Mathematical Structures in Computer Science (2024), **34**, pp. 45–62
doi:10.1017/S0960129523000403

PAPER

The order-K-ification monads*

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(Received 14 October 2022; revised 14 November 2023; accepted 23 November 2023;
first published online 21 December 2023)

Abstract

Monads prove to be useful mathematical tools in theoretical computer science, notably in denoting different effects of programming languages. In this paper, we investigate a type of monads which arise naturally from Keimel and Lawson's **K**-ification.

A subcategory of $\mathbf{TOP_0}$ is called of type K* if it consists of monotone convergence spaces and is of type K in the sense of Keimel and Lawson. Each such category induces a canonical monad $\mathcal{K}$ on the category **DCPO** of dcpos and Scott-continuous maps, which is called the order-**K**-ification monad in this paper. First, for each category of type K*, we characterize the algebras of the corresponding monad $\mathcal{K}$ as k-complete posets and algebraic homomorphisms as k-continuous maps, from which we obtain that the order-**K**-ification monad gives the free k-complete poset construction over the category $\mathbf{POS_d}$ of posets and Scott-continuous maps. In addition, we show that all k-complete posets and Scott-continuous maps form a Cartesian closed category. Moreover, we consider the strongness of the order-**K**-ification monad and conclude with the fact that each order-**K**-ification monad is always commutative.

Keywords: dcpos; order-**K**-ification monad; Eilenberg-Moore algebras; k-complete posets

1. Introduction

Non-determinism is an important semantic concept in Theoretical Computer Science and domain theory. This concept offers new insights in designing more powerful programming languages. Pioneering mathematical, in particular, domain-theoretic models considered for non-determinism were due to Plotkin and Smyth in Plotkin (1976) and Smyth (1976). In order to capture the possibilities of multiple outputs in non-deterministic computations, concrete power-domains have been introduced by Hennessy and Plotkin in (1979), and each of these constructions gives rise to a monad. Nowadays, it has been routine to use monads to give denotational semantics to computational effects like non-determinism, and different powerdomain constructions have been proposed, for example, the Hoare, Plotkin, Smyth and Probabilistic powerdomain constructions in domain theory, to name a few.

Generalizing Hennessy and Plotkin's work, in Schalk (1993), Andrea Schalk studied the Hoare power construction on the category **DCPO** of all dcpos and Scott-continuous maps, a general and versatile setting for denotational semantics. She proved that for a dcpo D, the Hoare powerdomain $\mathcal{H}(D)$ of D, comprised of all Scott closed subsets of D under set inclusion, is the free inflationary semilattice of D and useful in modeling the so-called angelic non-determinism.

*This work is supported by the National Natural Science Foundation of China (No.12231007).

The Hoare powerdomain construction itself, seemingly being order-theoretic, can be factored through its topological counterpart $\mathcal{H}_t$. For a topological space X, $\mathcal{H}_t(X)$ is the set of all closed subsets of X equipped with the lower Vietoris topology. The construction $\mathcal{H}_t$ is restricted to an endofunctor on the category **MCS** of monotone convergence spaces, and then, $\mathcal{H}$ can be realized as the composite $\Omega \circ \mathcal{H}_t \circ \Sigma$, where Σ assigns to each dcpo L the topological space $(L, \sigma(L))$ ($\sigma(L)$ is the Scott topology on L), and Ω assigns to each monotone convergence space X the dcpo $(X, \leq)$ with $\leq$ being the specialization order on X. Both Σ and Ω leave morphisms intact. It is easy to see that Σ is left adjoint to Ω.

$$
\begin{tikzcd}
\mathbf{DCPO} \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathbf{MCS} \arrow[l, bend left, "\Omega"] \arrow[loop above, "\mathcal{H}_t"]
\end{tikzcd}
$$

It is interesting to see that many other constructions in domain theory actually arise in a similar fashion. For example, replacing $\mathcal{H}_t$ by the sobrification monad $\mathcal{S}_t$ on **MCS** (this makes sense as all sober spaces are monotone convergence spaces), the composite $\Omega \circ \mathcal{S}_t \circ \Sigma$ gives the so-called order-sobrification monad $\mathcal{S}$ on **DCPO** (Ho et al. 2018). While $\mathcal{H}$ is useful in denotational semantics, $\mathcal{S}$ is employed heavily in giving a satisfactory answer to the Ho-Zhao problem by Ho et al. (2018). That useful application also motivated Jia to systematically investigate the order-sobrification monad $\mathcal{S}$ in Jia (2020).

Canonical categorical reasoning tells us that for each monad $\mathcal{T}$ on **MCS**, the composite $\Omega \circ \mathcal{T} \circ \Sigma$ actually gives rise to a monad on the category **DCPO**. In this paper, we mainly investigate the monads of this specific form, with $\mathcal{T}$ a reflector on the category **MCS**. As the sobrification functor is a reflector on **MCS**, the order-sobrification monad considered in Ho et al. (2018) and Jia (2020) will be subsumed under our work. Indeed, inspired by the reflectivity of **SOB** in $\mathbf{TOP_0}$, reflectors on $\mathbf{TOP_0}$ or equivalently, reflective subcategories of T_0 topological spaces have been studied extensively in domain theory. In particular, it was Keimel and Lawson who first tried to find a class of reflective subcategories in a unified form. They identified in Keimel and Lawson (2009) the following four properties and proved that each subcategory **K** of $\mathbf{TOP_0}$ satisfying them is actually reflective (the objects of **K** are called **K**-spaces).

- (K_1) Homeomorphic copies of **K**-spaces are **K**-spaces.
- (K_2) All sober spaces are **K**-spaces.
- (K_3) In a sober space S, the intersection of any family of **K**-subspaces is a **K**-space.
- (K_4) Continuous maps $f : S \to T$ between sober spaces S and T are **K**-continuous, that is, for every **K**-subspace K of T, the inverse image $f^{-1}(K)$ is a **K**-subspace of S.

Later, Xu focused on the subcategories satisfying (K_2), proposed the concept of adequateness, and proved that each adequate category **K** is reflective in $\mathbf{TOP_0}$ (Xu 2020). More recently, Ershov raised the concept of wide categories and defined **K**-completions in them, in which he also introduced the notion of ample **K**-precompletion. He proved that each wide category admitting an ample **K**-precompletion is reflective in $\mathbf{Top_0}$ (Ershov 2022). Then, the existence of **D**-completions (Ershov 1999; Wyler 1979), $\mathbf{D}_b$-completions (Keimel and Lawson 2009), and **WF**-completions (Liu et al. 2020; Wu et al. 2020) can be realized as corollaries to the aforementioned results. Actually, we will see that the properties of being wide and of possessing an ample **K**-precompletion are not only sufficient conditions for a full subcategory **K** to be reflective but also the necessary ones. Indeed, different completions considered by Keimel and Lawson (2009), Xu (2020) and Ershov (2022) are equivalent.

Given a reflective subcategory **K** of $\mathbf{TOP_0}$, the corresponding reflector $\mathcal{K}_t$, sending each T_0 topological space X to its **K**-ification (many authors also call it the **K**-*space completion*), is a monad on $\mathbf{TOP_0}$ (modulo post-composing with the obvious inclusion functor). The **K**-ification

https://doi.org/10.1017/S0960129523000403 Published online by Cambridge University Press

proves to be useful in denotational semantics, for example, Jia, Lindenhovius, Mislove, and Zamdzhiev employed **K**-ifications to construct commutative probabilistic monads for probabilistic programming languages, solving a long-standing open problem in denotational semantics (Jia et al. 2021).

Starting with a monotone convergence space, however, it is not always the case that $\mathcal{K}_t$ would return a monotone convergence space (see Xu et al. 2020 for example); hence, $\mathcal{K}_t$ cannot be restricted on **MCS** in general, nor $\Omega \circ \mathcal{K}_t \circ \Sigma$ can be well-defined. To avoid that, we consider full subcategories satisfying (K_1), (K_2) and Xu's adequateness that are also contained in **MCS** to ensure that the resulting reflectors return monotone convergence spaces, and call such categories of type K*. Now $\mathcal{K}_t$ induces a monad $\mathcal{K}$ on **DCPO**, which is called *order-***K***-ification monad*. It can also be seen that the monad $\mathcal{K}$ refines $\mathcal{H}$, in the sense that for a dcpo D, $\mathcal{K}$ actually picks a certain subdcpo of $\mathcal{H}(D)$, according to the given category of type K*. Hence, like the Hoare powerdomain monad $\mathcal{H}$, monad $\mathcal{K}$ may also find its uses in semantics.

In this paper, we systematically investigate the order-**K**-ification monads induced by categories of type K*. For each category of type K*, we characterize the Eilenberg-Moore algebras of the resulting $\mathcal{K}$ and the corresponding algebraic homomorphisms, from which we obtain that the Eilenberg-Moore category is precisely **KCPO** of k-complete posets and k-continuous maps. In addition, we find that each category $\mathbf{KCPO}_\sigma$ consisting of all k-complete posets and Scott-continuous maps is Cartesian closed; thus, it could be a model for the λ-calculus. We also verify that $\mathcal{K}$ is always a commutative monad. Hence, each monad $\mathcal{K}$ in this form on the category of dcpos serves as a λ_c-model in the sense of Moggi (1989). In particular, when the category of type K* is chosen to be **SOB**, our order-**SOB**-ification monad is exactly the order-sobrification monad $\mathcal{S}$ proposed in Ho et al. (2018), and all of our results generalize that of Jia in (2020).

