frac{dt}{t^{n+5}} \lesssim\\
&\qquad \int_{|\widetilde{x}|/4}^{\infty} \int_{B(0,t)} \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n-2}}\,dy\,\frac{dt}{t^{n+5}} \sim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n-2}} \int_{|\widetilde{x}|/4}^{\infty} \frac{1}{t^5}\,dt \sim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n+2}}.
\end{aligned}$$

When $|y| < |\widetilde{x}|/4$, we have $|\widetilde{x} + y - \theta\widetilde{z}| > |\widetilde{x}|/4$ for all $\theta \in [0,1]$. Similar to the above estimate, we have

$$\big|I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})\big| \lesssim \frac{|\widetilde{z}|}{|\widetilde{x}|^{n-1}},$$

and hence

$$\int_{|\widetilde{x}|/4}^{\infty} \int_{|y|<|\widetilde{x}|/4} |I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})|^2 \, dy \, \frac{dt}{t^{n+5}} \lesssim \int_{|\widetilde{x}|/4}^{\infty} \int_{|y|<|\widetilde{x}|/4} \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n-2}} \, dy \, \frac{dt}{t^{n+5}} \sim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{n-2}} \int_{|\widetilde{x}|/4}^{\infty} \frac{1}{t^{n+5}} \, dt \sim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^{2n+2}}.$$

Therefore, to obtain the desired estimate of J_1, we still need to estimate

$$J_3 := \int_{|\widetilde{x}|/4}^{\infty} \int_{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}|} |I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})|^2 \, dy \, \frac{dt}{t^{n+5}},$$

and this estimate depends on the dimension n. We consider the following three cases.

We first consider the case $n = 1$. In this case, $I_2(x) = \frac{1}{2}|x|$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Observe that $|I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})| \lesssim |\widetilde{z}|$ for all $y \in \mathbb{R}^n$. Then

$$J_3 \le |\widetilde{z}|^2 \int_{|\widetilde{x}|/4}^{\infty} \int_{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}|} dy \, \frac{dt}{t^6} \lesssim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^4}, \tag{2.8}$$

which is a desired estimate.

We now consider the case $n = 2$. In this case, $I_2(x) = -\frac{1}{2\pi}\log|x|$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Again, since I_2 is a radial function, by the mean value theorem, we find that

$$|I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})| \lesssim \frac{|\widetilde{z}|}{|\widetilde{x} + y - \widetilde{z}|} \quad \text{for all} \quad |\widetilde{x} + y - \widetilde{z}| \le |\widetilde{x} + y|. \tag{2.9}$$

From this, we deduce that

$$\begin{aligned}
&\int_{|\widetilde{x}|/4}^{\infty} \int_{\substack{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}| \\ |\widetilde{x}| \le |\widetilde{x}+y-\widetilde{z}| \le |\widetilde{x}+y|}} |I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})|^2 \, dy \, \frac{dt}{t^7} \\
&\quad \lesssim |\widetilde{z}|^2 \int_{|\widetilde{x}|/4}^{\infty} \int_{\substack{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}| \\ |\widetilde{x}| \le |\widetilde{x}+y-\widetilde{z}| \le |\widetilde{x}+y|}} \frac{1}{|\widetilde{x} + y - \widetilde{z}|^2} \, dy \, \frac{dt}{t^7} \\
&\quad \lesssim |\widetilde{z}|^2 \int_{|\widetilde{x}|/4}^{\infty} \int_{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}|} \frac{1}{|\widetilde{x}|^2} \, dy \, \frac{dt}{t^7} \lesssim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^6}.
\end{aligned}$$