2. Preliminaries

Let us introduce the concepts and notions to be used in this paper.

Let P be a poset. A subset A of P is called an *upper set* (resp., a *lower set*) if $A = \uparrow A$ (resp., $A = \downarrow A$), where $\uparrow A = \{x \in P : x \geq a$ for some $a \in A\}$ (resp., $\downarrow A = \{x \in P : x \leq a$ for some $a \in A\}$). A nonempty subset D of P is said to be *directed* if for each finite subset $F \subseteq D$ there exists some $d \in D$ such that $F \subseteq \downarrow d$. Then P is *directed complete* (or a *dcpo*) if every directed subset D of P has a least upper bound, that is, $\sup D$ exists in P. Let $\sigma(P)$ denote the Scott topology on P, where every U in $\sigma(P)$, called *Scott open*, satisfies $U = \uparrow U$ and for any directed subset D for which $\sup D$ exists, $\sup D \in U$ implies $D \cap U \neq \emptyset$. Correspondingly, $A \subseteq P$ is *Scott closed* if $A = \downarrow A$ and for any directed subset D of P contained in A, $\sup D \in A$ when $\sup D$ exists. Until it is otherwise stated, we always equip posets with the Scott topology, and $cl_\sigma(A)$ or $\overline{A}$ is used to denote the closure of $A \subseteq P$ with respect to the Scott topology.

For a T_0 topological space X, the partial order $\leq_X$, defined by $x \leq y$ iff x is contained in the closure of y, is called the *specialization order*. We have that for any $x \in X$, $\downarrow x = cl(\{x\})$ and a continuous map f between two T_0 spaces is always order-preserving. X is called a *monotone convergence space* (or a *d-space*) if every subset D directed relative to the specialization order has a supremum, and the relation $\sup D \in U$ for any open set U of X implies $D \cap U \neq \emptyset$. Let $C(X)$ denote the set of all closed subsets of X. The *lower Vietoris topology* on $C(X)$ is the topology generated by $\{\Diamond U : U \in \mathcal{O}(X)\}$ as a subbase, where $\Diamond U = \{A \in C(X) : A \cap U \neq \emptyset\}$, and the resulting space denoted by $\mathcal{H}_t(X)$ is called *the Hoare power space*, here t in the subscript refers to the fact that the construction acts on *topological* spaces. $A \in C(X)$ is called *irreducible* if for any $B, C \in C(X)$, $A \subseteq B \cup C$ implies that $A \subseteq B$ or $A \subseteq C$. X is called *sober* if every nonempty irreducible closed set is the closure of a point. From Gierz et al. (2003, Exercise V-4.9) we know that there is a sobrification $(\mathbb{S}(X), s_X)$ for each T_0 space X, where the standard construction for the sobrification is to set

$$\mathbb{S}(X) := \{A \subseteq X : A \text{ is closed and irreducible}\}$$

topologized by open sets $U^s := \{A \in \mathbb{S}(X) : A \cap U \neq \emptyset\}$ for each open subset U of X and s_X is a topological embedding from X to $\mathbb{S}(X)$ defined by $s_X(x) = cl(\{x\})$ for each $x \in X$. We call it standard sobrification.

For a general full subcategory **K** of a category **C**, **K** is called *reflective* if the inclusion functor has a left adjoint, which is then called a *reflector* and exhibited in the following way:

Definition 2.1. (Keimel and Lawson 2009) A morphism $\mu : C \to \widetilde{C}$ of an object C in **C** to an object $\widetilde{C}$ is called a *universal* **K**-*ification* if it satisfies the following universal property:

For every object K in **K** and every map $f : C \to K$ in **C**, there is a unique morphism $\tilde{f} : \widetilde{C} \to K$ in **K** such that $\tilde{f} \circ \mu = f$:

$$\begin{tikzcd} C \arrow[r, "\mu"] \arrow[d, "f"'] & \widetilde{C} \arrow[dl, "\tilde{f}"] \\ K & \end{tikzcd}$$

We call $\widetilde{C}$ together with the universal **K**-ification μ a **K**-ification of C. It was Keimel and Lawson who first showed that a full subcategory **K** is reflective in $\mathbf{TOP_0}$ if it satisfies (K_1) to (K_4) mentioned in the Introduction. In this sense of universal **K**-ifications, Xu in (2020) provided another approach to constructing **K**-ifications of T_0 spaces.

Definition 2.2 (Xu 2020). Fix a subcategory **K** of $\mathbf{TOP_0}$ that satisfies (K_2). A subset A of a T_0 space X is called a **K**-*set*, provided for any continuous map $f : X \to Y$ to a **K**-space Y (i.e., Y is an object in **K**), there exists a unique $y_A \in Y$ such that $cl(f(A))$=$cl(\{y_A\})$. Denote by $\mathbf{K}(X)$ the set of all closed **K**-sets of X.

In Xu (2020), Xu called a full subcategory **K** of $\mathbf{TOP_0}$ *adequate* if for any T_0 space X, $\mathcal{K}(X)$, the space obtained by endowing $\mathbf{K}(X)$ with the lower Vietoris topology, is a **K**-space. He proved that when **K** satisfies (K_2) and is adequate, the pair $\langle \mathcal{K}(X), \zeta_X \rangle$, where $\zeta_X : X \to \mathcal{K}(X) : x \mapsto cl(\{x\})$, is a **K**-ification of X.

Recently, Ershov called a full subcategory **K** of $\mathbf{TOP_0}$ *wide* if for any T_0 space X there exists an extension $Y \geq X$ such that $Y \in \mathbf{K}$. For a wide category **K**, he endowed new definitions for **K**-subspaces and **K**-completions and used them to offer sufficient conditions for the existence of **K**-ification of an arbitrary T_0 space (Ershov 2022).

Definition 2.3 (Ershov 2022). Let **K** be a wide subcategory of $\mathbf{TOP_0}$. We say that a continuous map $f : X \to Y$ between two T_0 spaces is **K**-*precomplete* if for the inclusion functor $i : \mathbf{K} \hookrightarrow \mathbf{Top_0}$, the natural transformation $(-) \circ f : \mathbf{Top_0}(Y, i(-)) \to \mathbf{Top_0}(X, i(-))$ is invertible.

Definition 2.4 (Ershov 2022). Let **K** be a wide category. An arbitrary subspace $Z \leq \mathbb{S}(X)$ containing X, for which the natural embedding $s_X : X \to Z$ is **K**-precomplete, is called a **K**-*subspace* for X.

Ershov denoted by $X^{\mathbf{K}}$ the greatest **K**-subspace for X, whose existence is guaranteed by Ershov (2022, Theorem 2.2). In addition, he called $X^{\mathbf{K}}$ a **K**-*completion* of a T_0 space X if $X^{\mathbf{K}} \in \mathbf{K}$.

Definition 2.5. (Ershov 2022) Let **K** be a wide category. A functor $F : \mathbf{TOP_0} \to \mathbf{TOP_0}$ together with a natural transformation $\eta : Id \to F$ is called a **K**-*precompletion* if the map $\eta_X : X \to F(X)$ is **K**-precomplete for any T_0 space X.

https://doi.org/10.1017/S0960129523000403 Published online by Cambridge University Press

A **K**-precompletion (F, η) is referred to as *ample* if, for any T_0 space X, the fact that $\eta_X : X \to F(X)$ is an identity map implies the inclusion $X \in \mathbf{K}$.

Ershov has proved that for a wide category **K**, the existence of an ample **K**-precompletion guarantees the existence of the **K**-completion of an arbitrary T_0 space X (Ershov 2022, Theorem 4.3), in other words, he provided a sufficient condition to make the category **K** reflective in $\mathbf{TOP_0}$. Next we will show that this condition is also necessary.

Proposition 2.6. *Let* **K** *be a wide category. Then, the* **K***-completion exists for each T_0 space X if and only if there is an ample* **K***-precompletion (F, η).*

Proof. "If" is clear from Ershov (2022, Theorem 4.3). Now assume that the **K**-completion exists for each T_0 space X, i.e., $X^{\mathbf{K}} \in \mathbf{K}$. Define $F : \mathbf{TOP_0} \to \mathbf{TOP_0}$ as the form of Ershov (2022, Theorem 3.7) and $\eta_X : X \to F(X) = X^{\mathbf{K}}$ as s_X. Since $X^{\mathbf{K}}$ is a **K**-subspace of X, s_X is **K**-precomplete, so is η_X. If η_X is an identity map, i.e., $X \cong X^{\mathbf{K}}$, then from Ershov (2022, Theorem 3.8), we know the existence of **K**-completions indicates that (K_1) is satisfied. So $X^{\mathbf{K}} \in \mathbf{K}$ implies $X \in \mathbf{K}$. Hence, (F, η) is an ample **K**-precompletion. □

Lemma 2.7 (Xu 2020). *Let* **K** *be a subcategory of* $\mathbf{TOP_0}$ *that satisfies (K_1) and (K_2). If it is adequate, then the following conditions are equivalent for each T_0 space X:*

(1) *X is a* **K***-space.*
(2) $\mathbf{K}(X) = \{\downarrow x : x \in X\}$, *that is, for each $A \in \mathbf{K}(X)$, there exists an $x \in X$ such that $A = cl(\{x\})$.*

Lemma 2.8. *Let Y be a T_0 space and X a subspace of Y. If $A \subseteq X$, then $cl_Y(cl_X(A)) = cl_Y(A)$.*

Proof. It is clear that $cl_X(A) = cl_Y(A)$ for any $A \subseteq X$. So $cl_Y(cl_X(A)) = cl_Y(A)$. □

Proposition 2.9. *If* **K** *is a subcategory of* $\mathbf{TOP_0}$ *satisfying the property (K_2), then it satisfies properties (K_1) to (K_4) if and only if it is closed and adequate.*

Proof. Theorem 5.17 in Xu (2020) told us each subcategory satisfying (K_1) to (K_4) is adequate. Thus, we just need to prove the reverse; that is, **K** satisfies (K_3) and (K_4) if it is closed and adequate.

For (K_3), let $\{X_i : i \in I\}$ be a family of **K**-subspaces of S, where **K**-subspaces mentioned in (K_3) are subspaces of S that are **K**-spaces in the relative topology. Suppose $A \subseteq \bigcap_{i\in I} X_i$ is a **K**-set. Then, A is a **K**-set of S, which implies that $cl_S(A) = cl_S(\{a\})$ for some $a \in S$ since the sober space S is a **K**-space. Meanwhile, A is also a **K**-set of every **K**-space X_i; thus, there is an $a_i \in X_i$ such that $cl_{X_i}(A) = cl_{X_i}(\{a_i\})$. By Lemma 2.8, we have $cl_S(cl_{X_i}(A)) = cl_S(A)$, which means $cl_S(cl_{X_i}(\{a_i\})) = cl_S(\{a\})$. By the fact that $cl_S(cl_{X_i}(\{a_i\})) = cl_S(\{a_i\})$, we have $cl_S(\{a_i\}) = cl_S(\{a\})$. Thus, $a_i = a$ and $a \in \bigcap_{i\in I} X_i$. Therefore, $cl_{\bigcap_{i\in I} X_i}(A) = cl_{\bigcap_{i\in I} X_i}(\{a\})$. So relying on the adequateness, by Lemma 2.7, we have that $\bigcap_{i\in I} X_i$ is a **K**-space.

For (K_4), let S, T be sober spaces and $f : S \to T$ a continuous map. Assume that X is a **K**-subspace of T. For any **K**-set $A \subseteq f^{-1}(X)$, A is also a **K**-set of S. Then, there is an $a \in S$ such that $cl_S(A) = cl_S(\{a\})$. Besides, by the continuity of f, we know $f(A) \subseteq X$ is a **K**-set of T and one can verify that $f(A)$ is also a **K**-set of X. Thus there are two points $a_1 \in X$ and $a_2 \in T$ such that $cl_X(f(A)) = cl_X(\{a_1\})$ and $cl_T(f(A)) = cl_T(\{a_2\})$, respectively. By Lemma 2.8, $cl_T(cl_X(f(A))) = cl_T(f(A))$, thus $a_1 = a_2$. Now we have $f(a) = f(\sup_S A) = \sup_T f(A) = a_2$. As $a_1 = a_2$ and $a_1 \in X$, $f(a) \in X$, that is, $\sup_S A = a \in f^{-1}(X)$. This means $\sup_{f^{-1}(X)} A$ exists and equals to $\sup_S A$. Thus, $cl_{f^{-1}(X)}(A) = cl_{f^{-1}(X)}(\{\sup_{f^{-1}(X)} A\}) = cl_{f^{-1}(X)}(\{\sup_S A\}) = cl_{f^{-1}(X)}(\{a\})$. Therefore, using the adequateness and Lemma 2.7 again, we have that $f^{-1}(X)$ is a **K**-subspace of S. □

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Given that Ershov has shown that a full subcategory $\mathbf{K}$ of $\mathbf{TOP_0}$ satisfies the properties (K_1) to (K_4) if and only if $\mathbf{K}$ is wide and $\mathbf{K}$-completion exists for each T_0 space (see Ershov 2022, Theorem 3.8), then together with Propositions 2.6 and 2.9, we could draw the following conclusion.

Theorem 2.10. *Let* $\mathbf{K}$ *be a full subcategory of* $\mathbf{TOP_0}$. *Then, the following statements are equivalent.*

(1) $\mathbf{K}$ *is a wide category and there exists an ample* $\mathbf{K}$*-precompletion* (F, η) *on the category* $\mathbf{TOP_0}$.
(2) $\mathbf{K}$ *satisfies the properties* (K_1)-(K_4).
(3) $\mathbf{K}$ *satisfies the properties* (K_1), (K_2) *and is adequate.*

It can be seen from the above theorem that the $\mathbf{K}$-ifications of a T_0 space constructed by Keimel and Lawson, Xu, and Ershov respectively are consistent. In our paper, we will mainly use Xu's description for $\mathbf{K}$-ifications. This is because, when constructing an order-$\mathbf{K}$-ification monad on **DCPO** and further examining its algebras, it benefits us a lot if we know concretely what composes such a completion.

3. Categories of Type K*

In what follows, a *category of type* K is defined by satisfying Properties (K_1), (K_2) mentioned in the Introduction and the adequacy property. Categories of type K were initially considered by Keimel and Lawson in (2009), and they are also called K-categories in Jia and Mislove (2022).

Definition 3.1. A full subcategory of $\mathbf{TOP_0}$ of type K is said to be *of type* K* if its objects, called $\mathbf{K}^*$-spaces, also satisfy the following property:

(K_5) All $\mathbf{K}^*$-spaces are monotone convergence spaces.

Remark 3.2. Given a category of type K, the full subcategory of all monotone convergence $\mathbf{K}$-spaces, denoted by $\mathbf{K}^*$, is of type K*. In the following, the category $\mathbf{K}^*$ is always constructed in this way from a category $\mathbf{K}$ of type K.

Example 3.3. **SOB** and **MCS** have been shown to be categories of type K* in Gierz et al. (2003), Keimel and Lawson (2009). Wu et al. proved that **WF** of all well-filtered spaces and continuous maps satisfies the properties (K_1) to (K_4) and thus a category of type K (Wu et al. 2020). As Xi and Lawson in Xi nad Lawson (2017) illustrated that each well-filtered space is a monotone convergence space, **WF** is also a category of type K*, which lies between **SOB** and **MCS**.

The following example distinguishes the category of type K* from that of type K.

Example 3.4. In Xu et al. (2020), Xu introduced the ω-well-filtered spaces and illustrated that an ω-well-filtered space may not be a monotone convergence space. The reader is referred to Xu et al. (2020, Example 4.3) for details. It was proved that the category ω-**WF** of all ω-well-filtered spaces and continuous maps is reflective in $\mathbf{TOP_0}$, which indicates that ω-**WF** satisfies the properties (K_1) to (K_4) by Shen et al. (2021, Theorem 2.16). So ω-**WF** is of type K, but not of type K*.

Definition 3.5. Let X be a T_0 space. $A \subseteq X$ is a $\mathbf{K}^*$-*set* if for any $\mathbf{K}^*$-space Y and any continuous map $f : X \to Y$, there exists a unique element $y_0 \in Y$ such that $cl(f(A)) = cl(\{y_0\})$.

Let $\mathbf{K}^*(X)$ denote the set of all closed $\mathbf{K}^*$-sets of X. Then, A is a $\mathbf{K}^*$-set iff $cl(A) \in \mathbf{K}^*(X)$. In particular, when $\mathbf{K}^*$ is **SOB** or **WF**, a $\mathbf{K}^*$-set of a T_0 space X is indeed an irreducible set or a

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well-filtered determined set defined by Xu in Xu and Zhao (2020), respectively. We will use $\mathbf{WF}(X)$ to denote the family consisting of all closed well-filtered determined sets of X.

Lemma 3.6. *Let X, Y be T_0 spaces and $f : X \to Y$ a continuous map. If $A \subseteq X$ is a $\mathbf{K}^*$-set, then $f(A)$ is a $\mathbf{K}^*$-set of Y.*

Proof. The proof is similar to that of Xu (2020, Lemma 3.11). □

Definition 3.7 (Zhang and Li 2017). A subset A of a space X is called *tapered* if for any continuous map $f : X \to Y$ with Y a monotone convergence space, $\sup f(A)$ always exists in Y.