If $|\widetilde{x} + y - \widetilde{z}| < |\widetilde{x}| \le |\widetilde{x} + y|$, then $|\widetilde{x} + y - \widetilde{z}| > |\widetilde{x}|/2$ since $|\widetilde{z}| < |\widetilde{x}|/2$, and hence by (2.9), we have

$$\begin{aligned}
&\int_{|\widetilde{x}|/4}^{\infty} \int_{\substack{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}| \\ |\widetilde{x}+y-\widetilde{z}| < |\widetilde{x}| \le |\widetilde{x}+y|}} |I_2(\widetilde{x} + y) - I_2(\widetilde{x} + y - \widetilde{z})|^2 \, dy \, \frac{dt}{t^7} \\
&\quad \lesssim |\widetilde{z}|^2 \int_{|\widetilde{x}|/4}^{\infty} \int_{\substack{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}| \\ |\widetilde{x}+y-\widetilde{z}| < |\widetilde{x}| \le |\widetilde{x}+y|}} \frac{1}{|\widetilde{x} + y - \widetilde{z}|^2} \, dy \, \frac{dt}{t^7} \\
&\quad \lesssim |\widetilde{z}|^2 \int_{|\widetilde{x}|/4}^{\infty} \int_{|\widetilde{x}|/4 \le |y| \le 2|\widetilde{x}|} \frac{1}{|\widetilde{x}|^2} \, dy \, \frac{dt}{t^7} \lesssim \frac{|\widetilde{z}|^2}{|\widetilde{x}|^6}.
\end{aligned}$$

Finally, if $|\widetilde{x}+y|<|\widetilde{x}|$, then

$$|I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})|\sim\Big|\log\frac{|\widetilde{x}+y|}{|\widetilde{x}|}-\log\frac{|\widetilde{x}+y-\widetilde{z}|}{|\widetilde{x}|}\Big|\lesssim\log\frac{|\widetilde{x}|}{|\widetilde{x}+y-\widetilde{z}|}.$$

Fix $\beta\in(0,1)$. Notice that $F(x):=\log(x)/|x|^\beta$ is bounded on $[1,\infty)$. Then by (2.9), we find that

$$\begin{aligned}
&\int_{|\widetilde{x}|/4}^{\infty}\int_{\substack{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|\\|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|<|\widetilde{x}|}}\big|I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})\big|^2\,dy\,\frac{dt}{t^7}\\
&\quad\lesssim|\widetilde{z}|\int_{|\widetilde{x}|/4}^{\infty}\int_{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|}\log\Big(\frac{|\widetilde{x}|}{|\widetilde{x}+y-\widetilde{z}|}\Big)\frac{1}{|\widetilde{x}+y-\widetilde{z}|}\,dy\,\frac{dt}{t^7}\\
&\quad\lesssim|\widetilde{z}|\int_{|\widetilde{x}|/4}^{\infty}\int_{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|}\frac{|\widetilde{x}|^\beta}{|\widetilde{x}+y-\widetilde{z}|^{\beta+1}}\,dy\,\frac{dt}{t^7}\\
&\quad\lesssim|\widetilde{z}||\widetilde{x}|^\beta\int_{|\widetilde{x}|/4}^{\infty}\int_{B(0,\frac32|\widetilde{x}|)}\frac{1}{|y|^{\beta+1}}\,dy\frac{dt}{t^7}\\
&\quad\sim|\widetilde{z}||\widetilde{x}|^\beta\int_{|\widetilde{x}|/4}^{\infty}\int_0^{\frac32|\widetilde{x}|}r^{-\beta}\,dr\,\frac{dt}{t^7}\sim\frac{|\widetilde{z}||\widetilde{x}|^\beta|\widetilde{x}|^{1-\beta}}{|\widetilde{x}|^6}\sim\frac{|\widetilde{z}|}{|\widetilde{x}|^5}.
\end{aligned}$$

Combining the above estimates, we obtain

$$\int_{|\widetilde{x}|/4}^{\infty}\int_{\substack{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|\\|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|}}\big|I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})\big|^2\,dy\,\frac{dt}{t^7}\lesssim\frac{|\widetilde{z}|}{|\widetilde{x}|^5}.$$

By a similar argument, we know that the above estimate remains true if we replace $|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|$ by $|\widetilde{x}+y-\widetilde{z}|>|\widetilde{x}+y|$. Altogether, we obtain

(2.10) $$\mathrm{J}_3\lesssim\frac{|\widetilde{z}|}{|\widetilde{x}|^5},$$

which is a desired estimate for J_3.