Lemma 3.8 (Zhang and Li 2017). *Let X be a monotone convergence space. If $A \subseteq X$ is tapered and closed, then $A = \Downarrow(\bigvee A)$.*

Clearly, each directed subset is tapered, which together with the above lemma guarantees the following result:

Lemma 3.9. *Let X be a T_0 space. Then, the following conditions are equivalent:*

(1) *X is a monotone convergence space.*
(2) *For any tapered and closed subset $A \subseteq X$, $A = cl(\{x_0\})$ for some $x_0 \in X$.*

Lemma 3.10. *Let X be a T_0 space. Then, we have*

(1) *Every $\mathbf{K}$-set of X is a $\mathbf{K}^*$-set.*
(2) *Every tapered set of X is a $\mathbf{K}^*$-set.*

Proof. (1): Let Y be a $\mathbf{K}^*$-space and $f : X \to Y$ a continuous map. By definition of a $\mathbf{K}^*$-space, we know Y is a $\mathbf{K}$-space; thus for any $\mathbf{K}$-set $A \subseteq X$, there exists a unique element y such that $cl(f(A)) = cl(\{y\})$ by Lemma 2.7. It follows that A is a $\mathbf{K}^*$-set.

(2): Similarly, we can prove that each tapered set is also a $\mathbf{K}^*$-set by Lemma 3.9. □

Lemma 3.11. *For a T_0 space X, the following conditions are equivalent:*

(1) *X is a $\mathbf{K}^*$-space.*
(2) *For each $\mathbf{K}^*$-set $A \subseteq X$, there exists an element a_0 such that $cl(A) = cl(\{a_0\})$.*

Proof. (1) $\Rightarrow$ (2): Let $id : X \to X$ be the identity map. Its continuity makes a fact that there is an element a_0 such that $cl(A) = cl(\{a_0\})$.

(2) $\Rightarrow$ (1): For any $\mathbf{K}$-set A of X, by Lemma 3.10, A is a $\mathbf{K}^*$-set, thus $cl(A) = cl(\{a_0\})$ for some $a_0 \in A$ by (2). So we conclude that X is a $\mathbf{K}$-space by Lemma 2.7. Similarly, we could show that X is also a monotone convergence space. □

Theorem 3.12. *Let X be a T_0 space. Then, $\mathcal{K}_t(X)$, i.e., $\mathbf{K}^*(X)$ endowed with the lower Vietoris topology is a $\mathbf{K}^*$-space.*

Proof. Assume that $\mathcal{A}$ is a closed $\mathbf{K}^*$-set of $\mathbf{K}^*(X)$. We claim that $\bigcup \mathcal{A}$ is a $\mathbf{K}^*$-set of X. Let Y be a $\mathbf{K}^*$-space and $f : X \to Y$ a continuous map. Then, define a map

$$g : \mathbf{K}^*(X) \to \mathbf{K}^*(Y) : C \mapsto cl(f(C)),$$

https://doi.org/10.1017/S0960129523000403 Published online by Cambridge University Press

whose rationality is guaranteed by Lemma 3.6. For each open set $\Diamond U$ of $\mathbf{K}^*(Y)$, where $U \in \mathcal{O}(Y)$, $g^{-1}(\Diamond U) = \Diamond f^{-1}(U)$, so g is continuous. Since each $A \in \mathcal{A}$ belongs to $\mathbf{K}^*(X)$, there exists a $y_A \in Y$ such that $cl(f(A)) = \downarrow y_A$. It follows that $g(\mathcal{A}) = \{\downarrow y_A : A \in \mathcal{A}\}$ is a $\mathbf{K}^*$-set of $\mathbf{K}^*(Y)$. As Y is a $\mathbf{K}^*$-space, by Lemma 3.11, we could define a map

$$h : \mathbf{K}^*(Y) \to Y : E \mapsto \sup E,$$

one can easily verify that h is continuous. Then, by Lemma 3.6 again, $h(\{\downarrow y_A : A \in \mathcal{A}\}) = \{y_A : A \in \mathcal{A}\}$ is a $\mathbf{K}^*$-set of Y. Hence, $\sup\{y_A : A \in \mathcal{A}\} = y_0$ exists. Then, we have

$$cl(f(\bigcup \mathcal{A})) = cl(\bigcup\{f(A) : A \in \mathcal{A}\}) = cl(\bigcup\{\downarrow y_A : A \in \mathcal{A}\}) = cl(\{y_0\}),$$

which entails that $\bigcup \mathcal{A}$ is a $\mathbf{K}^*$-set. So $\mathcal{A} = cl(\{\bigcup \mathcal{A}\})$ and $\mathcal{K}_t(X)$ is a $\mathbf{K}^*$-space by Lemma 3.11. □

Theorem 3.13. *Let X be a T_0 space. Then, the pair $(\mathcal{K}_t(X), \eta_X)$, where $\eta_X\colon X \to \mathcal{K}_t(X)$, $x \mapsto cl(\{x\})$, is a $\mathbf{K}^*$-ification of X.*

Proof. The proof is similar to that of Xu (2020, Theorem 4.6) which proves that $\langle X^k = \mathcal{K}(X), \zeta_X\rangle$ is a $\mathbf{K}$-ification of X. □

Remark 3.14. (Xu 2020, Theorem 5.14) When a category $\mathbf{K}^*$ of type K^* is specifically taken as **WF**, $\mathcal{K}_t(X)$ is $\mathcal{WF}_t(X)$, i.e., $\mathbf{WF}(X)$ endowed with the lower Vietoris topology, and $\eta_X : X \to \mathcal{WF}_t(X)$ is defined as

$$\text{for any } x \in X,\ \eta_X(x) = \downarrow x.$$

Then, $(\mathcal{WF}_t(X), \eta_X)$ is a well-filtered reflection of X.

4. The Order-K-ification Monad

A *monad* on a category $\mathbf{C}$ consists of an endofunctor $\mathcal{T}$ on $\mathbf{C}$ together with natural transformations $\eta : Id_{\mathbf{C}} \to \mathcal{T}$ and $\mu : \mathcal{T}^2 \to \mathcal{T}$ such that $\mu_A \circ \mathcal{T}\eta_A = Id_{\mathcal{T}A} = \mu_A \circ \eta_{\mathcal{T}A}$ and $\mu_A \circ \mathcal{T}\mu_A = \mu_A \circ \mu_{\mathcal{T}A}$ (Mac Lane 1998).

Let $\mathbf{K}^*$ be a category of type K^*, as in Remark 3.2, determined by certain category of type K. Theorems 3.12 and 3.13 tell us the corresponding reflector $\mathcal{K}_t$, sending each T_0 topological space X to its $\mathbf{K}^*$-*ification* (or the $\mathbf{K}^*$-*space completion*), composing the inclusion functor *Inc* is not only a monad on $\mathbf{TOP_0}$ but can be restricted to **MCS**. Now compose them with the pair of functors Ω and Σ:

$$\begin{tikzcd} \mathbf{DCPO} \arrow[r, bend left, "\Sigma"] \arrow[r, phantom, "\perp"] & \mathbf{MCS} \arrow[l, bend left, "\Omega"] \arrow[r, bend left, "\mathcal{K}_t"] \arrow[r, phantom, "\perp"] & \mathbf{K}^* \arrow[l, bend left, "Inc"] \end{tikzcd}$$

where Σ assigns to each dcpo L the topological space $(L, \sigma(L))$, and Ω assigns to each monotone convergence space X the dcpo $(X, \leq)$ with $\leq$ the specialization order on X. We write $\mathcal{K}_t \circ \Sigma$ as $\mathcal{K}_d$ and $Inc \circ \Omega$ as Inc. Then $\mathcal{K}_d$ is left adjoint to Inc. By the fact that each adjoint pair determines a monad, one can refer to Borceux (1994, Proposition 4.2.1), we know the triple $(\mathrm{Inc} \circ \mathcal{K}_d, \eta, \mathrm{Inc} \circ \varepsilon \circ \mathcal{K}_d)$, where η and ε are the unit and counit respectively, turns into a monad on **DCPO**.

We denote $\mathrm{Inc} \circ \mathcal{K}_d$ with $\mathcal{K}$, for any dcpo L, $\mathcal{K}(L)$ is a dcpo consisting of all closed $\mathbf{K}^*$-sets of $(L, \sigma(L))$ ordered by set inclusion. For each directed family $\mathcal{C}$ of $\mathcal{K}(L)$, one can verify that $\bigcup \mathcal{C}$ is a $\mathbf{K}^*$-set of L, so the supremum of $\mathcal{C}$ in $\mathcal{K}(L)$ is the Scott closure of $\bigcup \mathcal{C}$. Meanwhile, we calculate that $\mathrm{Inc} \circ \varepsilon \circ \mathcal{K}_d$ (replaced by μ_L when it works on a dcpo L) is a natural transformation from $\mathcal{K}(\mathcal{K}(L))$ to $\mathcal{K}(L)$ that maps $\mathcal{A}$ to $\sup_{\mathcal{K}(L)} \mathcal{A}$.

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Lemma 4.1. *Let L be a* dcpo *and $\mathcal{A}$ a Scott closed* $\mathbf{K}^*$*-set of* $\mathcal{K}(L)$. *Then* $\bigcup \mathcal{A} \in \mathcal{K}(L)$.