Finally, we consider the case $n=3$. In this case, $I_2(x)=c_3|x|^{-1}$ for all $x\in\mathbb{R}^n\setminus\{0\}$. Since I_2 is a radial function, when $|\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|$, we know that

(2.11) $$\big|I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})\big|\lesssim\frac{1}{|\widetilde{x}+y-\widetilde{z}|},$$

and by the mean value theorem, we have

(2.12) $$\big|I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})\big|\lesssim\frac{|\widetilde{z}|}{|\widetilde{x}+y-\widetilde{z}|^2}.$$

Fix $\alpha \in (0,1)$. By (2.11) and (2.12), we see that

(2.13)
$$\begin{aligned}
&\int_{\substack{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|\\ |\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|}} |I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})|^2\,dy\\
&\quad=\int_{\substack{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|\\ |\widetilde{x}+y-\widetilde{z}|\le|\widetilde{x}+y|}} |I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})|^{2-\alpha}|I_2(\widetilde{x}+y)-I_2(\widetilde{x}+y-\widetilde{z})|^{\alpha}\,dy\\
&\quad\lesssim|\widetilde{z}|^{\alpha}\int_{|\widetilde{x}|/4\le|y|\le2|\widetilde{x}|}\frac{1}{|\widetilde{x}+y-\widetilde{z}|^{2-\alpha+2\alpha}}\,dy\lesssim|\widetilde{z}|^{\alpha}\int_{B(0,\frac{7}{2}|\widetilde{x}|)}\frac{1}{|y|^{2+\alpha}}\,dy\\
&\quad\sim|\widetilde{z}|^{\alpha}\int_0^{\frac{7}{2}|\widetilde{x}|}r^{-\alpha}\,dr\sim\frac{|\widetilde{z}|^{\alpha}}{|\widetilde{x}|^{\alpha-1}}.
\end{aligned}$$

The above estimate also holds true when $|\widetilde{x}+y-\widetilde{z}|>|\widetilde{x}+y|$ due to a similar argument. Hence,

$$\mathrm{J}_3\lesssim\frac{|\widetilde{z}|^{\alpha}}{|\widetilde{x}|^{\alpha-1}}\int_{|\widetilde{x}|/4}^{\infty}\frac{dt}{t^8}\sim\frac{|\widetilde{z}|^{\alpha}}{|\widetilde{x}|^{\alpha+6}}\tag{2.14}$$

as desired.

Therefore, by (2.6), (2.7), and (2.8) for $n=1$, (2.10) for $n=2$, or (2.14) for $n=3$, we have (2.5). This finishes the proof of Lemma 2.5. ∎

Remark 2.6 Observe that there exists a restriction $n\in\{1,2,3\}$ in Lemma 2.5. This restriction comes from its proof, which fails when $n\ge4$. Indeed, if $n\ge4$, the integral

$$\int_{B(0,\frac{7}{2}|\widetilde{x}|)}1/|y|^{2+\alpha}\,dy$$

appearing in the proof of Lemma 2.5 in the case when $n=3$ (see (2.13)) should be replaced by

$$\int_{B(0,\widetilde{c}|\widetilde{x}|)}1/|y|^{2(n-2)+\alpha}\,dy,$$

which is infinity when $n\ge4$ and $\alpha\in(0,2)$, where $\widetilde{c}$ is a positive constant independent of $\widetilde{x}$.

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 The equivalence between (ii) and (iii) of Theorem 1.1 is from Lemma 2.2. This equivalence holds true for all $p\in(1,\infty)$ and $n\in\mathbb{N}$.

To complete the proof, it suffices to prove the equivalence between (i) and (ii) of Theorem 1.1. Clearly, (i) ⇒ (ii) follows, respectively, from Lemma 2.4 when $p\in[2,\infty)$ and $n\in\mathbb{N}$, and from Lemma 2.5 when $p\in(1,2)$ and $n\in\{1,2,3\}$. Moreover, in either case, we have

$$\|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)}\lesssim\|\Delta f\|_{L^p(\mathbb{R}^n)},\qquad f\in W^{2,p}(\mathbb{R}^n).\tag{2.15}$$

Now we show (ii) ⇒ (i). Suppose that n and p satisfy either $p\in[2,\infty)$ and $n\in\mathbb{N}$ or $p\in(1,2)$ and $n\in\{1,2,3\}$, and $f,\,g\in L^p(\mathbb{R}^n)$ satisfy that $\mathcal{S}(f,g)\in L^p(\mathbb{R}^n)$. We shall prove that g coincides with Δf modulus a positive constant almost everywhere.

Indeed, let ϕ be a non-negative radial C^∞ function supported in $B(0,1)$ and $\|\phi\|_{L^1(\mathbb{R}^n)}=1$ and, for $x\in\mathbb{R}^n$ and $\epsilon\in(0,\infty)$, let $\phi_\epsilon(x):=\epsilon^{-n}\phi(x/\epsilon)$. Clearly, for

$\epsilon \in (0,\infty)$, $f_\epsilon := f * \phi_\epsilon$ and $g_\epsilon := g * \phi_\epsilon$ are both in $W^{2,p}(\mathbb{R}^n)$. Thus, by the above proved conclusion (i) ⇒ (ii), we know that $\mathcal{S}(f_\epsilon) \in L^p(\mathbb{R}^n)$. By the Minkowski inequality, for all $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
&\mathcal{S}(f_\epsilon, g_\epsilon)(x)\\
&= \Big\{ \int_0^\infty \int_{B(x,t)} \Big| \Big[\frac{B_t f - f}{t^2} - B_t g \Big] * \phi_\epsilon(y) \Big|^2 \, dy \, \frac{dt}{t^{n+1}} \Big\}^{\frac12}\\
&= \Big\{ \int_0^\infty \int_{B(x,t)} \Big| \int_{\mathbb{R}^n} \Big[\frac{B_t f(y-z) - f(y-z)}{t^2} - B_t g(y-z) \Big] \phi_\epsilon(z)\, dz \Big|^2 \, dy \, \frac{dt}{t^{n+1}} \Big\}^{\frac12}\\
&\le \int_{\mathbb{R}^n} \Big\{ \int_0^\infty \int_{B(x+z,t)} \Big| \frac{B_t f(y) - f(y)}{t^2} - B_t g(y) \Big|^2 \, dy \, \frac{dt}{t^{n+1}} \Big\}^{\frac12} \phi_\epsilon(z)\, dz\\
&= \int_{\mathbb{R}^n} \mathcal{S}(f,g)(x+z) \phi_\epsilon(z)\, dz = \mathcal{S}(f,g) * \phi_\epsilon(x).
\end{aligned}$$

For all $x \in \mathbb{R}^n$ and $\epsilon \in (0,\infty)$, define

$$\begin{aligned}
D_\epsilon(x) &:= \Big\{ \int_0^\infty \int_{B(x,t)} \Big| B_t g_\epsilon(y) - \frac{1}{2n+4} B_t(\Delta f_\epsilon)(y) \Big|^2 \, dy \, \frac{dt}{t^{n+1}} \Big\}^{\frac12}\\
&= |B(0,1)|^{\frac12} \Big\{ \int_0^\infty \fint_{B(x,t)} \Big| \fint_{B(y,t)} \Big[g_\epsilon(z) - \frac{1}{2n+4} \Delta f_\epsilon(z) \Big]\, dz \Big|^2 \, dy \, \frac{dt}{t} \Big\}^{\frac12}.
\end{aligned}$$