Proof. The proof of $\bigcup \mathcal{A}$ being a $\mathbf{K}^*$-set is similar to that in Theorem 3.12 and the Scott closure of $\bigcup \mathcal{A}$ one can easily verify. □

This lemma tells us for each $\mathcal{A} \in \mathcal{K}(\mathcal{K}(L))$, $\sup_{\mathcal{K}(L)} \mathcal{A} = \bigcup \mathcal{A}$. Now we conclude that

Theorem 4.2. *The endofunctor $\mathcal{K}$ together with the unit η and the multiplication μ forms a monad, called an order-*$\mathbf{K}$*-ification monad, on* **DCPO**. *Concretely, $\mathcal{K}$ associates with a* dcpo *L the* dcpo *$\mathcal{K}(L)$ and with a morphism $f : L \longrightarrow M$ in* **DCPO** *the map $\mathcal{K}(f) : \mathcal{K}(L) \longrightarrow \mathcal{K}(M)$, which is defined by*

$$\forall A \in \mathcal{K}(L),\ \mathcal{K}(f)(A) = \overline{f(A)};$$

$\eta_L : L \longrightarrow \mathcal{K}(L)$ and $\mu_L : \mathcal{K}(\mathcal{K}(L)) \longrightarrow \mathcal{K}(L)$ are defined by

$$\forall x \in L,\ \eta(x) = \overline{\{x\}},$$

and

$$\forall \mathcal{A} \in \mathcal{K}(\mathcal{K}(L)),\ \mu(\mathcal{A}) = \bigcup \mathcal{A},$$

respectively.

Remark 4.3. When the category **K** is specifically taken as **SOB** or **WF**, the inducing order-**SOB**-ification monad or order-**WF**-ification monad is denoted by $\mathcal{S}$ or $\mathcal{W}$, where $\mathcal{S}$ is actually the order-sobrification monad constructed by Ho et al. to solve the Ho-Zhao problem in Ho et al. (2018).

5. The Eilenberg-Moore Algebras of $\mathcal{K}$

Recall that a $\mathcal{T}$*-algebra* (Mac Lane 1998) of a monad $(\mathcal{T}, \eta, \mu)$ is a pair (C, ξ), where C is an object of **C** and $\xi : \mathcal{T}C \to C$ is a morphism in **C**, that satisfies $\xi \circ \mu_C = \xi \circ \mathcal{T}\xi$ and $\xi \circ \eta_C = id_C$. In this case, ξ is called a *structure map*. If both (A, ξ_A) and (B, ξ_B) are $\mathcal{T}$-algebras, a $\mathcal{T}$*-algebra homomorphism* from (A, ξ_A) to (B, ξ_B) is an arrow $h : A \to B$ satisfying $h \circ \xi_A = \xi_B \circ \mathcal{T}h$.

For a monad $(\mathcal{T}, \eta, \mu)$ on the category **C**, the category of $\mathcal{T}$-algebras determines with respect to what algebraic structures the functor $\mathcal{T}$ can be understood to be universal. More precisely, for each morphism f in **C** which maps A to a $\mathcal{T}$-algebra (C, ξ), there exists a unique $\mathcal{T}$-algebra homomorphism $h : \mathcal{T}A \to C$ such that $f = h \circ \eta_A$. Identifying such structures is not only an interesting mathematical problem, but also could be useful for semantics. One can refer to Jia et al. (2022) for example, where the authors proved that the category of algebras of the subprobability valuation monad on the category of domains is isomorphic to the category of so-called continuous Kegelspitzen, and this result plays a crucial role in giving an adequate semantics to variational quantum programming languages.

Let $\mathbf{C}^{\mathcal{T}}$ denote the category of all $\mathcal{T}$-algebras and their homomorphisms, which is also called the Eilenberg-Moore category of $\mathcal{T}$ over **C**. We proceed to characterize the $\mathcal{K}$-algebras over **DCPO**.

Definition 5.1.
(1) A poset P is called *k-complete* if for every Scott closed $\mathbf{K}^*$-set A of P, $\sup A$ exists.
(2) Let L and M be posets. A map $f : L \to M$ is called *k-continuous* if for any $A \in \mathcal{K}(L)$ whose supremum exists, $f(\sup A) = \sup f(A)$.

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When $\mathbf{K}^*$ is **SOB**, k-complete posets are precisely strongly complete (Ho et al. 2018, Definition 2.1); when $\mathbf{K}^*$ is **WF**, we call a k-complete poset P *w-complete*, that is, every Scott closed well-filtered determined subset of P has a supremum in P.

Since directed subsets are tapered, they are $\mathbf{K}^*$-sets by Lemma 3.10. Then the following results are immediately obtained.

Proposition 5.2. *(1) Every k-complete poset is a dcpo.*
(2) Each k-continuous map between two posets is Scott-continuous.

Readers will be referred to Escardó (1998) for the notions mentioned in the following.

A category $\mathbf{C}$ is called *poset-enriched* if the set of its hom-sets is a poset and its composition operation is monotone. A *poset-functor* between poset-enriched categories is a functor which is monotone on hom-posets. One can easily see that **DCPO** is a poset-enriched category and $\mathcal{K}$ is a poset-functor. If there is a pair of arrows $l : X \to Y$ and $r : Y \to X$ in $\mathbf{C}$ such that $l \circ r \leq \mathrm{id}_Y$ and $r \circ l \geq \mathrm{id}_X$, then l is said to be a *left adjoint* of r, denoted by $l \dashv r$. The adjunction $l \dashv r$ is *reflective* if $l \circ r = \mathrm{id}_Y$. A monad $\mathcal{T} = (\mathcal{T}, \eta, \mu)$ on a poset-enriched category $\mathcal{C}$ is called a *left* KZ*-monad* if and only if $\mathcal{T}$ is a poset-functor and $\mu_C \dashv \eta_{\mathcal{T}C}$ for all $C \in \mathcal{C}$. In addition, if $\mathcal{F} : \mathbf{C} \to \mathbf{C}$ is a poset-functor, we shall say that a map $f : X \to Y$ in $\mathbf{C}$ is a *left $\mathcal{F}$-embedding* if $\mathcal{F}f$ has a right adjoint and the adjunction is reflective.

Proposition 5.3. *$\mathcal{K}$ is a left* KZ*-monad over the poset-enriched category* **DCPO**.

Proof. We just need to prove that $\mu_L \dashv \eta_{\mathcal{K}(L)}$ for any dcpo L. On the one hand, it follows immediately from the monad law that $\mu_L \circ \eta_{\mathcal{K}(L)} = \mathrm{id}_{\mathcal{K}(L)}$ holds. On the other hand, for any $\mathcal{A} \in \mathcal{K}(\mathcal{K}(L))$,

$$\eta_{\mathcal{K}(L)} \circ \mu_L(\mathcal{A}) = \eta_{\mathcal{K}(L)}(\bigcup \mathcal{A}) = \downarrow\{\bigcup \mathcal{A}\} \supseteq \mathcal{A},$$

which means $\eta_{\mathcal{K}(L)} \circ \mu_L \geq \mathrm{id}_{\mathcal{K}(\mathcal{K}(L))}$. So $\mu_L \dashv \eta_{\mathcal{K}(L)}$; hence, $\mathcal{K}$ is a left KZ-monad over **DCPO**. □

We fully characterize the Eilenberg-Moore algebras of the monad $\mathcal{K}$ in the following theorem.

Theorem 5.4. *Let L be a* dcpo*. The following statements are equivalent.*

1. *There exists a structure map $\alpha : \mathcal{K}(L) \to L$ such that (L, α) is a $\mathcal{K}$-algebra.*
2. *L is an injective object over left $\mathcal{K}$-embeddings.*
3. *L is a k-complete poset.*

Proof. (1) ⇒ (2): It is immediate by Escardó (1998, Theorem 4.2.2).

(2) ⇒ (3): One can derive from the monad law that $\mu_L \circ \mathcal{K}\eta_L = id_{\mathcal{K}(L)}$, which reveals that η_L is a left embedding. So there is an extension $m : \mathcal{K}(L) \to L$ of the identity of L along η_L by (2). This entails that $m \circ \eta_L = id_L$. Let A be a Scott closed $\mathbf{K}^*$-set of L. We claim that $\sup A = m(A)$ exists. For each $a \in A$, $\downarrow a \subseteq A$. Since m is monotone, $m(\downarrow a) \leq m(A)$, which means $a = m \circ \eta(a) \leq m(A)$. If b is another upper bound of A, then $A \subseteq \downarrow b$. By the monotonicity of m again, we have $m(A) \leq m(\downarrow b) = b$. Thus, $\sup A = m(A)$ exists and L is a k-complete poset.