Then, by the Minkowski inequality, we know that for $x \in \mathbb{R}^n$,

$$D_\epsilon(x) \le \mathcal{S}(f_\epsilon)(x) + \mathcal{S}(f_\epsilon, g_\epsilon)(x) \le \mathcal{S}(f_\epsilon)(x) + \mathcal{S}(f,g) * \phi_\epsilon(x).$$

Thus, $D_\epsilon \in L^p(\mathbb{R}^n)$ for $\epsilon \in (0,\infty)$ and, in particular, for all $\epsilon \in (0,\infty)$, $D_\epsilon(x) < \infty$ almost everywhere. By the Hölder inequality, we find that

$$\begin{aligned}
E_\epsilon(x) &:= |B(0,1)|^{\frac12} \Big\{ \int_0^\infty \Big| \fint_{B(x,t)} \fint_{B(y,t)} \Big[g_\epsilon(z) - \frac{1}{2n+4} \Delta f_\epsilon(z) \Big]\, dz\, dy \Big|^2 \frac{dt}{t} \Big\}^{\frac12}\\
&\le |B(0,1)|^{\frac12} \Big\{ \int_0^\infty \fint_{B(x,t)} \Big| \fint_{B(y,t)} \Big[g_\epsilon(z) - \frac{1}{2n+4} \Delta f_\epsilon(z) \Big]\, dz \Big|^2 \, dy \, \frac{dt}{t} \Big\}^{\frac12}\\
&= D_\epsilon(x).
\end{aligned}$$

Thus, $E_\epsilon(x) < \infty$ almost everywhere. From the Taylor expansion, we deduce that for $\epsilon \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\lim_{t\to 0^+} \fint_{B(x,t)} \fint_{B(y,t)} \Big[g_\epsilon(z) - \frac{1}{2n+4} \Delta f_\epsilon(x) \Big]\, dz\, dy = g_\epsilon(x) - \frac{1}{2n+4} \Delta f_\epsilon(x);$$

here and hereafter, $t \to 0^+$ means that $t \in (0,\infty)$ and $t \to 0$. Thus, for almost every $x \in \mathbb{R}^n$,

$$g_\epsilon(x) - \frac{1}{2n+4} \Delta f_\epsilon(x) = 0. \tag{2.16}$$

By the continuity of g_ϵ and Δf_ϵ, for any $\epsilon \in (0,\infty)$, (2.16) holds true for every $x \in \mathbb{R}^n$. Hence, $\frac{1}{2n+4}\Delta f_\epsilon \to g$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0^+$. Since $f_\epsilon \to f$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0^+$, it follows that $\Delta f_\epsilon \to \Delta f$ as $\epsilon \to 0^+$ in the sense of distribution. Therefore, $\frac{1}{2n+4}\Delta f = g$ almost everywhere, and hence $f \in W^{2,p}(\mathbb{R}^n)$. This proves (ii) ⇒ (i).

Finally, we prove the inverse inequality of (2.15) by borrowing some ideas from [4]. Notice that

$$\begin{aligned}(2.17)\qquad \|\Delta f\|_{L^p(\mathbb{R}^n)} &\sim \|\Delta f\|_{F^0_{p,2}(\mathbb{R}^n)} \sim \|f\|_{\dot F^2_{p,2}(\mathbb{R}^n)}\\ &\sim \Big\|\Big\{\int_0^\infty\Big[\fint_{B(\cdot,t)}|\phi_t * f(y)|\,dy\Big]^2\frac{dt}{t^5}\Big\}^{\frac12}\Big\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

where $\dot F^0_{p,2}(\mathbb{R}^n)$ and $\dot F^2_{p,2}(\mathbb{R}^n)$ denote Triebel–Lizorkin spaces, the second equality in (2.17) is due to the well-known lifting property of Triebel–Lizorkin spaces, and the third one follows from the Lusin area function characterization of Triebel–Lizorkin spaces (see, for example, [12, Theorem 2.8] and its proof). Here we take $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\widehat{\phi}\subset\big\{\xi\in\mathbb{R}^n : 2^{k_0-1}\le|\xi|\le 2^{k_0+1}\big\}$$

and $|\widehat{\phi}(\xi)| \ge$ constant > 0 when $\frac35 2^{k_0} \le |\xi| \le \frac53 2^{k_0}$ for some $k_0 \in \mathbb{Z}$ which will be determined later. (It is well known that different k_0 gives equivalent quasi-norms of $\|f\|_{\dot F^2_{p,2}(\mathbb{R}^n)}$.) Here and hereafter, $\widehat{\phi}$ denotes the Fourier transform of a Schwartz distribution ϕ, and $\check{\phi}$ its inverse Fourier transform. On the other hand, for all $\xi\in\mathbb{R}^n$,

$$\Big(B_t f - f - \frac{t^2}{2n+4}B_t(\Delta f)\Big)^\wedge(\xi) = \Big[m(t\xi) - 1 - \frac{t^2|\xi|^2}{2n+4}m(t\xi)\Big]\widehat{f}(\xi) =: \widetilde{A}(t|\xi|)\widehat{f}(\xi),$$

where

$$\begin{aligned} m(\xi) &:= 1 - 2\gamma_n\int_0^1(1-u^2)^{\frac{n-1}{2}}\Big(\sin\frac{u|\xi|}{2}\Big)^2\,du, \quad \xi\in\mathbb{R}^n,\\ \gamma_n &:= \Big[\int_0^1(1-u^2)^{\frac{n-1}{2}}\,du\Big]^{-1},\\ \widetilde{A}(s) &:= 2\gamma_n\Big(1-\frac{s^2}{2n+4}\Big)\int_0^1(1-u^2)^{\frac{n-1}{2}}\Big(\sin\frac{us}{2}\Big)^2\,du - 1, \quad s\in(0,\infty);\end{aligned}$$

see [4, Lemma 2.1]. Therefore,

$$\phi_t * f = [\widehat{\phi}(t\,\cdot)\widehat{f}(\cdot)]^\vee = \Big[\frac{\widehat{\phi}(t\,\cdot)}{\widetilde{A}(t|\cdot|)}\widetilde{A}(t|\cdot|)\widehat{f}(\cdot)\Big]^\vee =: [\eta(t\,\cdot)\widetilde{A}(t|\cdot|)\widehat{f}(\cdot)]^\vee.$$

Here, $\eta(\xi) := \frac{\widehat{\phi}(\xi)}{\widetilde{A}(|\xi|)}$ for all $\xi\in\mathbb{R}^n$. For all $t\in(0,\infty)$ and $x\in\mathbb{R}^n$, let

$$h_t(x) := B_t f(x) - f(x) - \frac{t^2}{2n+4}B_t(\Delta f)(x) = [\widetilde{A}(t|\cdot|)\widehat{f}]^\vee(x).$$

Then

$$\phi_t * f = [\eta(t\,\cdot)]^\vee * h_t = \check{\eta}_t * g,$$

where $\check{\eta}_t(x) := t^{-n}\check{\eta}(x/t)$ for all $t\in(0,\infty)$ and $x\in\mathbb{R}^n$. Since

$$\Big(1-\frac{s^2}{2n+4}\Big)\int_0^1(1-u^2)^{\frac{n-1}{2}}\Big(\sin\frac{us}{2}\Big)^2\,du \to 0, \quad s\to 0^+,$$

it follows that, when s is small enough, then $|\widetilde{A}(s)| > \frac12$. Thus, we can take k_0 small enough such that $\eta\in C_c^\infty(\mathbb{R}^n)$, and hence $\check{\eta}$ is a Schwartz function. Then, for any