(3) ⇒ (1): Since $\sup A$ exists for each $A \in \mathcal{K}(L)$, we could define a map $\alpha : \mathcal{K}(L) \to L$ by

$$\forall A \in \mathcal{K}(L), \alpha(A) = \sup A.$$

Then, one can easily verify that α is Scott-continuous, besides, $\alpha \circ \eta = \mathrm{id}_L$ and $\alpha \circ \mu = \alpha \circ \mathcal{K}\alpha$ hold. So (L, α) is a $\mathcal{K}$-algebra. □

https://doi.org/10.1017/S0960129523000403 Published online by Cambridge University Press

By definitions of the $\mathcal{K}$-algebra homomorphisms and the k-continuity, the following conclusion is obtained immediately, which will help us characterize the Eilenberg-Moore category of $\mathcal{K}$.

Proposition 5.5. *Let L, M be dcpos. If (L, α_L) and (M, α_M) are $\mathcal{K}$-algebras, then $f : L \to M$ is a $\mathcal{K}$-algebra homomorphism if and only if f is k-continuous.*

Now let $\mathbf{DCPO}^{\mathcal{K}}$ denote the category of all $\mathcal{K}$-algebras and all $\mathcal{K}$-algebra homomorphisms, that is the Eilenberg-Moore category of $\mathcal{K}$. Then obviously, $\mathcal{K}$ produces a monadic adjunction:

$$\mathcal{F} : \mathbf{DCPO} \to \mathbf{DCPO}^{\mathcal{K}}, \qquad \mathcal{U} : \mathbf{DCPO}^{\mathcal{K}} \to \mathbf{DCPO},$$

where $\mathcal{F}$ assigns each dcpo L to $(\mathcal{K}(L), \mu_L)$ and each map $f : L \to M$ from L to dcpo M to $\mathcal{K}(f) : (\mathcal{K}(L), \mu_L) \to (\mathcal{K}(M), \mu_M)$, and $\mathcal{U}$ as a forgetful functor is the right adjoint of $\mathcal{F}$. As the characterizations of $\mathcal{K}$-algebras and their homomorphisms imply that $\mathbf{DCPO}^{\mathcal{K}}$ is equivalent to the category **KCPO**, which has the k-complete posets as objects and the k-continuous maps as morphisms, we reach the following conclusion:

Corollary 5.6. *$\mathcal{K}$ gives a free k-complete poset construction over* **DCPO**.

In Zhao and Fan (2010), Zhao and Fan have proved that the category **DCPO** is a reflective full subcategory of the category $\mathbf{POS_d}$ of posets and Scott-continuous maps. Thus combining the above corollary, the following result can be derived immediately:

Corollary 5.7. *$\mathcal{K}$ gives a free k-complete poset construction over* $\mathbf{POS_d}$.

Let $\mathbf{KCPO}_\sigma$ be the full subcategory of **DCPO** which has k-complete posets as objects and Scott-continuous maps as morphisms.

It is a truism that Cartesian closed categories give rise to the models of various typed and untyped λ-calculi and functional programming languages (Lambek 1985; Scott 1976; Streicher 2006). We will see in the following theorem that similar to the category **DCPO**, $\mathbf{KCPO}_\sigma$ is Cartesian closed for each category $\mathbf{K}^*$ of type K*.

Proposition 5.8. *The category* $\mathbf{KCPO}_\sigma$ *is Cartesian closed.*

Proof. Since $\mathbf{KCPO}_\sigma$ is a full subcategory of **DCPO**, we only need to prove that for any k-complete posets L and M, $L \times M$ and $[L \to M]$ (the set of all Scott-continuous maps between L and M) are still k-complete posets.

Claim 1: $L \times M$ is a k-complete poset.

Let $A \subseteq L \times M$ be a Scott closed $\mathbf{K}^*$-set. Then, A is a $\mathbf{K}^*$-set in $\Sigma L \times \Sigma M$. Since the projections π_L and π_M are continuous, $\pi_L(A)$ and $\pi_M(A)$ are $\mathbf{K}^*$-sets of L and M, respectively. The fact that L and M are k-complete posets implies the existence of $\sup(\pi_L(A))$ and $\sup(\pi_M(A))$. One can verify that $(\sup(\pi_L(A)), \sup(\pi_M(A)))$ is the supremum of A. So $L \times M$ is a k-complete poset.

Claim 2: $[L \to M]$ is a k-complete poset.

Let $\{f_i : i \in I\} \subseteq [L \to M]$ be a $\mathbf{K}^*$-set. We define $g : L \to M$ by

$$\forall x \in L, g(x) = \sup\{f_i(x) : i \in I\}.$$

By Lemma II-2.8 and Lemma II-2.9 in Gierz et al. (2003), we know the map $eval_x : [L \to M] \to M$ defined by $eval_x(h) = h(x)$ is continuous. Thus, $eval_x(\{f_i : i \in I\}) = \{f_i(x) : i \in I\}$ is a $\mathbf{K}^*$-set of M.

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Since M is k-complete, $\sup\{f_i(x) : i \in I\}$ exists in M. This means g is well-defined. Obviously, g is Scott-continuous and $g = \sup\{f_i : i \in I\}$. Thus, $[L \to M]$ is k-complete.

In conclusion, $\mathbf{KCPO}_\sigma$ is Cartesian closed. □

6. $\mathcal{K}$ is a Commutative Monad

Recall that a *monoidal category* (see Borceux 1994, Definition 6.1.1) is a category $\mathbf{C}$ equipped with an object $\top$ in $\mathbf{C}$ called *unit*, a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ called *tensor product*, and the natural isomorphisms α, r and l defined as the following forms (for the objects A, B and C):

(i) $\alpha_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$,
(ii) $r_A : A \otimes \top \to A$,
(iii) $l_A : \top \otimes A \to A$,

such that certain equations hold. It will be a *symmetric monoidal category* if in addition it has the natural isomorphism s defined as the form:

(iv) $s_{A,B} : A \otimes B \to B \otimes A$,

such that certain equations hold. It was Kock who first defined strong monads over symmetric monoidal categories (Kock 1972). In this section we will first investigate the strongness of the monad $\mathcal{K}$ induced by the category $\mathbf{K}^*$ of type K^* on **DCPO**, where **DCPO** is clearly a Cartesian monoidal category. In this case, the tensor product $\otimes$ is precisely the categorical product and $\top$ is the terminal object, we use $\times$ and 1 to represent them respectively. Moreover, $\alpha(A, B, C)$, r_A, l_A are defined concretely as follows:

(i) $\alpha_{A,B,C} = ((\pi_1, \pi_1 \circ \pi_2), \pi_2 \circ \pi_2))$,
(ii) $r_A = \pi_1$,
(iii) $l_A = \pi_2$,

where $\pi_i (i = 1, 2)$ denotes the projection onto the ith component.

Definition 6.1. A *strong monad* over a category $\mathbf{C}$ with an object 1 and finite products is a monad $(\mathcal{T}, \eta, \mu)$ together with a natural transformation $t'' : (-) \times \mathcal{T}(=) \to \mathcal{T}(- \times =)$ such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{T}A & \\
1 \times \mathcal{T}A \arrow[u, "l_{\mathcal{T}A}"] \arrow[r, "t''_{1,A}"'] & \mathcal{T}(1 \times A) \arrow[ul, "\mathcal{T}l_A"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[d, "id_A \times \eta_B"'] \arrow[dr, "\eta_{A\times B}"] & \\
A \times \mathcal{T}B \arrow[r, "t''_{A,B}"] & \mathcal{T}(A \times B)
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times B) \times \mathcal{T}C \arrow[rr, "t''_{A\times B,C}"] \arrow[d, "\alpha_{A,B,\mathcal{T}C}"'] & & \mathcal{T}((A \times B) \times C) \arrow[d, "\mathcal{T}\alpha_{A,B,C}"] \\
A \times (B \times \mathcal{T}C) \arrow[r, "id_A \times t''_{B,C}"] & A \times \mathcal{T}(B \times C) \arrow[r, "t''_{A,B\times C}"] & \mathcal{T}(A \times (B \times C))
\end{tikzcd}$$

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$$
\begin{tikzcd}
A \times TB \arrow[rr, "t''_{A,B}"] & & \mathcal{T}(A \times B) \\
A \times \mathcal{T}^2 B \arrow[u, "id_A \times \mu_B"] \arrow[r, "t''_{A,\mathcal{T}B}"] & \mathcal{T}(A \times \mathcal{T}B) \arrow[r, "\mathcal{T}t''_{A,B}"] & \mathcal{T}^2(A \times B) \arrow[u, "\mu_{A\times B}"']
\end{tikzcd}
$$

in which the natural transformation t'' is called *tensorial strength*. Moreover, $t' = \mathcal{T}s \circ t'' \circ s : \mathcal{T}(-) \times (=) \to \mathcal{T}(- \times =)$ is called *cotensorial strength*.