$N \in \mathbb{N}$ and $x \in \mathbb{R}^n$, $|\check{\eta}(x)| \lesssim (1+|x|)^{-N}$, and we see that for all $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
\fint_{B(x,t)} |\phi_t * f(y)|\,dy &= \fint_{B(x,t)} |\check{\eta}_t * h_t(y)|\,dy \le \int_{\mathbb{R}^n} |\check{\eta}(u)| \fint_{B(u-x,t)} |h_t(y)|\,dydu \\
&\lesssim \int_{\mathbb{R}^n} \frac{t^{-n}}{(1+|u+x|/t)^N} \fint_{B(u,t)} |h_t(y)|\,dydu \\
&\sim \int_{|u+x|\le t} \frac{t^{-n}}{(1+|u+x|/t)^N} \fint_{B(u,t)} |h_t(y)|\,dy\,du \\
&\quad + \sum_{k=0}^{\infty} \int_{2^{k-1}t<|u+x|\le 2^k t} \frac{t^{-n}}{(1+|u+x|/t)^N} \fint_{B(u,t)} |h_t(y)|\,dy\,du \\
&\lesssim \fint_{|u+x|\le t} \fint_{B(u,t)} |h_t(y)|\,dy\,du \\
&\quad + \sum_{k=0}^{\infty} 2^{nk} 2^{-N(k-1)} \fint_{|u+x|\le 2^k t} \fint_{B(u,t)} |h_t(y)|\,dy\,du \\
&\lesssim M\Big(\fint_{B(\cdot,t)} |h_t(y)|\,dy \Big)(-x)\Big[1+\sum_{k=0}^{\infty} 2^{-(N-n)k}\Big] \\
&\sim M\Big(\fint_{B(\cdot,t)} |h_t(y)|\,dy \Big)(-x),
\end{aligned}$$

where M denotes the Hardy–Littlewood maximal operator and we took $N > n$. Recall that, for any $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$Mg(x) := \sup_{r\in(0,\infty)} B_r(|g|)(x),$$

where $B_r(|g|)$ is as in (1.1) with t and g replaced, respectively, by r and $|g|$. Therefore, by (2.17), the Fefferman–Stein vector-valued inequality (see [5]), and the Hölder inequality, we have

$$\begin{aligned}
\|\Delta f\|_{L^p(\mathbb{R}^n)} &\lesssim \Big\| \Big(\int_0^\infty \Big[M\Big(\fint_{B(\,\cdot\,,t)} |h_t(y)|\,dy \Big) \Big]^2 \frac{dt}{t^5} \Big)^{\frac12} \Big\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \Big\| \Big\{ \int_0^\infty \Big[\fint_{B(\,\cdot\,,t)} |h_t(y)|\,dy \Big]^2 \frac{dt}{t^5} \Big\}^{\frac12} \Big\|_{L^p(\mathbb{R}^n)} \\
&\lesssim \Big\| \Big\{ \int_0^\infty \int_{B(\,\cdot\,,t)} |h_t(y)|^2\,dy \frac{dt}{t^{n+5}} \Big\}^{\frac12} \Big\|_{L^p(\mathbb{R}^n)} \sim \|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

This finishes the proof of Theorem 1.1. ∎

Remark 2.7 By the last part of the proof of Theorem 1.1, we know that there exists a positive constant C such that for all $n \in \mathbb{N}$, $p \in (1,\infty)$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\Delta f \in L^p(\mathbb{R}^n)$,

$$\|\Delta f\|_{L^p(\mathbb{R}^n)} \le C\|\mathcal{S}(f)\|_{L^p(\mathbb{R}^n)}. \tag{2.18}$$

The *inverse inequality* of (2.18) is only proved when $p \in [2,\infty)$ and $n \in \mathbb{N}$, or $p \in (1,2)$ and $n \in \{1,2,3\}$, while the case when $p \in (1,2)$ and $n \in [4,\infty) \cap \mathbb{N}$ is unknown. We also point out that (2.18) when $p \in (1,2)$ and $n \in \mathbb{N}$ can also be obtained via (2.3), the

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polarization and a well known duality argument, together with (2.15) with $p \in (2, \infty)$ (see [6, p. 507, Remark 5.6]).

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