Lemma 6.2. *Let P and Q be posets. Then, $cl_\sigma(A) \times B = cl_\sigma(A \times B)$ for any $A \subseteq P$ and $B \in \Gamma(Q)$.*

Proof. It is obvious that $cl_\sigma(A \times B) \subseteq cl_\sigma(A) \times B$ since $cl_\sigma(A) \times B$ is closed in $\Sigma(P \times Q)$. Now see the reverse. For any $(a, b) \in cl_\sigma(A) \times B$ and $U \in \sigma(P \times Q)$ with $(a, b) \in U$. Set $U_b = \{x \in P : (x, b) \in U\}$. Clearly, $a \in U_b$ and it is an upper set. Let $D \subseteq P$ be a directed subset with $\sup D \in U_b$, that is, $(\sup D, b) \in U$. Then, we have $(d_0, b) \in U$ for some $d_0 \in D$ by the Scott openness of U, which implies that $d_0 \in U_b$. Thus, $U_b \in \sigma(P)$. Then, $A \cap U_b \neq \emptyset$ by $a \in cl_\sigma(A)$, so there is an $a_0 \in A \cap U_b$. It follows that $(a_0, b) \in (A \times B) \cap U$. Therefore, $(a, b) \in cl_\sigma(A \times B)$ and $cl_\sigma(A) \times B \subseteq cl_\sigma(A \times B)$. □

Proposition 6.3. *$\mathcal{K}$ is a strong monad over* **DCPO**.

Proof. Obviously, the direct product $L_1 \times \cdots \times L_n$ of finitely many dcpos $L_1, ..., L_n$ is still a dcpo. Given $\{L, M, N\} \subseteq$ **DCPO**, we define l_L as π_2 and $\alpha_{L,M,N} : (L \times M) \times N \to L \times (M \times N)$ as $(\pi_1 \circ \pi_1, (\pi_2 \circ \pi_1, \pi_2))$. Define $t''_{L,M} : L \times \mathcal{K}(M) \longrightarrow \mathcal{K}(L \times M)$ by

$$\forall (x, A) \in L \times \mathcal{K}(M),\ t''_{L,M}(x, A) = {\downarrow} x \times A.$$

Claim 1: $t''_{L,M}$ is well-defined.

It is sufficient to show that ${\downarrow} x \times A \in \mathcal{K}(L \times M)$. Obviously, ${\downarrow} x \times A$ is closed in $\Sigma(L \times M)$. Now we prove that ${\downarrow} x \times A$ is a $\mathbf{K}^*$-set in $\Sigma(L \times M)$. For any $\mathbf{K}^*$-space Y and the continuous map $f : L \times M \to Y$, we fix $x \in L$ and define $f_x : M \to Y$ as

$$\forall a \in M, f_x(a) = f(x, a),$$

which is well-defined. Now claim that f_x is continuous, i.e., $f_x^{-1}(U) \in \sigma(M)$ for each $U \in \mathcal{O}(Y)$. For any $a_1 \in f_x^{-1}(U)$ and $a_2 \in M$ with $a_1 \leq a_2$, $f_x(a_1) = f(x, a_1) \in U$, that is, $(x, a_1) \in f^{-1}(U) \in \sigma(L \times M)$. The fact $(x, a_2) \geq (x, a_1)$ implies $(x, a_2) \in f^{-1}(U)$, and so $f_x(a_2) \in U$. Thus $f_x^{-1}(U)$ is an upper set. Now let $D \subseteq M$ be a directed subset with $\sup D \in f_x^{-1}(U)$. Then, $(x, \sup D) \in f^{-1}(U)$. It follows that there exists a $d \in D$ such that $(x, d) \in f^{-1}(U)$, that is, $d \in f_x^{-1}(U)$. Thus, f_x is continuous, which guarantees that $f_x(A)$ is a $\mathbf{K}^*$-set of Y by Lemma 3.6. Since Y is a $\mathbf{K}^*$-space, $\overline{f_x(A)} = \overline{\{f(x, a) : a \in A\}} = \overline{\{y_0\}}$ for some $y_0 \in Y$. This means $\overline{f({\downarrow} x \times A)} = \overline{\{y_0\}}$, and hence, ${\downarrow} x \times A$ is a $\mathbf{K}^*$-set of $L \times M$.

Claim 2: $t''_{L,M}$ is Scott-continuous.

We just need to prove $t''_{L,M}$ is Scott-continuous in each component. Fixed $A \in \mathcal{K}(M)$, for any directed subset $D \subseteq L$, we have

$$
\begin{aligned}
t''_{L,M}(\sup D, A) &= {\downarrow}\sup D \times A \\
&= \overline{\bigcup_{d \in D} {\downarrow} d} \times A \\
&= \overline{\bigcup_{d \in D} ({\downarrow} d \times A)} \\
&= \sup\{t''_{L,M}(d, A) : d \in D\},
\end{aligned}
$$

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where Lemma 6.2 guarantees the third equation. It follows that $t''_{L,M}$ is Scott-continuous in the first component. Next, given $x \in L$, let $\{A_i : i \in I\}$ be a directed family of $\mathcal{K}(M)$. We have

$$\begin{aligned} t''_{L,M}(x, \sup_{i\in I} A_i) &= \downarrow x \times \sup_{i\in I} A_i \\ &= \downarrow x \times \overline{\bigcup_{i\in I} A_i} \\ &= \overline{\bigcup_{i\in I} (\downarrow x \times A_i)} \\ &= \sup_{i\in I} t''_{L,M}(x, A_i). \end{aligned}$$

So $t''_{L,M}$ is Scott-continuous in the second component, and it is concluded that $t''_{L,M}$ is Scott-continuous.

Claim 3: $t''_{L,M}$ is a natural transformation, that is, the following diagram commutes:

$$\begin{tikzcd} L \times \mathcal{K}(M) \arrow[r, "t''_{L,M}"] \arrow[d, "f\times\mathcal{K}(g)"'] & \mathcal{K}(L \times M) \arrow[d, "\mathcal{K}(f\times g)"] \\ L' \times \mathcal{K}(M') \arrow[r, "t''_{L',M'}"] & \mathcal{K}(L' \times M') \end{tikzcd}$$

where L', M' are dcpos and $f : L \to L'$, $g : M \to M'$ are Scott-continuous maps.

Now pick $(m, A) \in L \times \mathcal{K}(M)$ to prove that $(\mathcal{K}(f \times g) \circ t''_{L,M})(m, A) = (t''_{L',M'} \circ f \times \mathcal{K}(g))(m, A)$. By the facts that

$$(\mathcal{K}(f \times g) \circ t''_{L,M})(m, A) = \mathcal{K}(f \times g)(\downarrow m \times A) = \overline{(f \times g)(\downarrow m \times A)} = \overline{f(\downarrow m) \times g(A)} \text{ and,}$$
$$(t''_{L',M'} \circ f \times \mathcal{K}(g))(m, A) = t''_{L',M'}(f(m), \overline{g(A)}) = \downarrow f(m) \times \overline{g(A)},$$

we just need to show $\overline{f(\downarrow m) \times g(A)} = \downarrow f(m) \times \overline{g(A)}$. Obviously, $\overline{f(\downarrow m) \times g(A)} \subseteq \downarrow f(m) \times \overline{g(A)}$. On the contrary, for any $x \in \overline{g(A)}$ and $U \in \sigma(L' \times M')$ with $(f(m), x) \in U$. Set $U_{f(m)} = \{y \in M' : (f(m), y) \in U\}$. Then, $x \in U_{f(m)}$ and $U_{f(m)} \in \sigma(M')$. As $x \in \overline{g(A)}$, $U_{f(m)} \cap g(A) \neq \emptyset$, which implies that there exists a $y_0 \in U_{f(m)} \cap g(A)$, that is, $(f(m), y_0) \in (f(\downarrow m) \times g(A)) \cap U \neq \emptyset$. This means $(f(m), x) \in \overline{f(\downarrow m) \times g(A)}$, so $\downarrow f(m) \times \overline{g(A)} \subseteq \overline{f(\downarrow m) \times g(A)}$ holds.

Claim 4: The four diagrams given in Definition 6.1 commute when replacing $\mathcal{T}$ with $\mathcal{K}$ and A, B, C with dcpos L, M, N, respectively.

The proof of the following equations is similar to that of the above, so we omit it.

(i) For any $A \in \mathcal{K}(L)$,

$$(\mathcal{K}l_L \circ t''_{\{1\},L})(1, A) = r_{\mathcal{K}(L)}(1, A),$$

where 1 is the terminal object in **DCPO**.

(ii) For any $(a, b) \in L \times M$,

$$(t''_{L,M} \circ \mathrm{id}_L \times \eta_M)(a, b) = \eta_{L\times M}(a, b).$$

(iii) For any $((a, b), A) \in (L \times M) \times \mathcal{K}(N)$,

$$(\mathcal{K}\alpha_{L,M,N} \circ t''_{L\times M,N})((a, b), A) = (t''_{L,M\times N} \circ \mathrm{id}_L \times t''_{M,N} \circ \alpha_{L,M,\mathcal{K}(N)})((a, b), A).$$

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(iv) For any $(a, \mathcal{A}) \in L \times \mathcal{K}(\mathcal{K}(M))$,

$$(\mu_{L\times M} \circ \mathcal{K}t''_{L,M} \circ t''_{L,\mathcal{K}(M)})(a, \mathcal{A}) = (t''_{L,M} \circ \mathrm{id}_L \times \mu_M)(a, \mathcal{A}). \qquad \square$$

From the construction of the $s_{L,M}$ above, we can obtain the following result.

Corollary 6.4. *$\mathcal{K}(L) \times \mathcal{K}(M) \subseteq \mathcal{K}(L \times M)$, where $\mathcal{K}(L) \times \mathcal{K}(M) = \{A \times B : A \in \mathcal{K}(L), B \in \mathcal{K}(M)\}$.*

Proof. For any $(A, B) \in \mathcal{K}(L) \times \mathcal{K}(M)$, we define the map $t''^{B}_{L,M} : L \to \mathcal{K}(L \times M)$:

$$\forall x \in L,\ t''^{B}_{L,M}(x) = t''_{L,M}(x, B).$$

By the proof of Proposition 6.3, we know $t''^{B}_{L,M}$ is well-defined and Scott-continuous. Thus, $\overline{t''^{B}_{L,M}(A)} = \overline{\{\downarrow x \times B : x \in A\}} \in \mathcal{K}(\mathcal{K}(L \times M))$. It follows that $A \times B = \mu(\overline{\{\downarrow x \times B : x \in A\}}) \in \mathcal{K}(L \times M)$. $\square$

For dcpos L and M, we calculate the cotensorial strength $t'_{L,M} : \mathcal{K}(L) \times M \to \mathcal{K}(L \times M)$ as $t'_{L,M}((C, x)) = C \times \downarrow x$ for any $(C, x) \in \mathcal{K}(L) \times M$. Following the standard categorical terminology (see Kock 1970), the strong monad $(\mathcal{K}, \eta, \mu)$ on **DCPO** is *commutative* if ψ and $\widetilde{\psi}$ agree, where $\psi, \widetilde{\psi} : \mathcal{K}(L) \times \mathcal{K}(M) \to \mathcal{K}(L \times M)$ are defined as follows:

$$\psi_{L,M} = \mu_{L\times M} \circ \mathcal{K}t''_{L,M} \circ t'_{L,\mathcal{K}(M)},$$
$$\widetilde{\psi}_{L,M} = \mu_{L\times M} \circ \mathcal{K}t'_{L,M} \circ t''_{\mathcal{K}(L),M}.$$

It is folklore that computationally, strongness of a monad together with adequacy of the corresponding denotational semantics can be used to establish contextual equivalences for effectful programs (Moggi 1991; Plotkin and Power 2001). Particularly, if $\mathcal{K}$ is commutative, it will carry the structure ψ making $(\mathcal{K}, \eta, \mu)$ into a symmetric monoidal monad on **DCPO**. With this stronger commutative property, we would know that it does not matter which order two instances of the effect appear in programs.

Theorem 6.5. *$\mathcal{K}$ is a commutative monad on* **DCPO**.

Proof. We just need to prove that for any dcpos L, M and $(A, B) \in \mathcal{K}(L) \times \mathcal{K}(M)$, $(\mu_{L\times M} \circ \mathcal{K}t''_{L,M} \circ t'_{L,\mathcal{K}(M)})(A, B) = (\mu_{L\times M} \circ \mathcal{K}t'_{L,M} \circ t''_{\mathcal{K}(L),M})(A, B)$, that is,

$$\bigcup \overline{\{\downarrow a \times B' : a \in A, B' \subseteq B\}} = \bigcup \overline{\{A' \times \downarrow b : A' \subseteq A, b \in B\}}.$$

For convenience, let *lhs* denote the left hand side of the equation and *rhs* the right hand side. Consider each $a \in A$ and $B' \subseteq B$. For any $b' \in B'$, $b' \in B$. Then, $\downarrow a \times \downarrow b' \in \{A' \times \downarrow b : A' \subseteq A, b \in B\}$, which implies $\downarrow a \times \downarrow b' \subseteq rhs$. It follows that $\downarrow a \times B' = \bigcup\{\downarrow a \times \downarrow b' : b' \in B'\} \subseteq rhs$, in other words, $\downarrow a \times B' \in \downarrow rhs$. Then $\overline{\{\downarrow a \times B' : a \in A, B' \subseteq B\}} \subseteq \downarrow rhs$, which means $lhs \subseteq rhs$. $rhs \subseteq lhs$ can be proved similarly. $\square$

Remark 6.6. When $\mathcal{K}$ is specifically taken as $\mathcal{S}$, i.e., the order-sobrification monad proposed by Ho et al. (2018), then $\mathcal{S}$ is commutative, and the conclusion (Jia 2020, Theorem 3.6) given by Jia is generalized by Theorem 6.5.

We conclude our paper with a brief discussion of the advantage of semantic applications of $\mathcal{K}$. We have proved that each $\mathcal{K}$ is a strong monad over the category **DCPO**, this gives rise to the structures of λ_c-models considered by Moggi (1989).

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Definition 6.7. (Moggi 1989) A λ_c-*model* over a category **C** with finite products is a strong monad $(\mathcal{T}, \eta, \mu)$ together with a $\mathcal{T}$-exponential for every pair (A, B) of objects in **C**, i.e., a pair

$$\langle (\mathcal{T}B)^A, eval_{A,\mathcal{T}B} : (\mathcal{T}B)^A \times A \to \mathcal{T}B \rangle$$

satisfying the universal property that for any object C and $f : C \times A \to \mathcal{T}B$, there exists a unique $h : C \to (\mathcal{T}B)^A$, denoted by $\Lambda_{A,\mathcal{T}B}(f)$ s.t.

$$f = eval_{A,\mathcal{T}B} \circ (\Lambda_{A,\mathcal{T}B}(f) \times Id_A).$$

From the above definition, we could clearly see that each strong monad on a Cartesian closed category is a λ_c-model. So each order-**K**-ification monad $\mathcal{K}$ is a λ_c-model by its strongness and the Cartesian closedness of **DCPO**.

Moggi gave the interpretation of a formal system called λ_c-calculus in a λ_c-model, which is sound and complete with respect to the interpretation. In our case, λ_c-calculus is interpreted as morphisms of the Kleisli category for $\mathcal{K}$.

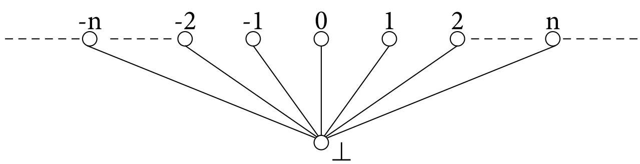
Recall that the Hoare power construction $\mathcal{H}$ on the category **DCPO** is useful for modeling the angelic non-determinism. In particular, one can verify that $\mathcal{H}$ is a strong monad on **DCPO**, so naturally, it is also a λ_c-model. As a refinement of $\mathcal{H}$, there are fewer morphisms in the Kleisli category of each monad $\mathcal{K}$ than that of $\mathcal{H}$. So some useless semantic trash could be culled if we consider using the Kleisli category of $\mathcal{K}$ as semantic categories, to provide a more accurate interpretation for the programming language at hand.

We would end this paper with an example to illustrate that there are indeed fewer morphisms in the Kleisli category of the monad $\mathcal{K}$ for some particular categories of type K, than that of $\mathcal{H}$.

Example 6.8. Let's take $\mathcal{K}$ as the order-sobrification monad $\mathcal{S}$, which assigns $(Irr\Gamma(L), \subseteq)$ to each dcpo L, i.e., the poset of all irreducible closed subsets of ΣL with the inclusion order. Clearly, $\mathcal{S}(L) \subseteq \mathcal{H}(L)$. We take a concrete dcpo $L = \mathbb{Z} \cup \{\perp\}$ with the order $x \leq y$ defined as $x = \perp, y \in \mathbb{Z}$ or $x = y \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, and a dcpo $M = \{0, 1\}$ with the usual order $0 \leq 1$. In denotational semantics, L is the semantic of the type **int** and M is that of the type **unit**. Note that $\Gamma(L) = \{\downarrow A : A \subseteq \mathbb{Z}\} \cup \{\perp\}$, $Irr\Gamma(L) = \{\downarrow n : n \in \mathbb{Z}\} \cup \{\perp\}$. We define a map $f : M \to \mathcal{H}(L)$ as

$$f(m) = \begin{cases} \perp, & \mathrm{m} = 0 \\ \downarrow\{1, 2\}, & \mathrm{m} = 1. \end{cases}$$

The map f is Scott-continuous since it is monotone and M is finite; hence, it is in $\mathbf{DCPO}_{\mathcal{H}}$. However, there is no Scott-continuous map $g : M \to \mathcal{S}(L)$ in $\mathbf{DCPO}_{\mathcal{S}}$ such that $f = i \circ g$, where $i : \mathcal{S}(L) \to \mathcal{H}(L)$ is the canonical inclusion map. This is obvious because the images of $i \circ g$ consist of principal ideals, while $f(1)$ is not a principal ideal.



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https://doi.org/10.1017/S0960129523000403 Published online by Cambridge University Press

Acknowledgements. The authors would like to thank the referee for the numerous and very helpful suggestions that have improved this paper substantially.

Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Cite this article: Hou H, Miao H and Li Q (2024). The order-**K**-ification monads. *Mathematical Structures in Computer Science* **34**, 45–62. https://doi.org/10.1017/S0960129523000403